A dimensionality paradigm
for surface interrogations

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Abstract. We propose a paradigm for analyzing problems involving complex curved surfaces in a manner suitable for practical implementation. Rather than deriving closed-form expressions for certain surfaces and problems, we reformulate the problem in a higher-dimensional space with more variables but simpler equations, thus avoiding complex symbolic manipulation and numerically delicate operations. In particular, we consider the formation of offsets, spherical blends of fixed or variable radius, and Voronoi surfaces.

1. Introduction

Many technical difficulties must be faced when creating the algorithmic infrastructure needed for solid modeling with curved surfaces. Specifically, a number of desirable surface operations, including offsetting, spherical blending, and the formation of Voronoi surfaces, raise difficult mathematical problems that must be solved in order to represent and interrogate the resulting surfaces. While exact, closed-form representations could be derived in principle, it is often not feasible to do so because of time and space requirements of the needed calculations. For this reason, a number of approximation schemes have been advocated in the literature, including [Chandru et al. '90, Farouki '86b, Rossignac & Requicha '84].

It would appear that there is little hope to construct exact representations of offsets, spherical blends, or Voronoi surfaces, except in very limited settings. However, when these operations are reformulated in different spaces, then they become much simpler. But these spaces have more than three dimensions, and the question arises whether the reformulation of the problem can be used directly as a surface representation in lieu of an equivalent parametric or implicit representation in familiar three space. There are indications that this is indeed the case, and that doing so realizes certain advantages. These advantages include:

1. The higher-dimensional representation has lower algebraic degree than an equivalent representation in three space would have, and should therefore be easier to query numerically. Moreover, certain costly elimination steps that would be needed to find the representation in three space become unnecessary.

2. When creating the higher-dimensional representation, fewer operations are performed on the numerical input data. Therefore, the resulting set of equations has coefficients that are expected to be more precise, and therefore should yield results of greater accuracy than could be obtained by deriving closed-form representations first.

3. In the higher-dimensional problem formulation, additional structural properties of the
gometric operation are explicit. These properties help in applications such as motion
planning or machining. For example, when offsetting a parametric surface \( f \), we obtain
simultaneously a projection function that associates with each point \( p \) on the offset surface
its projection \( p_f \) on \( f \) and the parametric coordinates of \( p_f \).

In this paper, we examine the construction of offset surfaces, of Voronoi surfaces, and of
variable-radius blending surfaces as examples of this approach, and demonstrate that the
higher-dimensional formulation can be used directly for surface intersection. The higher-dimen-
sional formulation has been used earlier by Morgan [Morgan '81], for finding intersections
between lines and fixed-radius tubular surfaces.

There are several open issues related with our approach, including the following:
1. finding points on the surface,
2. analyzing the local surface behavior, and
3. excluding extraneous points from the surface formulation.
We are working on solutions for these and related problems, and make some comments
concerning them at the end of this paper.

2. Surface intersections

Geisow [Geisow '86] argues that the evaluation of surface intersection is reducible to
evaluating a plane curve \( h(u, v) = 0 \). In particular, the intersection of the implicit surface
\[ f(x, y, z) = 0 \]
with the parametric surface \( g \) given by
\[ x = g_1(u, v), \quad y = g_2(u, v), \quad z = g_3(u, v) \]
is readily seen to be in birational correspondence with the plane curve
\[ f(g_1, g_2, g_3) = h(u, v) = 0. \]
See, e.g., [Farouki '86a, Pratt & Geisow '86]. The intersection of two parametric surfaces can be
similarly mapped to a plane curve after first implicitizing one of the surfaces, e.g., [Sederberg
'83], a procedure always possible albeit not necessarily efficient. Even the intersection of two
implicit surfaces \( f \) and \( g \) can be so approached, by finding in the ideal of the two surfaces a
monoid \( \mathcal{G} \) that contains the intersection and is readily parameterized, [Hoffmann '87]. Related
approaches to the intersection of two implicit surfaces are given in [Abhyankar & Bajaj '87,
Garrity & Warren '89].

