Lecture 16

Quiz Solve

$$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & -2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 39 \\ 3 \\ 2 \end{bmatrix}$$

How I learned this!

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \left[\begin{array}{ccc|c} 3 & 6 & 9 & 39 \\ 2 & 5 & -2 & 3 \\ 1 & 3 & -1 & 2 \end{array}\right] \longrightarrow$$

$$\begin{array}{c} \\ -\tfrac{2}{3}① + ② \rightarrow \\ -\tfrac{1}{3}① + ③ \rightarrow \end{array} \left[\begin{array}{ccc|c} 3 & 6 & 9 & 39 \\ 0 & 1 & -8 & -23 \\ 0 & 1 & -4 & -11 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc|c} 3 & 6 & 9 & 39 \\ 0 & 1 & -8 & -23 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

$-1②+③$

$$4x_3 = 12$$

$$x_3 = 3$$

$$x_2 \boxed{-8x_3} = -23$$

$$x_2 - 24 = -23$$

$$x_2 = 1$$

$$x_1 = 3$$

Key Idea:
Encode these Steps w/ Matrices!
The LU factorization

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}}_{M_1} \begin{pmatrix} 3 & 6 & 9 \\ 2 & 5 & -2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 9 \\ 0 & 1 & -8 \\ 0 & 1 & -4 \end{pmatrix} = A_1$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{M_2} A_1 = \begin{pmatrix} 3 & 6 & 9 \\ 0 & 1 & -8 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= U$$

Matrices M_1 and M_2 Encode steps of GE

$$\boxed{M_2 M_1} A = U$$

$$A = \boxed{M_1^{-1} M_2^{-1}} U$$

(=> a lower-triangular matrix)

$$A = L U$$

$$M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & & \\ 2/3 & 1 & \\ 1/3 & 1 & 1 \end{pmatrix}$$

Super-Special

Key idea: the steps of GE can be encoded as the product of a

lower · upper

L $\quad$ U

triangular matrix

How to solve $A\underline{x} = \underline{b}$ w/ LU (version I)

$$A\underline{x} = \underline{b}$$

! $$LU\underline{x} = \underline{b}$$

$$U\underline{x} = L^{-1}\underline{b} = y$$

① Solve $Ly = \underline{b}$

② Solve $U\underline{x} = y$

How to generalize?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_1 A = A_1$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}$$

(1st step of GE)

Goal: pick m_{21}, m_{31} s.t.

$$M_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$m_{31} = \frac{a_{31}}{a_{11}}$$

$$M_1 \underline{z} = \begin{bmatrix} z_1 \\ z_2 - m_{21} z_1 \\ z_3 - m_{31} z_1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$M_1 A$

$$M_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \\ -M_{n1} & 0 & \cdots & 0 \; 1 \end{bmatrix}$$

$$M_{i1} = \frac{a_{i1}}{a_{11}}$$

Step 2

$$M_1 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

$$a_{22}' = a_{22} - M_{21} a_{12}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -M_{32} & 1 \end{bmatrix}$$

$$M_{32} = \frac{a_{32}'}{a_{22}'}$$

$M_1 A =$

$a_{ij}' =$

$M_{i2} =$

$$\begin{bmatrix} a_{11} & & \overset{\text{change}}{} & a_{12} \\ 0 & & & \\ \vdots & & \boxed{\quad a_{ij} \quad} & \\ 0 & & & \end{bmatrix}$$

$$a_{ij} - M_{i1} a_{1j}$$

$$\frac{a_{i2}'}{a_{22}'}$$

In general

$$\boxed{M_{n-1} M_{n-2} - M_1} A = \begin{bmatrix} a_{11} & a_{12} & - & a_{1n} \\ 0 & a_{22}' & - & a_{2n}' \\ & 0 & a_{33}'' - & a_{3n}' \\ | & | & & \\ 0 & 0 & & \tilde{a}_{nn} \end{bmatrix}$$

(n-1)'s

$$M_{n-1} - M_1 =$$ lower triangular!