Lecture 30

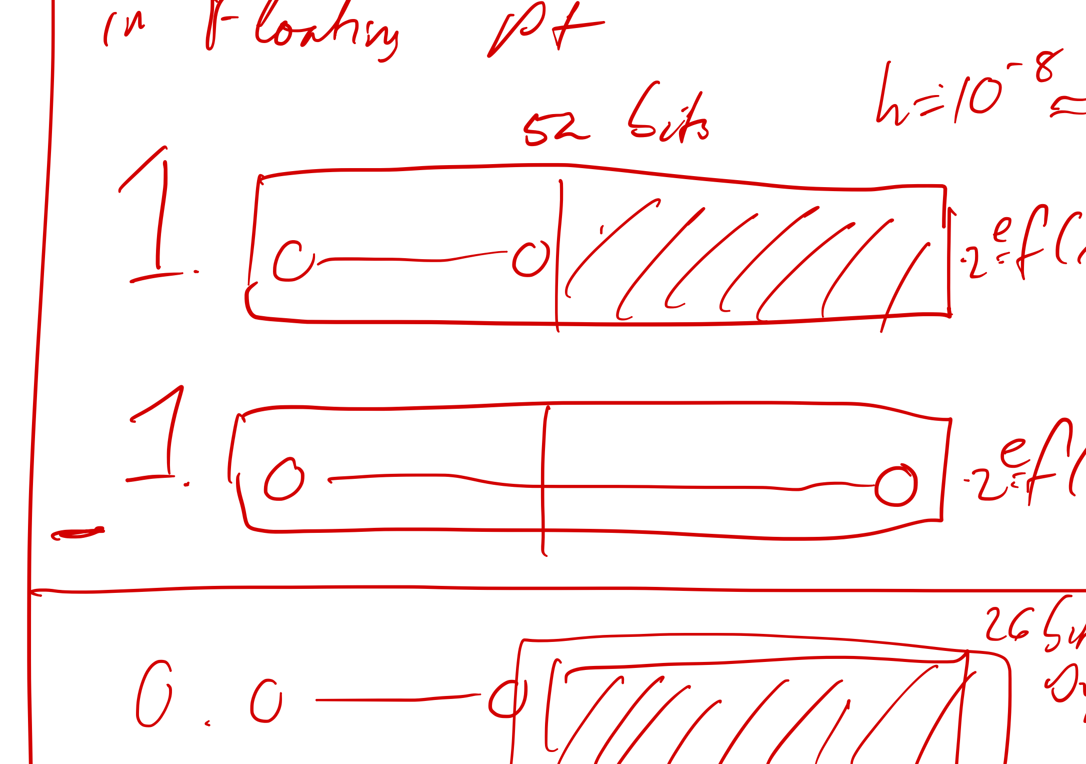
Forward Diff
$h \approx 10^{-8}$

Central Diff
$h = 10^{-5}$

Why?

Consider $\dfrac{f(x+h) - f(x)}{h}$

in Floating pt



Soln only has 26 bits of accuracy ☹

Why does Error increase:

Catastrophic cancellation

as $h \to 0$, # of bits left $\to 0$. too

☹

Numerical Analysis

$\hat{f}(x \oplus h) = \underbrace{f(x+h)(1+\delta_1)}_{\text{assume all floatn pt. Error in funct. Eval.}}$

$\hat{f}(x) = f(x)(1+\delta_2)$

$|\delta_1| \le \varepsilon$

Now assume Exact

FP Eval of $(\hat{f}(x+h) - \hat{f}(x))/h$

$\dfrac{\hat{f}(x+h) - \hat{f}(x)}{h} =$

$\dfrac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} =$

$\dfrac{f(x+h) - f(x)}{h} + \boxed{\dfrac{f(x+h)\delta_1 - f(x)\delta_2}{h}}$

Error! $\le \dfrac{|f(x+h) - f(x)| \cdot \max(|\delta_1|, |\delta_2|)}{h}$ $\le \varepsilon$

$|\text{Error}| \le |f(x+h) - f(x)| \cdot \dfrac{\varepsilon}{h}$

Error for FP Eval is

$O(\frac{\varepsilon}{h})$

as $h \to 0$, then

Error $\to \infty$.