While mathematically tidy and conceptually appealing, evaluating surface intersection in this
way is beset by a number of practical difficulties, including the following:

1. Implicitizing a parametrically defined surface entails substantial symbolic computation.
   Already in the case of curves of degree larger than 3, [Sederberg & Parry '88] counsel against
   elimination based on resultants. Moreover, in the absence of more sophisticated techniques
   such as [Buchberger '85, Buchberger et al. '88], the derived implicit form may have
   extraneous factors that are difficult to eliminate.

2. Substitution of the parametric functions into the implicit form, although conceptually
   straightforward, is numerically delicate, and can lead to substantial errors [Prakash &
   Patrikalakis '88].
3. By Bezout's theorem, the degree of \( h \) is equal to the product of the degrees of the intersecting surfaces. Thus, with the resulting high algebraic degrees numerical difficulties may arise even when evaluating \( h \) at some given point \( p \). See also [Farouki & Rajan '87]. Therefore, what looks attractive from a theoretical vantage point, may not work at all when implemented, and alternative approaches should be developed.

We propose an alternative a dimensionality paradigm that seeks to keep the problem manageable by reformulating the computation at hand. Specifically, we formulate an equivalent problem in higher-dimensional space, using more variables and more, but simpler equations. We demonstrate this paradigm intersecting a number of mathematically complicated surfaces.

In [Bajaj et al. '88] an algorithm was presented for tracing the intersection of two surfaces. The algorithm presented there in Section 3 can be applied without any essential modification to surface intersection in all dimensions. It repeats the following three steps, for \( i = 0, 1, 2, \ldots \):

\textbf{Step 1.} Given an initial point estimate \( q_i \) on the curve, refine it using Newton iteration to a point \( p_i \).

\textbf{Step 2.} At \( p_i \), construct a local approximant \( r(s) \), to second or third order.

\textbf{Step 3.} Choose adaptively a step size \( s_{i+1} \), and derive a new point estimate \( q_{i+1} = r(s_{i+1}) \).

In the examples of this paper, we compute a second order approximant to the curve and choose a step size such that the contribution of the second order term is not more than 10 percent. With this proviso, typically 2 or 3 Newton iteration steps are performed to refine the next curve point estimate to ten digits precision.

3. Offset surfaces

Given a surface \( f = 0 \), its \( r \)-offset consists of the points

\[ \text{Off}(f, r) = \{ p | d_f(p) = r \} \]

where \( d_f(p) \) is the Euclidean distance of the point \( p \) from the surface \( f = 0 \). Informally, at each regular surface point \( p \) of \( f \) we erect the surface normal and on it, at distance \( r \) in either direction, mark a point. Both points so obtained belong to \( \text{Off}(f, r) \).

One could define the distance function \( d_f(P) \) as the minimum Euclidean distance of \( p \) from a surface point \( q \), where \( q \) ranges over the entire surface. With such a definition, the offset of an algebraic surface is not, in general, algebraic, [Hoffmann '89b]. Here, we adopt the view that \( d_f(p) \) is a local minimum; that is, there is a point \( q \) of \( f \), at distance \( r \), such that the line \( \{ p, q \} \) is perpendicular to \( f \) at \( q \). Note that offset surfaces may contain self-intersections and other types of singularities.

In general, the offset surface has two sheets in real affine space, one sheet formed by the points offset from the surface in one direction, the other sheet formed by the points offset in the other normal direction. If \( f \) is an irreducible algebraic surface, both sheets together will in general be described by a single, irreducible algebraic equation, evidencing the fact that both belong to the same irreducible surface. Exceptions to this include the offsets of a sphere that consist of two spheres, each separately definable by an algebraic equation.

In the traditional offset formulation, e.g., [Farouki '86b], the second sheet does not appear to be constructed. The formulation involves a square root, to compute a unit normal vector, and,

\footnote{The nonalgebraic offset, defined with help of a global distance function, is actually a subset of our offset definition. It can be determined, in principle, from our offset surfaces by pruning certain surface portions adjacent to self-intersection curves.}
by convention, the positive root is assumed. We refer to this definition as the differential offset. In [Hoffmann '89b], we have demonstrated that the differential offset formulation can lead to qualitatively different offsets depending on whether the curve is given implicitly or parametrically, and that, in the differential formulation, the offset of an algebraic curve or surface is not necessarily algebraic.

