Polygon Triangulation (chapter 3)

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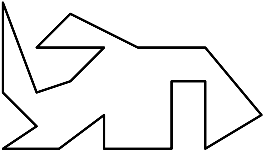
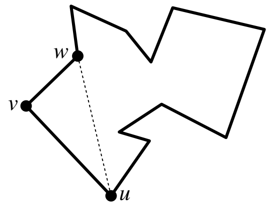
Polygon Triangulation



- Decompose a polygon into triangles.
- Applications: calculation, drawing, camera coverage.

Existence Proof

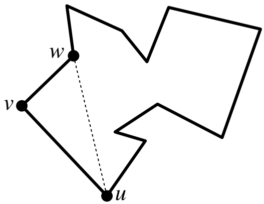


Theorem 3.1 Every n-vertex polygon has a triangulation and every triangulation has $n - 2$ triangles.

Proof

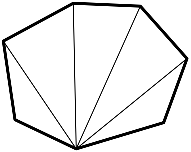
- A triangle ($n = 3$) is trivial, so consider $n > 3$.
- Lemma: there are vertices u and w with uw in the interior.
- This *diagonal* splits the polygon into polygons with m and $n - m + 2$ vertices.
- Triangulate them with $m - 2$ and $n - m$ triangles.
- The union is a triangulation with $n - 2$ triangles.
- The proof yields an $O(n^2)$ algorithm.

Proof of Lemma



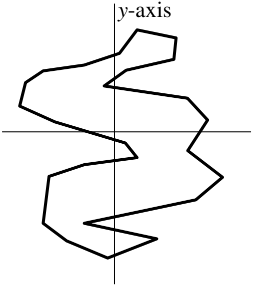
- Let v be the leftmost vertex with neighbors u and w.
- If uw is in the interior, we are done.
- Otherwise, an edge intersects uw.
- Consequently, a vertex is inside the triangle uvw.
- Let v' be the vertex in uvw farthest from uw.
- For vv' to intersect an edge, one endpoint must be farther from uw than v', which contradicts its definition.
- Hence, vv' is a diagonal.

Convex Decomposition



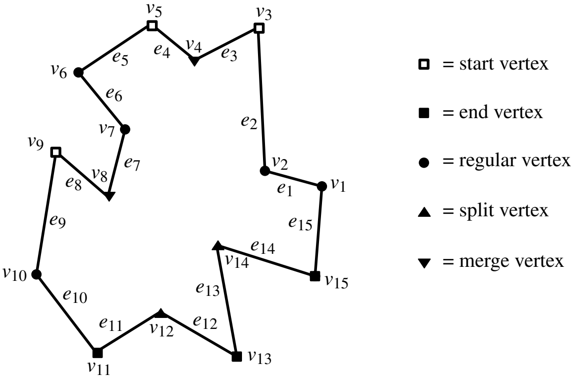
- Decompose the polygon into convex polygons.
- Triangulate each convex polygon with diagonals from one vertex to the others.
- Problem: convex decomposition is hard.

Monotone Decomposition



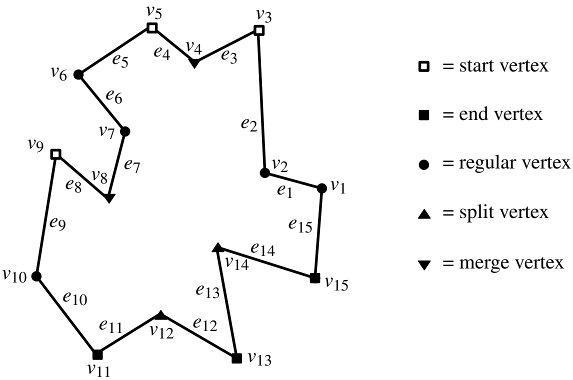
- A polygon is y-monotone if a horizontal line intersects it in a closed interval or in the empty set.
- The polygon consists of two y-monotone vertex chains that share a top and a bottom vertex.
- We decompose the input polygon into y-monotone polygons in $O(n \log n)$ time then triangulate them in $O(n)$ time.

Types of Vertices



- Start: maximum left turn (interior below).
- End: minimum left turn (interior above).
- Split: maximum right turn (the interior above).
- Merge: minimum right turn (interior below).
- Regular: increasing or decreasing (interior left or right).

