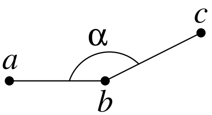
Predicates

Elisha Sacks

Planar Vector Geometry

- Vectors represent positions and directions.
- Vector u has Cartesian coordinates $u = (u_x, u_y)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector: $u/||u||$.
- Cross product: $u \times v = u_x v_y - u_y v_x$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos \alpha$.
- $u \times v = ||u|| \cdot ||v|| \cdot \sin \alpha$.

Predicates

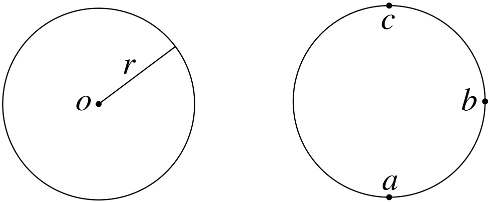


- A predicate is a polynomial in the parameters of objects.
- Our parameters are the Cartesian coordinates of points.
- We have already seen the left turn predicate for 2D points $\mathrm{LT}(a, b, c) = (c - b) \times (a - b)$.
- It has the same sign as $\sin \alpha$ with $\alpha = \angle(c - b, a - b)$.
- It can also be expressed as the determinant

$$\mathrm{LT}(a, b, c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix}$$

- Another simple predicate is the order of points a and b in direction u: $(b - a) \cdot u$ is positive if b comes after a.

Circles



- A circle can be represented by a center o and a radius r.
- A circle can also be represented by points a, b, and c.
- The first representation has three independent parameters.
- The second representation has six dependent parameters.
- Circle predicates depend on the choice of representation.
- A point p is outside an o, r circle if $||p - o|| - r$ is positive.
- The predicate can be rewritten without a square root as $(p - o) \cdot (p - o) - r^2$.

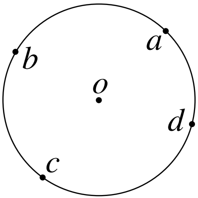
Point in Circle

- The predicate for a point p and an a, b, c circle is

$$\begin{vmatrix} a_x & a_y & a \cdot a & 1 \\ b_x & b_y & b \cdot b & 1 \\ c_x & c_y & c \cdot c & 1 \\ p_x & p_y & p \cdot p & 1 \end{vmatrix}$$

- The predicate is positive when p is outside the circle if a, b, c are in counterclockwise order around the circle.
- Replacing p with (x, y) and expanding along the last row yields $\mathrm{LT}(a, b, c)(x^2 + y^2) + ux + vy + w$.
- This is the equation of a circle after dividing by $\mathrm{LT}(a, b, c)$.
- It is the circle through a, b, c because the determinant is zero when p equals a, b, or c, since two rows are equal.
- It is positive for sufficiently large p because the LT is positive.

Angle Order



- Task: sort points counterclockwise around a point o.
- Need to define the order of points a and b around o.
- If $a_y > o_y$ and $b_y < o_y$, a is first.
- If $a_y < o_y$ and $b_y > o_y$, b is first.
- Otherwise, a is first if $\mathrm{LT}(a, o, b) < 0$.
- What are the degenerate cases?

Spatial Vector Geometry

- Vectors represent positions and directions.
- Vector u has coordinates $u = (u_x, u_y, u_z)$.
- Inner product: $u \cdot v = u_x v_x + u_y v_y + u_z v_z$.
- Vector length: $||u|| = \sqrt{u \cdot u}$.
- Unit vector: $u/||u||$.
- Cross product:
  $u \times v = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$
- Let α be the angle between u and v.
- $u \cdot v = ||u|| \cdot ||v|| \cdot \cos\alpha$.
- $u \times v = (||u|| \cdot ||v|| \cdot \sin\alpha)\, n$ with n a unit-vector perpendicular to u and v.

Predicates

- Point d is on the counterclockwise side of triangle abc if

$$\mathrm{LT}(a,b,c,d) = \begin{vmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ d_x & d_y & d_z & 1 \end{vmatrix} > 0.$$

- Point p is outside the sphere through points a, b, c, d with $\mathrm{LT}(a,b,c,d) > 0$ if

$$\begin{vmatrix} a_x & a_y & a_z & a \cdot a & 1 \\ b_x & b_y & b_z & b \cdot b & 1 \\ c_x & c_y & c_z & c \cdot c & 1 \\ d_x & d_y & d_z & d \cdot d & 1 \\ p_x & p_y & p_z & p \cdot p & 1 \end{vmatrix} > 0.$$