Offset surfaces can be defined mathematically with the envelope theorem from differential geometry [Spivak '75] that states that the envelope of a parameterized family $S(\alpha) = 0$ of surfaces satisfies the system of equations

$$S(\alpha) = 0,$$

$$\frac{\partial S(\alpha)}{\partial \alpha} = 0.$$  \hspace{1cm} (1)

With $\alpha$ a vector of $m$ independent parameters, the equations (2) are the $m$ first-order partial derivatives of $S(\alpha)$ by each component $u$ of $\alpha$. Thus the system consists of $m + 1$ equations.

If the parameters $\alpha$ are not all algebraically independent, e.g., are the coordinates of points on another surface, then the equations (2) must be replaced by the directional derivatives of $S(\alpha)$ and the defining dependency equations among the parameters must be added as additional constraints.

We formulate the $r$-offset of $f$ as the envelope of a family of spheres with radius $r$ whose centers are constrained to lie on the surface. Fig. 1 illustrates this concept in two dimensions. Assume that the surface $f$ is given in parametric form as

$$x = f_1(s, t), \quad y = f_2(s, t), \quad z = f_3(s, t).$$

Then its $r$-offset is described by the system

$$S(s, t) = 0, \quad \frac{\partial S(s, t)}{\partial s} = 0, \quad \frac{\partial S(s, t)}{\partial t} = 0$$

where

$$S(s, t) = (x - f_1(s, t))^2 + (y - f_2(s, t))^2 + (z - f_3(s, t))^2 - r^2.$$  \hspace{1cm} (3)

Next, assume that the surface is given by an implicit equation $f(x, y, z) = 0$. Let $p = (u_1, u_2, u_3)$ be a regular point of $f$, and let $\nabla f(p) = (a, b, c)$ be the surface gradient at $p$. We construct the equations for the $r$-offset by taking spheres of radius $r$ centered on the surface:

$$S: \quad (x - u_1)^2 + (y - u_2)^2 + (z - u_3)^2 - r^2 = 0,$$

$$f(u_1, u_2, u_3) = 0$$

and forming the directional derivatives by multiplying the vector of partials of $S$ by the $u_i$ with two linearly independent vectors perpendicular to the gradient at $p$. For example, with the two perpendiculars taken as $(-b, a, 0)$ and $(-c, 0, a)$, we obtain the directional derivatives

$$S_1 = \nabla S \cdot (-b, a, 0) \quad \text{and} \quad S_2 = \nabla S \cdot (-c, 0, a)$$

where

$$\nabla S = \left( \frac{\partial S}{\partial u_1}, \frac{\partial S}{\partial u_2}, \frac{\partial S}{\partial u_3} \right).$$
So, the $r$-offset of $f$ is described by the system

\[ S = 0, \quad f = 0, \quad S_1 = 0, \quad S_2 = 0. \]  \hfill (4)

We could have chosen a different pair of perpendiculars, e.g., $(b - c, a - b, c - a)$. Note, however, that the choice of perpendiculars may introduce extraneous solutions, [Hoffmann '89a], Ch. VII]. For example, the two peripherals $(-b, a, 0)$ and $(-a, 0, a)$ will introduce the extraneous factor $x$. This is the faithfulness problem of [Hoffmann '89b]. See also the remarks at the end of this paper.

In principle, we may recover the implicit equation of the $r$-offset from system (3) by eliminating both $s$ and $t$, and from system (4) by eliminating $u_1$, $u_2$, and $u_3$. If elimination is done using resultant methods, the final equation may contain additional extraneous factors. These factors may be difficult to find and eliminate. Alternatively, the elimination can be attempted using Gröbner basis techniques, e.g., as described in [Hoffmann '89a], avoiding the extraneous factor problem. Except for simple situations, however, one must expect that both approaches require very long running times. 2

An additional difficulty, not normally addressed in the literature, is the fact that we usually do not have exact surface equations to begin with. So, the symbolic computations either work with rational coefficient approximations of uncertain accuracy, or else incur arithmetic errors that have not been analyzed in the literature and thus cannot be predicted with confidence. In practical settings, therefore, even if the computation times were negligible, the value of eliminating the problem parameters would have to be carefully considered.