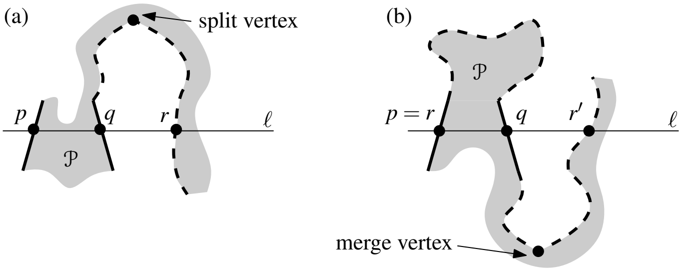
Monotonicity Condition



Lemma 3.4 A polygon P is y-monotone iff it has no split or merge vertices.

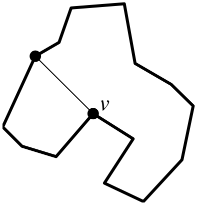
Proof Only if is easy: a horizontal just above a merge or just below a split intersects the polygon in at least two segments. To prove if, suppose P has no splits or merges yet is not y-monotone.

Proof of If



- There is a line ℓ that intersects P in multiple segments.
- Let pq be the leftmost segment.
- Follow the boundary from q with the interior on the left.
- Let r be the next intersection point with ℓ.
- (a) If $p \neq r$, the highest vertex between q and r is a split.
- (b) if $p = r$, follow the boundary in the other direction to r'.
- Since there are multiple components, $p \neq r'$ and the lowest vertex between q and r' is a merge.

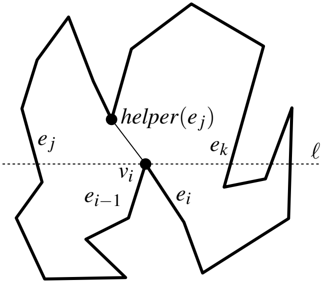
Decomposition Strategy



Insert a diagonal from each split/merge to a vertex above/below.

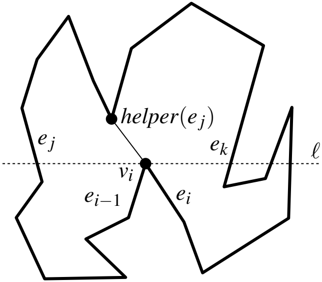
The merges and splits are monotone in the sub-polygons because the diagonals go in the opposite direction to their edges.

Algorithm



- Sweep the polygon from top to bottom with a horizontal line.
- The events are the vertices.
- Add an upward diagonal from each split vertex.
- Add a downward diagonal from each merge vertex.

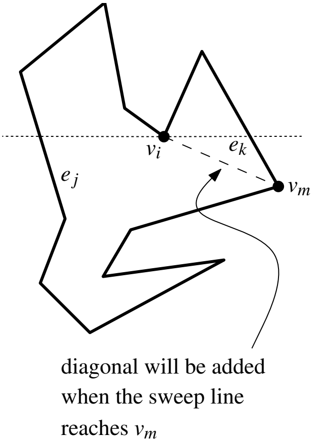
Split Vertices



The *helper* of an edge e is the lowest vertex v above the sweep line such that the horizontal segment from v to e is in the polygon.

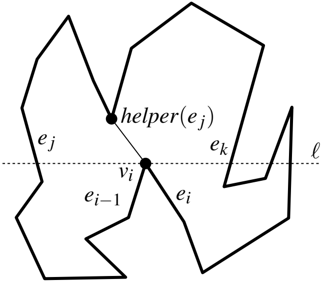
- A split vertex v_i connects to the helper of its left edge e_j.
- This occurs at the v_i event.

Merge Vertices



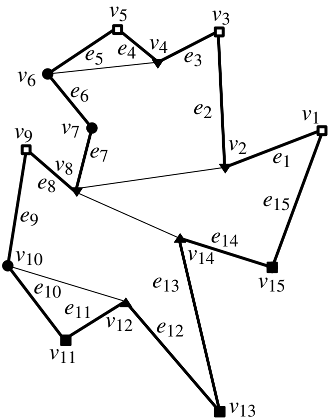
- A merge vertex v_i connects to the highest vertex below v_i that is a helper of its left edge e_j.
- The v_i event sets the helper of e_j to v_i.
- v_m is the next helper of e_j or its lower endpoint.