☹

Overall

Let's let

L be a forward diff approx of $f'(x)$

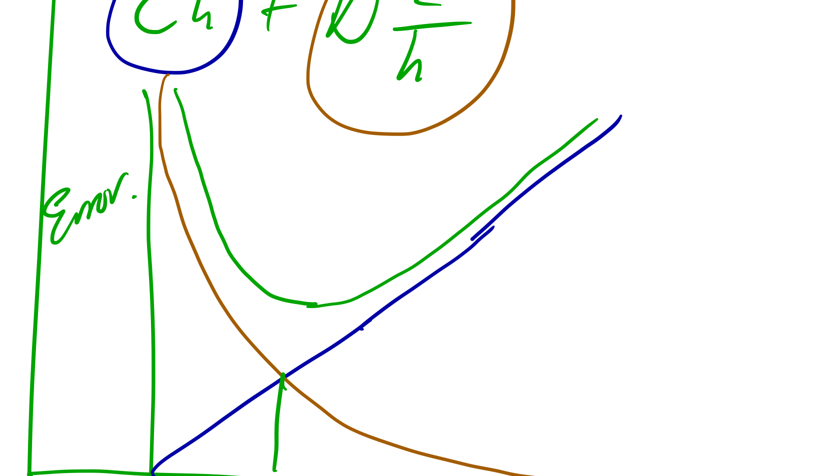
$L = f'(x) + O(h)$
$= f'(x) + Ch$

$\hat{L}$ = float pt. approx of L

$\hat{L} = L + O(\frac{\varepsilon}{h})$
$= L + D\frac{\varepsilon}{h}$

Overall Error:

$Ch + D\frac{\varepsilon}{h}$



Goal: set h s.t.

$Ch = D\frac{\varepsilon}{h}$

Suppose $C, D \approx 1 \Rightarrow$ ignore them!

$h = \frac{\varepsilon}{h} \Rightarrow$

$h^2 = \varepsilon \Rightarrow$

$h = \sqrt{\varepsilon}$

$\varepsilon \approx 10^{-16}$

$\Rightarrow$

$h = 10^{-8}$ ☺!

Next: Central!

$\cancel{f'(x)}$

L = Central Diff approx of $f'(x)$

$L = f'(x) + O(h^2)$
$= f'(x) + Ch^2$

$\hat{L} = L + O(\frac{\varepsilon}{h}) = L + D\frac{\varepsilon}{h}$

Quiz:

What is opt h for Central Diff?

Soln

Assume $C, D \approx 1$

$\frac{\varepsilon}{h} = h^2 \Rightarrow$

$\varepsilon = h^3 \Rightarrow$

$h = \sqrt[3]{\varepsilon}$

$= 10^{-16/3}$

$\approx 10^{-5}$

Richardson Extrapl

Idea: use Multiple h's! (Obvious!)

$f(x+h) = \cancel{f(x)} + hf'(x) + \cancel{\frac{h^2}{2}f''(x)} + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5)$

$-$

$f(x-h) = \cancel{f(x)} - hf'(x) + \cancel{\frac{h^2}{2}f''(x)} - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5)$

$= 0 + 2hf'(x) + \frac{2h^3}{6}f'''(x) + 0 + O(h^5)$

Central Diff $\Rightarrow$

$\dfrac{f(x+h) - f(x-h)}{2h} = f'(x) + \dfrac{h^2}{6}f'''(x) + O(h^4)$

Idea: use $h/2$ too well:

$\dfrac{f(x+h/2) - f(x-h/2)}{h} =$

$f'(x) + \dfrac{(\frac{h}{2})^2 f'''(x)}{6} + O(h^4)$

Idea: Call

$\dfrac{f(x+h) + f(x-h)}{2h} = L_1$

$\dfrac{f(x+h/2) + f(x-h/2)}{h} = L_2$

$\frac{4}{3}L_1 + \frac{1}{3}L_2$,