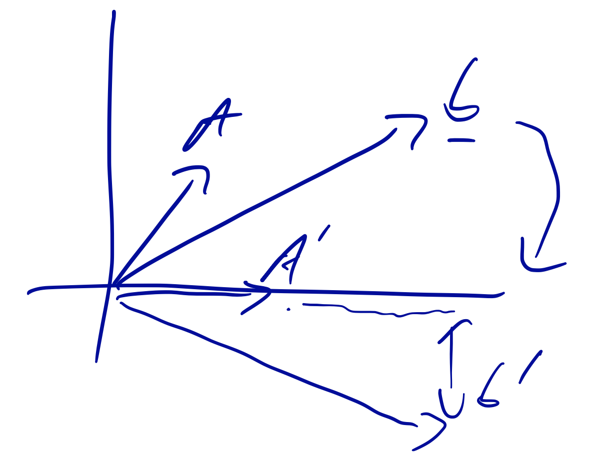
CS 314 Lecture 21

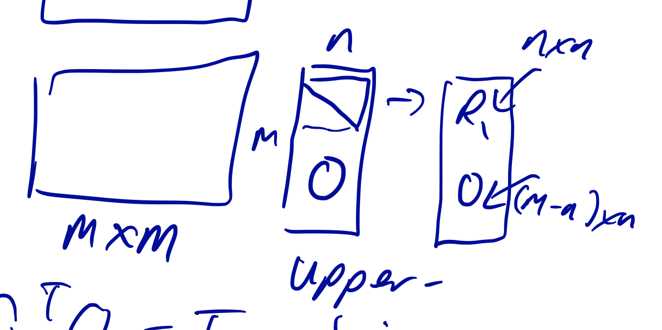
- QR + LS
- Conditioning + Stability

$\min \|Ax - b\|_2^2$



QR of A : m×n

$A = Q \quad R$



$Q^T Q = I$

Orthogonal Matrix

- $Q^T Q = I$
- Q has orthonormal columns
- Q is a "rotation"

Orthonormal Cols

$Q = \{V_1, V_2, \dots, V_n\}$

then

$V_i^T V_j = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$

Q is a rotation

FACT: $\|Qx\|_2^2 = \|x\|_2^2$

Proof:

$\|Qx\|_2^2 =$

$x^T Q^T (Qx) =$

$x^T I x =$

$x^T x = \|x\|_2^2$

∎

Least Sq w/ QR

$\min \|Ax - b\|_2^2$

$= \min \|QRx - b\|_2^2$

Now if Q is square + orthogonal, then so is Q^T

Idea: $Q Q^T = I$

$= \min \|Q^T [QRx - b]\|_2^2$

$= \min \|Rx - Q^T b\|_2^2$ (with $R = A'$, $Q^T b = b'$)

$= \min \left\| \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x - \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} \right\|_2^2$

Soln:

"$x = R_1^{-1} b_1$"

Solve ⟺

$R_1 x = b_1$

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Code for Least-Sq
Via QR

Input: A, b
n = size(A,2)
Q,R = qr(A)
b' = Q'*b
b1' = b'[1:n]
x = R\b1'
# in Julia R is n-by-n
# without the zeros
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Conditioning

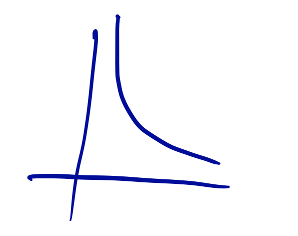
- a property of the Mathematical problem (not the algorithm)
- "sensitivity" to small changes
- ill-conditioned ⟺ highly sensitive to small changes.

Stability

- a property of the alg to evaluate the solution
- usually involve floating pt #'s

Suppose I want

$f(x) = \frac{1}{x}$ near $x = 0$



$f(x) = x - 5$ near $x = 5$

$f(x) = \sin(2^{60} x)$

Bad Alg

Goal $f(x) = x/2$

Alg:

for i = 1 to 60
  $x \leftarrow x \oplus 2$
for i = 1 to 59
  $x \leftarrow x \otimes 2$

Condition # of f @ x is

$K_f(x)$

$K(x) = \frac{x f'(x)}{f(x)}$

Goal: if $y = f(x)$
$\hat{y} = f(\hat{x})$
and $x \approx \hat{x}$
is $y \approx \hat{y}$?

Goal:

$\frac{|y - \hat{y}|}{|y|} = K(x) \frac{|x - \hat{x}|}{|x|}$

$\parallel$

$\frac{|f(x) - f(\hat{x})|}{|f(x)|}$

$\parallel$

$\frac{|f(x) - f(\hat{x})|}{|x - \hat{x}|} \cdot \frac{|x - \hat{x}|}{|x|} \cdot \frac{|x|}{|f(x)|}$

$\approx |f'(x)|$

$= \frac{|f'(x)||x|}{|f(x)|} \cdot \frac{|x - \hat{x}|}{|x|}$

$= K(x)$

$f(x) = 2x$

$K(x) = \frac{|2x|}{|2x|} = 1$

$f(x) = \sin(2^{60} x)$

$K(x) = \frac{|2^{60} \cos(2^{60} x)\, x|}{|\sin(2^{60} x)|}$