**Example 1.** We offset a quadric surface $f = 0$ by the distance 2 and intersect it with a quartic surface $h = 0$. The implicit equation of the offset of $f$ has degree 8, so the intersection curve with $h$, when properly mapped, must result in a plane curve of degree 22 which will be moderately hard to handle numerically. In contrast, the system to be formulated consists of equations with maximum degree 4 and is quite easy to treat numerically.

We formulate the offset equations first, with $f$ the ellipsoid $16x^2 + 36y^2 + 9z^2 - 144 = 0$.

The ellipsoid is centered at the origin, and its axes are 3, 2, and 4, respectively.

\[
(x - u_1)^2 + (y - u_2)^2 + (z - u_3)^2 - 4 = 0, \\
16u_1^2 + 36u_2^2 + 9u_3^2 - 144 = 0, \\
36u_2(x - u_1) - 16u_1(y - u_2) = 0, \\
9u_3(x - u_1) - 16u_1(z - u_3) = 0.
\]  \hfill (5)

To this system, we add the equation of the quartic surface

\[
(x^2 - 10x + y^2 + z^2 + 21)^2 - 20(x^2 - 10x + 29) - 16(y^2 - z^2) - 48x - 176 = 0.
\]

The shapes of the quartic and of the ellipsoid are shown in Fig. 2. Note that the quartic surface is a *ring cyclide*, [Chandru et al. '89], whose shape resembles a torus of uneven thickness.

We trace the intersection directly from the five equations. The trace not only constructs the intersection curve, as the points $(x, y, z)$, it also constructs simultaneously the projection $(u_1, u_2, u_3)$ of this curve onto the ellipsoid. This information is useful in compliant motion applications and in numerically controlled machining problems. A graphical rendering of both curves is shown in Fig. 3.  □

**Example 2.** We consider offsetting a Bézier bicubic patch $f$ by a distance 1 and intersecting it with a quadratic surface $h = 0$. The implicit equation of the bicubic patch is of degree 18. Assuming the case of offsetting rational curves is typical, [Farouki & Neff '89], the offset of the

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2 Practical experience with three different elimination algorithms has been reported in [Hoffmann '89b].
bicubic can be expected to have degree 72 or higher. Thus, the intersection curve would have to be mapped to a plane algebraic curve of degree 144 or higher and would pose very difficult numerical evaluation problems.

Let the bicubic patch be given by the control points

\[
\begin{pmatrix}
(0, 0, 0) & (-1, 1, 1) & (0, 2, 1) & (0, 3, 0) \\
(1, 0, 1) & (1, 1, 0) & (1, 2, 1) & (1, 4, -1) \\
(2, 0, 1) & (2, 1, -1) & (2, 2, 0) & (2, 3, -1) \\
(3, 0, 0) & (3, 1, 0) & (3, 2, 1) & (3, 3, 0)
\end{pmatrix}
\]

from which we obtain the following parametric equations

\[
x = f_1(s, t) = 3t(t - 1)^2(s - 1)^3 + 3s,
\]
\[
y = f_2(s, t) = 3s(s - 1)^2t^3 + 3t,
\]
\[
z = f_3(s, t) = -3s(s^2 - 5s + 5)t^3 - 3(s^3 + 6s^2 - 9s + 1)t^2 + (6s^3 + 9s^2 - 18s + 3)t - 3s(s - 1).
\]

this patch is shown graphically in Fig. 4. Its offset would be defined by

\[
(x - f_1)^2 + (y - f_2)^2 + (z - f_3)^2 - 1 = 0
\]

and its two partial derivatives by \( s \) and by \( t \). We choose for \( h \) the sphere \( 2(x^2 + y^2 + z^2) - 6x - 6y - 4z + 9 = 0 \), of radius 1 centered at \((1.5, 1.5, 1.0)\).

A trace of the resulting system is shown in Fig. 5. The trace also delivers \((s, t)\) values for each intersection point. These are the parameter coordinates of the projection of the intersection point onto the bicubic surface. \( \Box \)

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3 We know of no implemented elimination algorithm that is capable of determining the implicit equation of the offset of this bicubic. So, there seem to be only two viable approaches to interrogating such surfaces: (1) approximate the surface locally and interrogate the approximation, or (2) interrogate the higher-dimensional representation discussed here.
Fig. 3. Intersection curve of Example 1, and its projection onto ellipsoid.

