Lecture 22

Conditioning

How sensitive a problem is to changes in the input

Stability

How "good" is my algorithm for the problem.

Suppose $Alg(x) \approx f(x)$

Forward Error

$$\frac{|Alg(x) - f(x)|}{|f(x)|}$$

Backwards Error.

$Alg(x) = f(x + \delta x)$ for δx small

∘ the Alg is $\boxed{\text{Correct}}$ for a near by input.

Example

$a \oplus b = (a+b)(1+\delta_1)$

$= a(1+\delta_1) + b(1+\delta_1)$

$= \tilde{a} + \tilde{b}$

$x^T y = \sum x_i \otimes y_i$

An Alg is backwards Stable for f if $Alg(x) = f(x+\delta x)$

Matrix Norms

$|\alpha| = \max |\alpha x|$ where $|x| = 1$

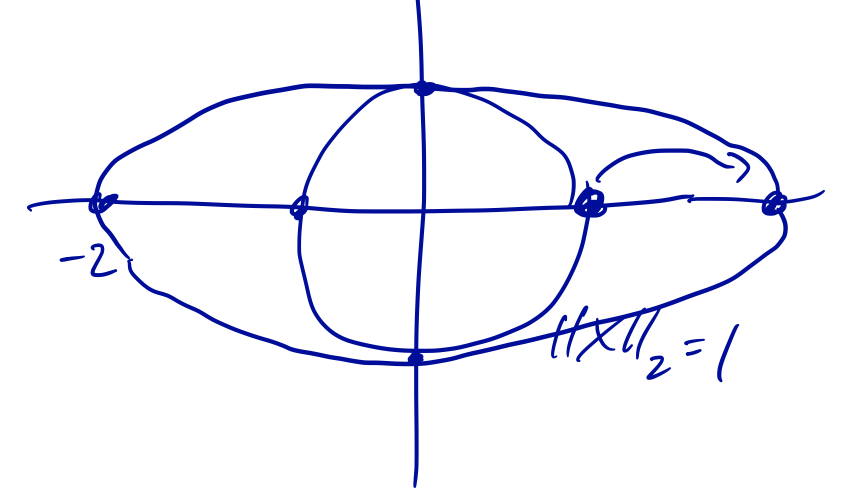
Matrix Norm via Vector norms

$\|A\|_2 = \max \|Ax\|_2$ when $\|x\|_2 = 1$

Example

$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

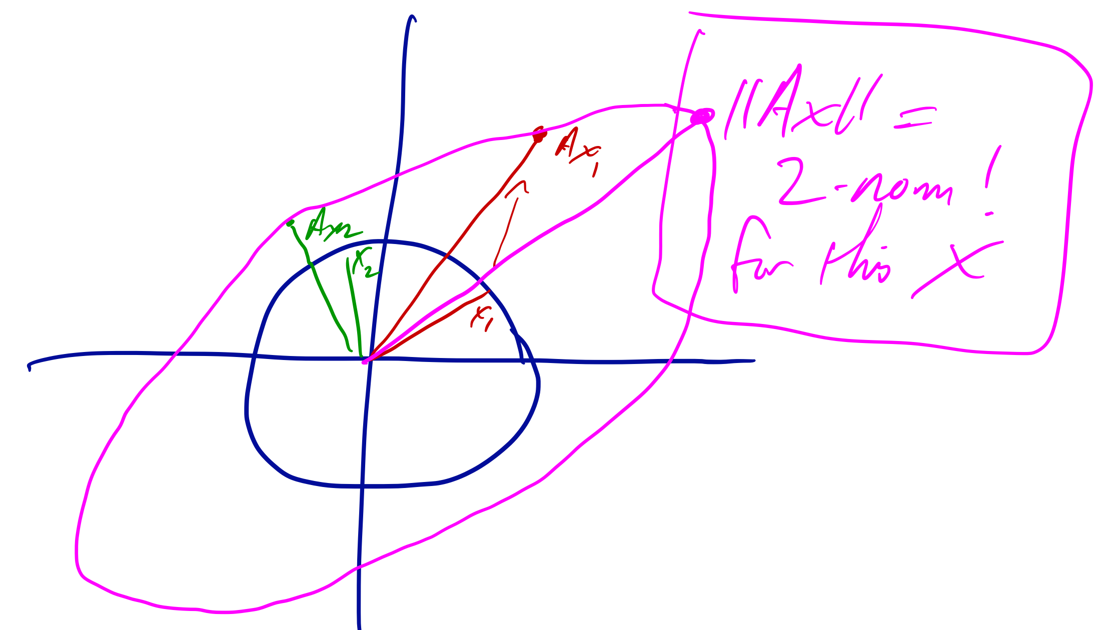
$\mathbb{R}^2$



Answers

- 2 ✓
- $\sqrt{5}$

In gener $\mathbb{R}^2$



$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$

$= \sigma_{max}(A)$

$A^T A$ for $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_{max}(A^T A) = 4 \Rightarrow$

$\|A\|_2 = 2.$

Properties of Matrix Norms

- $\|A\|_2 = 0$ only if $A = 0$
- $\|A\|_2 > 0$ if $A \neq 0$
- $\|\alpha A\| = |\alpha| \cdot \|A\|$
- $\|A + B\|_2 \le \|A\|_2 + \|B\|_2$
- $\|AB\|_2 \le \|A\|_2 \|B\|_2$

Condition of $A\underline{x} = \underline{b}$

Condition # of $f(x) = \dfrac{|x f'(x)|}{|f(x)|}$

$Ax = b$

$A\hat{x} = \hat{b}$ w/ $b \approx \hat{b}$

Assume A is Exact!

Goal

what goes here?

$$\frac{\|x - \hat{x}\|}{\|x\|} \le \boxed{\|A\| \cdot \|A^{-1}\|} \frac{\|b - \hat{b}\|}{\|b\|}$$

$$\frac{\|x - \hat{x}\|}{\|x\|} = \frac{\|A^{-1}b - A^{-1}\hat{b}\|}{\|x\|}$$

$$\le \frac{\|A^{-1}\| \cdot \|b - \hat{b}\|}{\|b\|} \boxed{\frac{\|b\|}{\|x\|}} \le \|A\|$$

$$\frac{\|b\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} \le \frac{\|A\| \cdot \cancel{\|x\|}}{\cancel{\|x\|}}$$

$\kappa(A) = \|A\| \cdot \|A^{-1}\|$ is the Condition # of a matrix!