CONDITIONING AND STABILITY

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There are two common ways for a numerical computation to go wrong.

CONDITIONING

The *problem* is sensitive to small changes.

STABILITY

The *algorithm* introduces large errors.

Definition

Let $y = f(x)$. The *relative condition number* of f tells us how much y changes if we change x by a tiny bit. Let $\hat{y} = f(\hat{x})$, where $\hat{x} \approx x$. Then the relative condition number is

$$\kappa(x) = \left| \frac{x f'(x)}{f(x)} \right|.$$

Why is this right? We want

$$\left| \frac{\hat{y} - y}{y} \right| \approx \kappa(x) \left| \frac{\hat{x} - x}{x} \right|,$$

and if $\hat{x}$ and x are close:

Example $f(x) = 2x$

Example $f(x) = \sin(2^{60} x)$.

Definition

forward error analysis

How much error do I have in solving problem $f(x)$ using algorithm `alg`?

$$\left| \frac{\texttt{alg}(x) - f(x)}{f(x)} \right|$$

backward error analysis

Does my algorithm solve a nearby problem exactly?

$$\texttt{alg}(x) = f(x + \delta x)$$

Backwards stable algorithms

1. inner products
2. solving $\boldsymbol{A}\mathbf{x} = \mathbf{b}$
3. computing $\boldsymbol{QR}$

Matrix norms

$$|\alpha| = \max |\alpha x| \text{ where } |x| = 1$$

how much it scales a unit quantity

$$\|\boldsymbol{A}\|_2 = \textit{how much it scales a unit vector}$$

$$= \max \|\boldsymbol{A}\mathbf{x}\|_2 \text{ where } \|\mathbf{x}\| = 1.$$

$$= \sqrt{\text{largest eigenvalues of } \boldsymbol{A}^T\boldsymbol{A}}\,.$$

Conditioning of linear systems

$$\boldsymbol{A}\mathbf{x} = \mathbf{b} \qquad \mathbf{b} \approx \hat{\mathbf{b}} \qquad \boldsymbol{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$$

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} =$$

in Julia: `cond(A)`