Diagonals



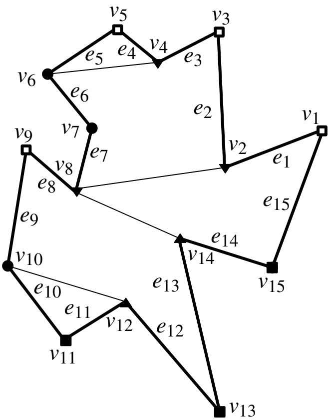
- Every diagonal goes up from a vertex v_i to the helper of its left edge e_j.
- Either v_i is a split vertex or the helper is a merge vertex.
- Both cases can occur together.

Algorithm Details



- The edges with the interior on the right that intersect the sweep line are stored in a tree T in left to right order.
- The tree and the helpers of its edges are updated at events.

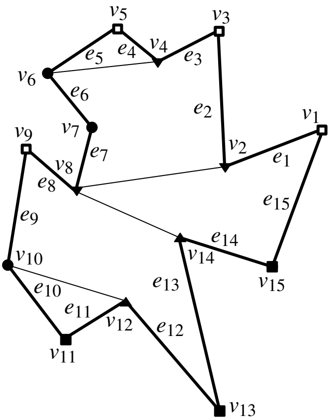
Start Vertex



1. Insert e_i in T and set $helper(e_i)$ to v_i.

Example v_5: insert e_5.

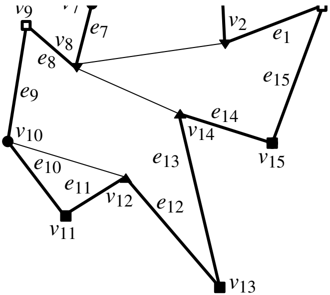
End Vertex



1. If $helper(e_{i-1})$ is a merge, add a diagonal from v_i to it.
2. Delete e_{i-1} from T.

Example v_{15}: $helper(e_{14}) = v_{14}$, which is not a merge.

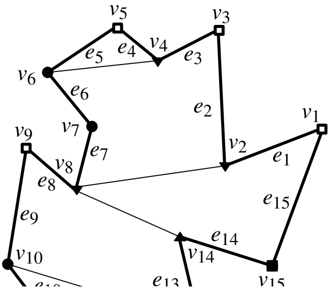
Split Vertex



1. Use T to find the edge e_j left of v_i.
2. Add a diagonal from v_i to $helper(e_j)$.
3. Set $helper(e_j)$ to v_i.
4. Insert e_i in T and set $helper(e_i)$ to v_i.

Example v_{14}: e_9 is the left edge with helper v_8. Add a diagonal from v_{14} to v_8.

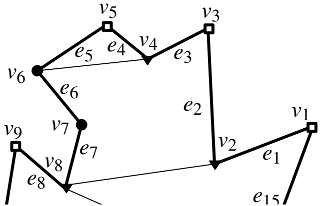
Merge Vertex



1. if $helper(e_{i-1})$ is a merge, add a diagonal from v_i to it.
2. Delete e_{i-1} from T.
3. Use T to find the edge e_j left of v_i.
4. If $helper(e_j)$ is a merge vertex, insert a diagonal from v_i to it.
5. Set $helper(e_j)$ to v_i.

Example v_8: the helper v_2 of e_7 is a merge, so add a diagonal from v_8 to v_2; the edge to the left is e_9 and its helper v_9 is a start vertex.

Regular Vertex

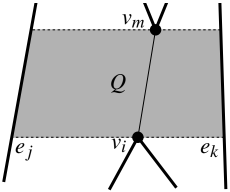


HANDLEREGULARVERTEX(v_i)

if the interior of $\mathcal{P}$ lies to the right of v_i
    then if $helper(e_{i-1})$ is a merge vertex
            then Insert the diagonal connecting v_i to $helper(e_{i-1})$ in $\mathcal{D}$.
          Delete e_{i-1} from $\mathcal{T}$.
          Insert e_i in $\mathcal{T}$ and set $helper(e_i)$ to v_i.
    else Search in $\mathcal{T}$ to find the edge e_j directly left of v_i.
          if $helper(e_j)$ is a merge vertex
            then Insert the diagonal connecting v_i to $helper(e_j)$ in $\mathcal{D}$.
          $helper(e_j) \leftarrow v_i$

Example: v_6 has the interior to the right: add a diagonal to v_4.

