Range Search (chapter 5)

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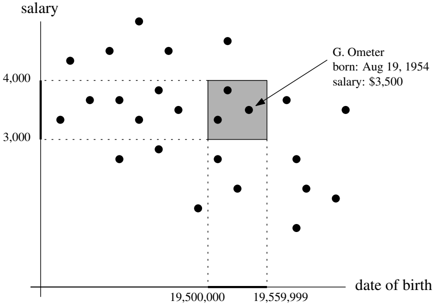
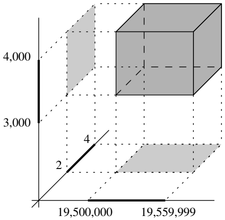
Range Search



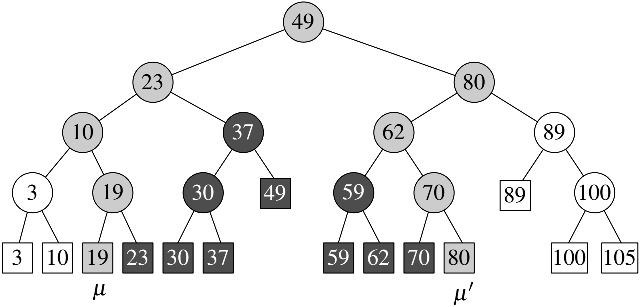
2D



3D

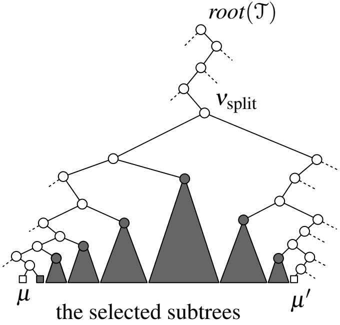
Find the points in an axis-aligned box.

1D Range Search



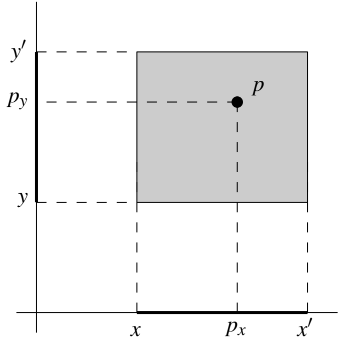
- Find the keys in a range $[x : x']$, for example $[18 : 77]$.
- Use a binary tree with keys at the leafs.
- Searches for x and x' end at leafs μ and μ'.
- The leafs between μ and μ' are in $[x : x']$.
- Need to test if μ and μ' are in $[x : x']$.

Algorithm



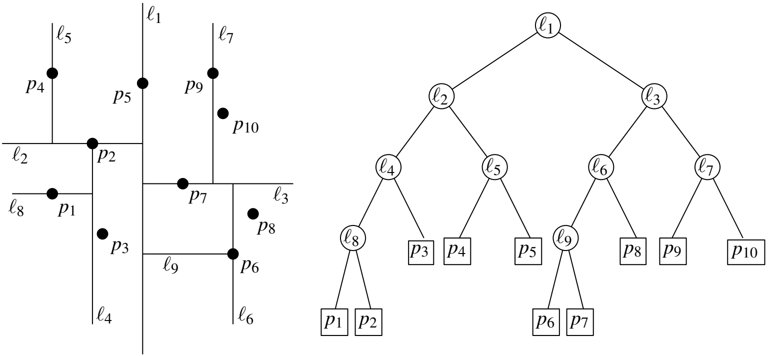
1. Set v to the root of the tree.
2. while v is not a leaf and $x' \leq x_v$ or $x > x_v$:
   if $x' \leq x_v$, replace v with its left child, else its right child.
3. Search for x in the v_{split} subtree.
4. Report right children where the search goes left.
5. Search for x' in the v_{split} subtree.
6. Report left children where the search goes right.

