COMPUTING EIGENVALUES AND EIGENVECTORS ON A COMPUTER

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1 WHAT ARE EIGENVALUES

$$\boldsymbol{A}\mathbf{x} = \lambda \mathbf{x}$$

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

2 WHY DO WE CARE ABOUT THEM?

Take good notes here! You'll be surprised when you need this information!

3 THE OLD WAY

$$\boldsymbol{A} = \begin{bmatrix} 1 & -10 \\ -5 & -4 \end{bmatrix}$$

We'll soon see a much better way to compute these eigenvalues and vectors

4 Properties of Eigenvalues and Vectors

Real-valued matrices can have

If $\boldsymbol{A}$ is symmetric

5 The Power Method

6 The Power Method in Practice

$$\mathbf{y} \approx$$

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x = rand
for i=1 to ...
  y =
  x =
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7 Small examples

$$A = 1/3\begin{bmatrix} 2.8 & 0.6 \\ 0.6 & 1.2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -10 \\ -5 & -4 \end{bmatrix}$$

$A^k =$

$\lambda =$

$\mathbf{x} =$

$\lambda =$

$\mathbf{x} =$

8 Convergence of the power method

Assumption 1 $\boldsymbol{A}$ is diagonalizable.

Assumption 2 There is one eigenvalue λ_1 with a magnitude larger than every other eigenvalue. Let $\mathbf{v}_1$ be the eigenvector.

Assumption 3 If we write the starting vector $\mathbf{x}$ in terms of eigenvectors of $\boldsymbol{A}$, then the component associated with $\mathbf{v}_1$ is non-zero.

Assumption 3 implies:

Assumption 1 implies:

Assumption 2 implies:

The rate of convergence for the power method is determined by the second largest magnitude eigenvalue of $\boldsymbol{A}$.

$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$

Then if $|\lambda_2|/|\lambda_1|$ is nearly 1, it'll converge slowly. And if $|\lambda_2|/|\lambda_1|$ is $\ll 1$, it'll converge quickly.

BEYOND A SINGLE VECTOR

How do we get multiple eigenvectors?

Property 1 The eigenvectors of a symmetric matrix are orthogonal.

Idea 1 If we find $\mathbf{x}_1, \lambda_1$, then we can *project* on the orthogonal subspace.

"we can get-rid-of $\mathbf{x}_1$ *in running the power method"*

This gives rise to a process called deflation!

Instead, let's just do them all at once.

Let $\boldsymbol{X}$ be a *block of orthogonal vectors*

Multiply $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$

Normalize