Lecture 29

Numerical Differentiation

given a computer representation of $f(x)$

Estimate, Compute or approx

$f'(x)$ at some point x

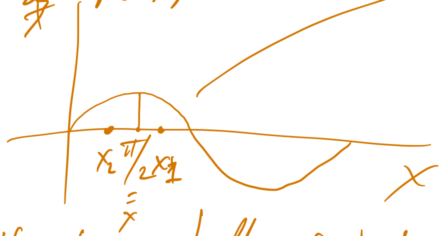
- * poly
- * grid of pts
- * sin / cosines
- * input/output

↳ subroutine

function ~~halloween~~ pu (x)
return sin(x)
End.

plot ~~halloween~~ (x)

~~f~~ pu(x)



How to use halloween(x) to Estimate

$\boxed{\text{~~halloween~~ pu}'(x)\ ?}$

~~halt'(x) ≈~~

Central Diff

$$pu'(x) \approx \frac{pu(x_1) - pu(x_2)}{x_2 - x_1}$$

$x_1 = x + \varepsilon$ ← machine

$x_2 = x - \varepsilon$

$$pu'(x) \approx \frac{pu(x+\varepsilon) - pu(x)}{\varepsilon}$$

Forward Diff

Exact

$pu'(\frac{\pi}{2}) = 0$

Forward Diff: $= 0$

Central $= 0$

@ $\pi/3$ w/ $\varepsilon = 0.$

Central $= 0.49$

Forward $= 0.495$

Exact $= 0.5$

01

99916 ..

6 ...

Summary

Forward

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

Central

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

Theorem: (Taylor)

Assume f is "smooth enough"

Taylor!

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{4!} f''''(x) + \ldots$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(\xi_x)$$

$$\xi_x \in [x, x+h]$$

→ Rewrite

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \boxed{\frac{h}{2} f''(\xi_x)}$$

⟶ Called Truncation Error!

This is called $O(h)$ truncation Error.

Central Diff:

Taylor:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(\xi_x)$$

also for f(x-h) too!

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(\xi_x)$$

$$\frac{f(x+h)-f(x-h)}{2h} = \underbrace{f'(x) + \frac{h^2}{12} f'''(\xi_x) + \frac{h^2}{12} f'''(\xi_x) + \frac{h^2}{6}(\text{ugly})}_{} \rightarrow O(h^2)$$