2D Range Search



- A 2D range consists of two 1D ranges.
- kd-trees treat the two ranges in the same way.
- Range trees make one primary and the other secondary.
- Assume for now that points have unique x and y coordinates.

kd-tree (originally called 2d-tree)

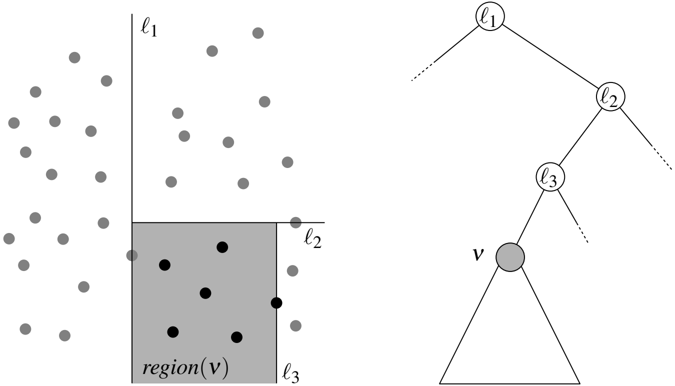


- A kd-tree has split lines at internal nodes and points at leafs.
- Split the points with a vertical line through their x median.
- Split each subset with a horizontal line through its y median.
  (Median points go left and down.)
- Repeat until every region contains one point.

Complexity of kd-tree Construction

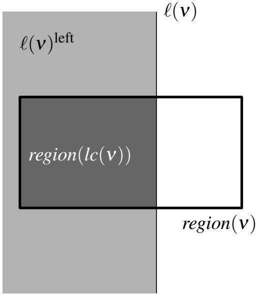
- Algorithm details
  - Sort the points on x and on y.
  - Obtain the x median x_m from the x sorted list.
  - Split both lists at x_m.
  - Pass the left/right halves to the left/right recursive call.
  - Likewise for y splits.
- Complexity
  - Sorting n points takes $O(n \log n)$ time.
  - Excluding sorting, $T(n) = 2T(n/2) + O(n)$ for $n > 1$.
  - $T(n) = O(n \log n)$.
  - Space complexity is $O(n)$ because every leaf stores a point.

Regions



- Each node defines a rectangular region.
- The region of the root is the plane.
- The regions of the children of a node are the intersections of its region with the half planes of its splitting line.
- They are open to the right or above the splitting line.

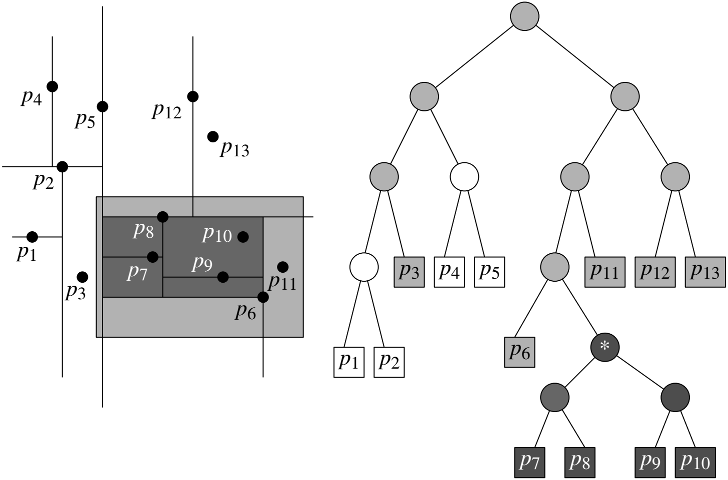
Range Search Algorithm



Search a tree with root v for points in Q (not shown).

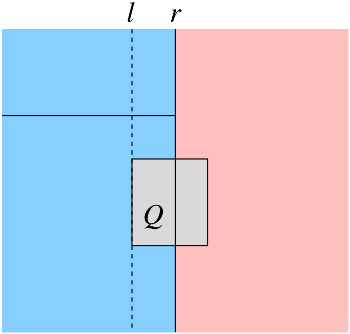
If v is a leaf, report its point if it is in Q.
else
  Split the region of v to compute the regions of $lc(v)$ and $rc(v)$.
  if the region of $lc(v)$ is contained in Q, report it.
    else if the region intersects Q, search $lc(v)$.
  if the region of $rc(v)$ is contained in Q, report it.
    else if the region intersects Q, search $rc(v)$.