Correctness



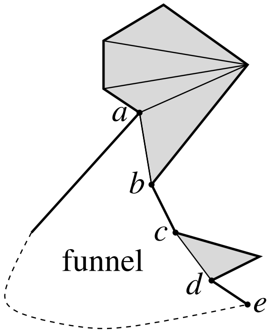
Lemma 3.5 The diagonals do not intersect each other or P.
Proof We will discuss a split vertex v_i; the other cases are similar.

- e_j and e_k are the closest edges left and right of v_i
- v_m is the helper of e_j, so the diagonal is $v_i v_m$.
- The rectangle Q contains no vertices.
- An edge of P that intersects $v_i v_m$ must intersect the horizontal that connects v_i to e_j or that connects v_m to e_j.
- This is impossible because e_j is directly left of v_i and v_m.
- Prior diagonals cannot intersect $v_i v_m$ because they are above v_i and cannot have an endpoint in Q.

Complexity

- The running time is $O(n \log n)$ for y sorting and T insertion.
- The space complexity is $O(n)$.

Monotone Polygon Triangulation

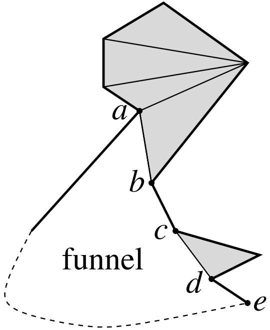


A monotone polygon is triangulated with a greedy algorithm that processes the vertices in decreasing y order.

The algorithm employs a stack of vertices in decreasing y-order with the lowest vertex at the top. Example: *abcde*.

The bottom vertex is from one side of the polygon and the rest of the stack is a portion of the other side that is concave upward.

Monotone Polygon Triangulation



The vertices of the stack form a polygonal chain, comprised of polygon edges and of diagonals, above which the polygon has already been triangulated.

The chain and the downward edge from the bottom vertex form a funnel.

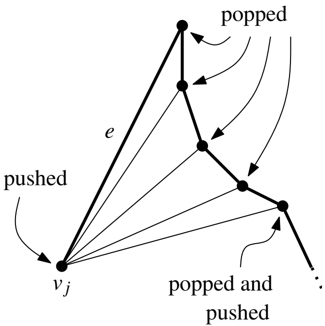
The triangulation is extended by inserting diagonals from the vertex below the stack to the stack vertices wherever possible.

Algorithm

1. Place the vertices in decreasing y order $v_1, \ldots, v_n$.
2. Initialize the stack to (v_1, v_2).
3. For $j = 3$ to $n - 1$
   - Case 1: v_j on the opposite chain from the stack top.
   - Case 2: v_J on the same chain as the stack top.
4. Insert a diagonal from v_n to each stack vertex, except for the first and last.

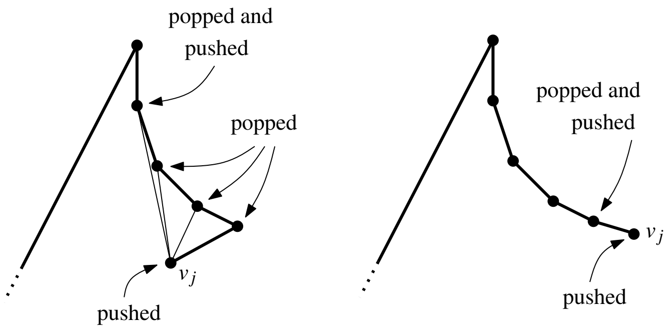
Why is the last vertex a special case?

Case 1



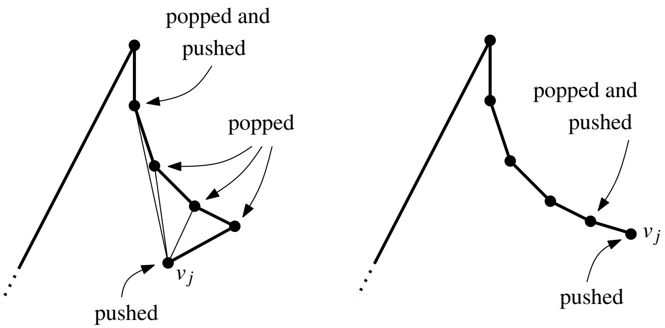
- The vertex v_j is on the opposite chain from the stack top.
- Generate triangles from v_j and the stack edges.
- The new stack is the old top then v_j.