Fig. 4. Bicubic patch of Example 2 to be offset.

Fig. 5. Offset of bicubic patch intersected with sphere, and its projection in parameter space.
Note that we can apply this method also to intersecting two offset surfaces, tracing simultaneously the intersection curve and its two projections.

4. Voronoi surfaces

We are given two surfaces \( f \) and \( g \), and consider the locus of all points in space that have equal distance from either surface. The resulting point set is the Voronoi surface of \( f \) and \( g \), formally defined by

\[
\text{Vor}(f, g) = \{ p = (x, y, z) | d_f(p) = d_g(p) \}.
\]

Note the analogy with Voronoi diagrams. We will consider these surfaces in the next section as a means for defining certain blending surfaces.

We adopt the following notational conventions: The surfaces to which we seek the Voronoi surface are \( f = 0 \) and \( g = 0 \). When a surface is given parametrically, we index the parametric functions by numbers and the parameters by the surface name. Thus, a parametric surface \( f \) would be given as

\[
x = f_1(s_f, t_f), \quad y = f_2(s_f, t_f), \quad z = f_3(s_f, t_f)
\]

Moreover, a sphere centered on a surface \( f \) is denoted \( S_f \).

We define the Voronoi surface with help of offsets from each surface by a variable distance \( r \). Thus, if the surfaces are given parametrically, as

\[
x = f_1(s_f, t_f), \quad y = f_2(s_f, t_f), \quad z = f_3(s_f, t_f),
\]

\[
x = g_1(s_g, t_g), \quad y = g_2(s_g, t_g), \quad z = g_3(s_g, t_g),
\]

then the Voronoi surface may be specified by

\[
\begin{align*}
S_f : & \quad (x - f_1)^2 + (y - f_2)^2 + (z - f_3)^2 - r^2 = 0, \\
& \quad \frac{\partial S_f}{\partial s_f} = 0, \quad \frac{\partial S_f}{\partial t_f} = 0, \\
S_g : & \quad (x - g_1)^2 + (y - g_2)^2 + (z - g_3)^2 - r^2 = 0, \\
& \quad \frac{\partial S_g}{\partial s_g} = 0, \quad \frac{\partial S_g}{\partial t_g} = 0.
\end{align*}
\]  \hspace{1cm} (6)

This system is the juxtaposition of the offsets from both \( f \) and \( g \) by a common but unspecified distance \( r \). With \( r \) ranging over all possible distances, we have a description of the Voronoi surface in an eight dimensional space. Similarly, when \( f \) and \( g \) are given implicitly, their Voronoi surface is described by eight equations in ten variables.

The implicit equation of the Voronoi surface could be recovered, in principle, by elimination of \( s_f, t_f, s_g, t_g, \) and \( r \) from the system, a proposition that is hardly practical for all but the simplest surfaces. Voronoi surfaces typically have high degree, but special situations can be identified in which the Voronoi surface is very simple. For instance, the Voronoi surface or two cylinders of equal radius with skew axes passing each other is a hyperbolic paraboloid.

Moreover, the Voronoi surface of a cylinder and an inclined plane consists of two elliptic cones.

Example 3. We intersect the Voronoi surface of \( f \) and \( g \) with a third surface \( h \). For \( f \), we choose the bicubic patch of Example 2 and use for \( g \) the ellipsoid \( 4(2x - 3)^2 + 9(2y - 3)^2 + \)
36(z - 1)^2 - 36 = 0, centered at (1.5, 1.5, 1.0) with axis lengths 1.5, 1.0, and 1. The Voronoi surface is thus described by

\[ S_f: \quad (x - f_1)^2 + (y - f_2)^2 + (z - f_3)^2 - r^2 = 0, \]
\[ \frac{\partial S_f}{\partial x} = 0, \quad \frac{\partial S_f}{\partial y} = 0, \]
\[ S_g: \quad (x - v_1)^2 + (y - v_2)^2 + (z - v_3)^2 - r^2 = 0, \]
\[ g: \quad 4(2v_1 - 3)^2 + 9(2v_2 - 3)^2 + 36(v_3 - 1)^2 - 36 = 0, \]
\[ 9(2v_2 - 3)(x - v_1) - 4(2v_1 - 3)(y - v_2) = 0, \]
\[ 36(v_3 - 1)(x - v_1) - 8(2v_1 - 3)(z - v_3) = 0. \]