Range Search Example



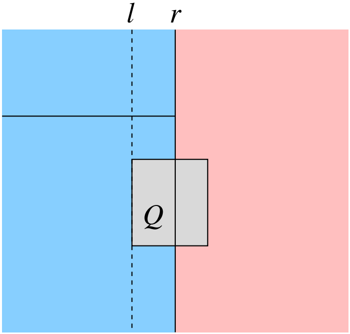
- This kd-tree is *not* constructed with our algorithm.
- The grey nodes are visited when Q is the grey rectangle.
- The region of the * node (dark grey) is contained in Q.

Complexity of kd-tree Search



- The k points in the regions contained in Q take $O(k)$ time.
- The regions of the other visited nodes intersect a side of Q.
- The number that intersect a side is bounded by the number that intersect its supporting line.
- We bound the number of x split nodes $V(n)$ whose regions intersect a vertical line l. The y split case is analogous.
- The bound is pessimistic because Q is usually small.

Complexity of kd-tree Search (continued)

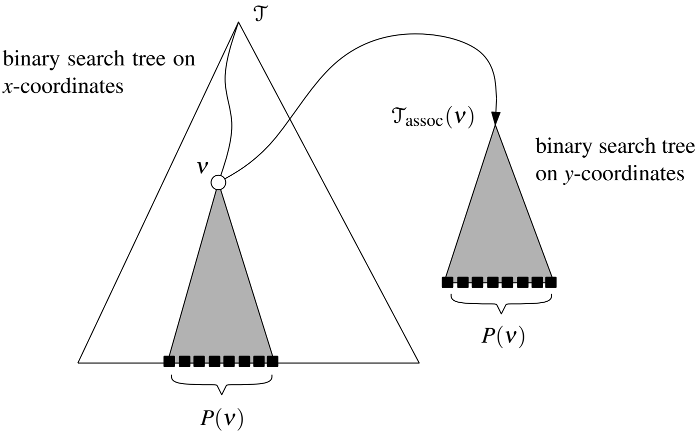


- The x split node r has n points in its region.
- The vertical line l intersects the region of one child of r.
- It intersects the regions of both children of this child.
- Each grandchild is an x split node with $n/4$ points.
- The recurrence is $V(n) = 2V(n/4) + 2$ for $n > 1$.
- The solution is $O(\sqrt{n})$, so the time complexity is $O(\sqrt{n} + k)$.

Higher Dimensional kd-trees

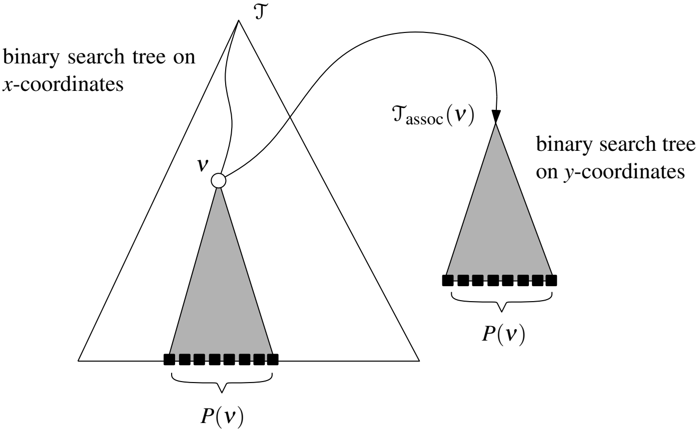
- kd-trees generalize to 3 dimensions.
- The split lines become planes.
- The construction cycles between x, y, and z splitting planes.
- Likewise in any dimension d.
- The worst-case query time is $O(n^{1-1/d} + k)$.
- They work well for small d in practice.