Case 2



- The vertex v_j is on the same side as the stack top.
- Generate triangles from v_j and the stack edges that it sees.
- The new stack is the other stack edges then v_j.

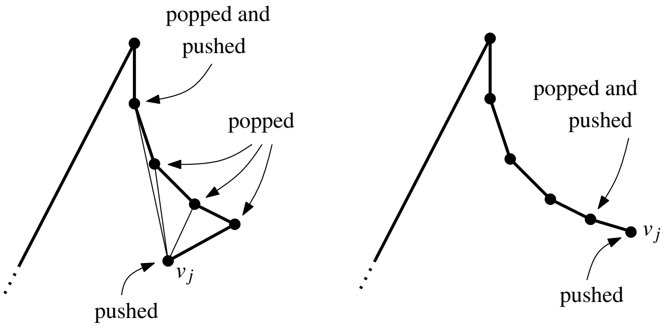
Case 2



- The vertex v_j is on the same side as the stack top.
- Generate triangles from v_j and the stack edges that it sees.
- The new stack is the other stack edges then v_j.

When does v_j see stack edge ab with $a_y < b_y$?

Case 2



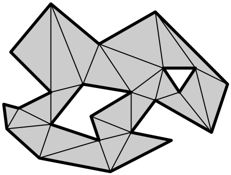
- The vertex v_j is on the same side as the stack top.
- Generate triangles from v_j and the stack edges that it sees.
- The new stack is the other stack edges then v_j.

When does v_j see stack edge ab with $a_y < b_y$?
abv_j is a left/right turn for ab a left/right edge.

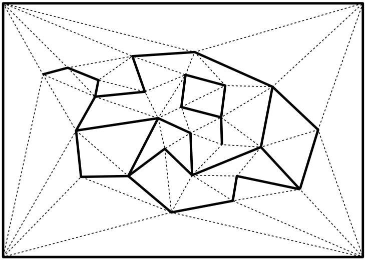
Degeneracy

- Vertices $a_1 = (x_1, y)$ and $a_2 = (x_2, y)$ are degenerate.
- The book specifies that a_1 is below a_2 if $x_1 > x_2$.
- This handles the degeneracies in the sweep algorithm.
- The greedy algorithm is degenerate when three consecutive vertices are collinear.
- The lowest vertex does not see the edge formed by the upper two vertices.

Generalizations

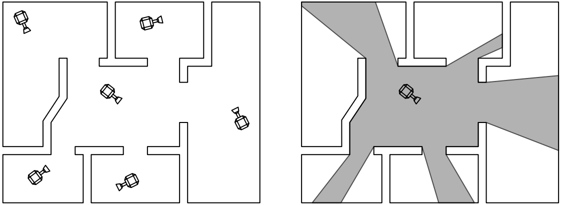


polygon with holes



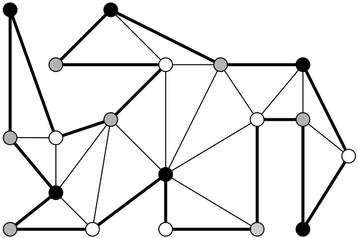
subdivision

Art Gallery Problem



Compute a minimal set of points inside a polygon from which every point in the polygon is visible.

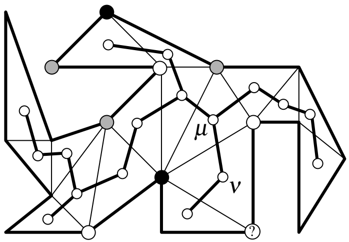
Solution Using Triangulation



1. Triangulate the polygon.
2. Compute a vertex 3-coloring of the triangulation.
3. Select the smallest color class.

- Conclusion: $\lfloor n/3 \rfloor$ of the n vertices suffice.
- The book shows a polygon for which this bound is tight.

Computing the 3-coloring



- The dual graph has a node for each triangle and an edge between triangles that share a diagonal.
- It is acyclic because every diagonal splits the polygon.
- Traverse the dual graph in depth-first order.
- Color the vertices of the first triangle.
- When crossing an edge into a triangle, color its third vertex differently from the edge vertices.