1 A BVP

In a two-point boundary value problem, we look for a function that satisfies a certain property, like

\[ u''(x) = f(x) \]

where \( x \) ranges between \([0, 1]\) and \( u(0) = \alpha \) and \( u(1) = \beta \).

This is different from a standard ODE because we have TWO constraints, one at \( u(0) \) and one at \( u(1) \).

In this intro, we’ll always assume that \( u(x) \) is a function that is defined where \( x \) is between \([0, 1]\). In general, these can get more complicated, but that’s for the future!

2 SOLUTION STRATEGY

The way we look for solutions of a BVP is to represent the function \( u \) via a regular grid of points. Then we can approximate the derivative terms via finite-difference approximations. This will turn into a linear system that we have to solve (which you now know how to solve!)

4 AN EXAMPLE

Let’s try and solve

\[ u''(x) = -2 \]

with \( u(0) = 1, u(1) = 1/2 \). Let’s represent \( u \) at exactly the points above, so \( u_1 = u(0), u_2 = u(0.1), \ldots, u_{10} = u(0.9), u_{11} = u(1) \).

This means that we know that \( u_1 = 1 \) and \( u_{N+1} = 1/2 \).

Now, we also know that \( u''(x) = -2 \). But, we can approximate:

\[ u''(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \]

if \( h = 0.1 \) and \( x = 0.1 \), then we have:

\[ u''(0.1) \approx \frac{u_1 - 2u_2 + u_3}{0.1^2} \]

In order to solve the BVP, we need:

\[ u''(0.1) = -2 \iff \frac{u_1 - 2u_2 + u_3}{0.1^2} = -2. \]
So we have one equation that defines a relationship between these three equations.
Let's repeat!

\[
\begin{align*}
    u''(0.2) &= -2 \iff \frac{u_2 - 2u_3 + u_4}{0.1^2} = -2. \\
u''(0.3) &= -2 \iff \frac{u_3 - 2u_4 + u_5}{0.1^2} = -2. \\
u''(0.4) &= -2 \iff \frac{u_4 - 2u_5 + u_6}{0.1^2} = -2. \\
u''(0.5) &= -2 \iff \frac{u_5 - 2u_6 + u_7}{0.1^2} = -2. \\
u''(0.6) &= -2 \iff \frac{u_6 - 2u_7 + u_8}{0.1^2} = -2. \\
u''(0.7) &= -2 \iff \frac{u_7 - 2u_8 + u_9}{0.1^2} = -2. \\
u''(0.8) &= -2 \iff \frac{u_8 - 2u_9 + u_{10}}{0.1^2} = -2. \\
u''(0.9) &= -2 \iff \frac{u_9 - 2u_{10} + u_{11}}{0.1^2} = -2. \\
\end{align*}
\]

We also know that \(u_1 = 1\) and \(u_{11} = 1/2\). Thus we have 11 equations and 11 unknowns!

It's a linear system

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 100 -200 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 -200 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 -200 & 100 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 -200 & 100 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 100 -200 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 100 -200 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 -200 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 -200 & 100 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
v_9 \\
v_{10} \\
v_{11} \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
-2 \\
1/2 \\
\end{bmatrix}.
\]

```plaintext
1 N = 10
2 xgrid = collect(0:1/N:1)
3 alpha = 1
4 beta = 0.5
5 f = x -> -2.0
6 h = 1/N
7
8 A = zeros(N+1,N+1)
9 b = zeros(N+1,1)
10 for i=1:N+1
11    if i == 1
12        A[i,i] = 1
13        b[i] = alpha
14    elseif i==N+1
15        A[i,i] = 1
16        b[i] = beta
17    else
18        A[i,i-1] = 1/h^2
19        A[i,i] = (-2)/h^2
20        A[i,i+1] = 1/h^2
21        b[i] = f(xgrid[i])
22    end
23 end
24 u = A\b
25 plot(xgrid, u, marker=:circle)
```
### 5 A MORE COMPLICATED EXAMPLE

This example models a rod that we are heating. (See the book for the physics.)

\[
\frac{\partial}{\partial x} \left[ p(x) \frac{\partial u}{\partial x} \right] = f(x)
\]

The function \(p(x)\) gives the conductivity of the bar. The function \(f(x)\) is the negative of the heat-input. So if \(f(x) = 0\), then there is no heat coming in.

The way to interpret this equation is that we are looking for a function \(u(x)\) that has the property that

\[
p(x)u''(x) + p'(x)u'(x) = f(x).
\]

But we need to know \(u(0)\) and \(u(1)\) too.

For this example, we'll use:

\[
\begin{align*}
u(0) &= -1, \\
u(1) &= 1, \\
p(x) &= 1 - 4(x - 1/2)^2, \\
f(x) &= -2 \sin(\pi x).
\end{align*}
\]

We discretize the derivative as follows: Let \(g(x) = \left[ p(x) \frac{\partial u}{\partial x} \right](x)\). That is, \(g(x)\) is the function we get from taking one derivative with respect to \(u\) and then multiplying by \(p(x)\). Then our BVP is:

\[
g'(x) = f(x).
\]

if we take a centered difference formula with \(h/2\), we get

\[
g'(x) = f(x) \approx \frac{g(x + h/2) - g(x - h/2)}{h} = f(x).
\]

Now, we need a formula for \(g(x + h/2)\). This we get from using another centered difference formula for \(\partial u/\partial x\):

\[
g(x + h/2) \approx p(x + h/2) \frac{u(x + h) - u(x)}{h}.
\]

Hence, we have the full discretization

\[
g(x) = \left[ p(x) \frac{\partial u}{\partial x} \right](x) \iff g(x) = \frac{p(x + h/2) \frac{u(x + h) - u(x)}{h} - p(x - h/2) \frac{u(x) - u(x - h)}{h}}{h}
\]

Or, for a grid of \(u_i\) values:

\[
g(x_i) = p(x_i + h/2) \frac{u_{i+1} - u_i}{h} - p(x_i - h/2) \frac{u_i - u_{i-1}}{h}
\]

This gives us a linear system that the following Julia code evaluates:

```julia
1  N = 10
2  xgrid = collect(0:1/N:1)
3  alpha = 1
4  beta = 1
5  f = x -> -2*2sin(pi*x)  # we are heating a rod in the middle
6  p = x -> 1+4*(x-0.5)^2  # the rod becomes more like an insulator towards the end.
7  h = 1/N
8  A = zeros(N+1,N+1)
9  b = zeros(N+1,1)
10 for i=1:N+1
11    if i == 1
12      A[1,i] = alpha
13      b[1] = alpha
14    elseif i == N+1
15      A[i,i] = beta
16     else
17       A[i,i] = p(xgrid[i]) - p(xgrid[i] - h)/h^2
18       A[i,i-1] = -p(xgrid[i] - h/2) - p(xgrid[i] - h)/h^2
19       b[i] = f(xgrid[i])
20     end
21  end
22  u = A\b
23  plot(xgrid, u)
```