Range Trees



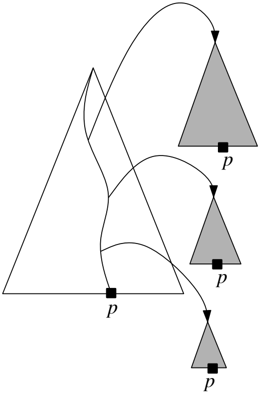
- Points are stored in a binary tree with x-coordinate keys.
- Each node v has an associated tree with y-coordinate keys.
- This tree contains the points $P(v)$ in the subtree of v.

Query Algorithm



- Search the primary tree for the x interval.
- The result is the primary nodes v with $P(v)$ in the x interval.
- Search their associated trees for the y interval.

Space Complexity



- A range tree for n points uses $O(n \log n)$ space.
- The primary tree has $O(n)$ nodes.
- A point is stored in the associated trees of the nodes on the primary tree path from the root to its leaf.
- This yields $O(n \log n)$ associated tree nodes.

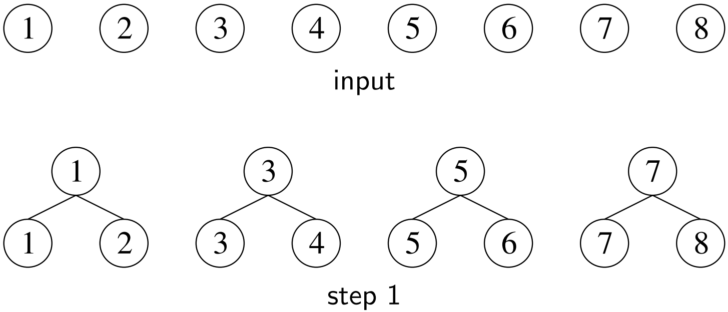
Construction Algorithm

buildRT(xpts, ypts)

1. Build the associated tree using ypts.
2. If xpts contains one point, the primary tree is a leaf.
3. Else
   3.1 Let point p have the median x coordinate.
   3.2 Split xpts into xpts1 and xpts2 at p_x.
   3.3 Split ypts into ypts1 and ypts2 at p_x.
   3.4 Set v_1 to buildRT(xpts1, ypts1).
   3.5 Set v_2 to buildRT(xpts2, ypts2).
   3.6 The primary tree has key p_x, left child v_1, and right child v_2.

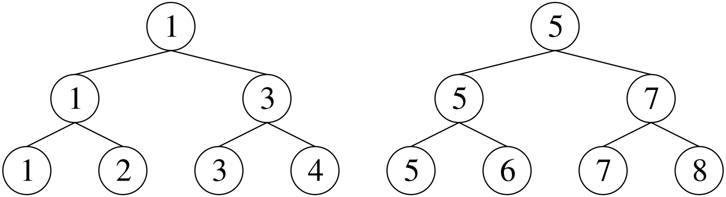
- xpts are presorted by x; ypts are presorted by y.
- Splitting maintains these orders.
- The associated tree is built bottom up.

Bottom Up Tree Construction

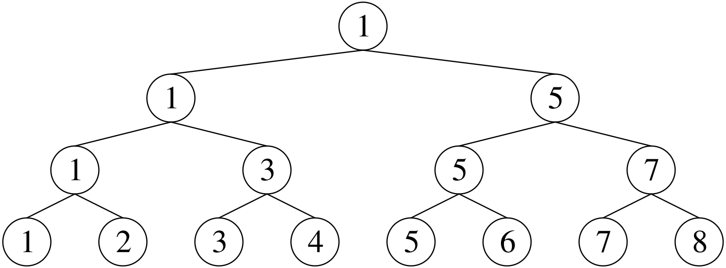


- Step i constructs level i above the leafs.
- Complexity: $n + n/2 + n/4 + \cdots < 2n$

Bottom Up Tree Construction



step 2

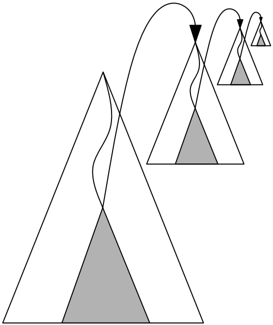


step 3

Time Complexity

- Construction
  - $O(n \log n)$ for sorting the points on x and on y.
  - $O(k)$ to build a k-point associated tree.
  - Time for primary node v is linear in the size of $P(v)$.
  - Construction time is linear in sum of $P(v)$.
  - It is $O(n \log n)$ because space is $O(n \log n)$.
- Search
  - $O(\log n)$ for the primary tree.
  - $O(\log n + k_v)$ for an associated tree with k_v outputs.
  - $O(\log^2 n + k)$ for k outputs.
  - Fractional cascading reduces this to $O(\log n + k)$.

Higher Dimensional Range Trees

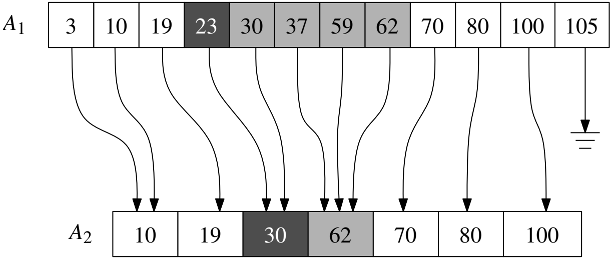


- Approach generalizes to dimension d.
- Time complexity is $O(\log^d n + k)$.
- Programming looks painful.
- Fractional cascading removes a $\log n$ factor.

General Sets of Points

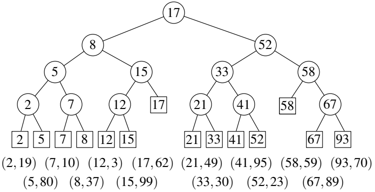
- The distinct coordinates assumption is easily removed.
- Define a composite number as a pair of numbers $(a|b)$.
- Define $(a|b) < (c|d)$ using lexicographic order.
- Replace a point p with $\hat{p} = ((p_x|p_y), (p_y|p_x))$.
- The resulting points have distinct coordinates.
- Form a kd-tree or a range tree for these points.
- Replace a query rectangle $[a, b] \times [c, d]$ with $[(a| - \infty), (b|\infty)] \times [(c| - \infty), (d|\infty)]$.
- The data need not be changed; just the comparison code.
- The method works in any dimension.

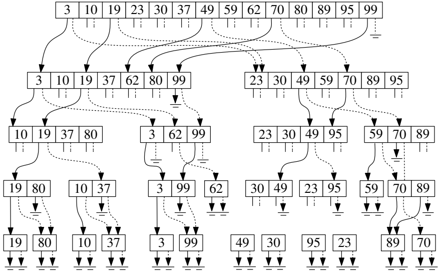
Fractional Cascading



- Remove a $\log n$ factor from searches of related ordered sets.
- A_1 and A_2 are sorted arrays with A_2 a subset of A_1.
- Each element of A_1 points to its least upper bound in A_2.
- The pointers are set in $O(n)$ time via array traversal.
- Search a range $[y : y']$ in A_1 and A_2 in $O(\log n + k)$ time.
  1. Find the first index $a \geq y$ in A_1 by binary search.
  2. Follow the a pointer to the first index $b \geq y$ in A_2.
  3. Traverse A_1 and A_2 until the indices are greater than y'.
- Example: $[20 : 65]$, $a = 23$, $b = 30$, grey elements returned.

Fractional Cascading in a 2D Range Tree





- Replace associated trees with sorted arrays.
- The array of primary node v stores pointers to its y greatest lower bounds in the arrays of $lc(v)$ and $rc(v)$.
- Search the primary tree as before.
- Compute the first array index with $p_y \geq y$ at the split node.
- Update this index at later nodes using the array pointers.
- For each node whose primary set is in the x range, traverse the associated array starting from its first array index.