We intersect the surface with the elliptic cylinder

\[ h: \quad 1250x^2 - 3750x + 4050y^2 - 10530y + 5607 = 0 \]

whose axis is parallel to the z-axis through the point (1.5, 1.3, 0). Fig. 6 shows a trace of the intersection curve and its two projections onto \( f \) and \( g \). \( \Box \)

5. Variable radius blends

Given two surfaces \( f \) and \( g \), a blending surface is a surface \( F \) that intersects both \( f \) and \( g \) tangentially along some curves. A constant radius blend is a blending surface that has a family of principal curvature lines consisting of circles of fixed radius. A variable radius blend, finally, is a blending surface that has a family of principal curvature lines consisting of circles whose radius may vary. Constant radius and variable radius blends are used in designing mechanical parts.

A canal surface is the envelope surface of a family of spheres with fixed or varying radius whose centers are constrained to lie on a space curve. The space curve is called the spine of the canal surface. Canal surfaces have been studied primarily in differential geometry, and a key theorem, due to Monge, implies that a variable radius blending surface is a canal surface, i.e., the envelope of a family of spheres whose centers are constrained to lie on a space curve.

Monge's theorem gives a way for defining constant radius blending surfaces with precision. Such a blend is simply the canal surface that envelopes the family of spheres of constant radius \( r \) whose centers lie on the curve \( \text{Off}(f, r) \cap \text{Off}(g, r) \). Such blending surfaces have been considered in [Rossignac & Requicha '84].
There has been no similarly precise definition for variable radius blends, primarily because of the difficulty to quantify the radius variation of such a surface. For example, the procedural definition given in [Pegna '88] is unsatisfactory because the algorithm for defining the radius variation contains unspecified degrees of freedom that affect the shape of the final surface. For this reason, we have made in [Chandru et al. '90] the following definition of a variable radius blend for the surfaces \( f \) and \( g \):

Consider the intersection \( \text{Vor}(f, g) \cap h \) of the Voronoi surface \( \text{Vor}(f, g) \) of \( f \) and \( g \) with a reference surface \( h \). Then a variable radius blend of \( f \) and \( g \) is the envelope of the family of spheres whose centers lie on \( \text{Vor}(f, g) \cap h \) and whose radii are such that each sphere touches both \( f \) and \( g \).

Hence, \( \text{Vor}(f, g) \cap h \) is the spine of the variable radius blend. Moreover, we can project each point \( p \) of the intersection onto the two surfaces, obtaining the points \( p_f \) and \( p_g \) at which the sphere centered at \( p \) touches \( f \) and \( g \). A section of the variable radius blend can then be constructed from the three points \( p \), \( p_f \), and \( p_g \) as explained below.

We construct a higher dimensional representation of a variable radius blending surface, from the definition given above. To this end, we will derive a system of equations defining the surface. The exact number of equations and variables depends on how the original surfaces were specified. See also the example below. It would be difficult to imagine that elimination of the additional variables could succeed in efficiently deriving an implicit equation for this blend. In [Chandru et al. '90] we have explored the alternative of approximating a variable radius blend with quartic surface elements.

As first step, we formulate the intersection curve of the Voronoi surface \( \text{Vor}(f, g) \) with some reference surface \( h \). We alter the system (6) by renaming the variables \( x \), \( y \), and \( z \) to \( u_1 \), \( u_2 \), and \( u_3 \), respectively. Assume that the reference surface is given by

\[
x = h_1(s_h, t_h), \quad y = h_2(s_h, t_h), \quad z = h_3(s_h, t_h)
\]

Then we adjoin to the system with the renamed variables the equations \( u_i = h_i(s_h, t_h) \). The enlarged system specifies that the point \( (u_1, u_2, u_3) \) must lie on the intersection \( \text{Vor}(f, g) \cap h \).

The next step in defining the variable radius blend is to formulate a description of the spheres that are centered on the intersection of the Voronoi and the reference surface, and have a radius such that they touch both \( f \) and \( g \). The center of a sphere in this family is simply the point \( (u_1, u_2, u_3) \). Its radius must be \( r \), since this is the distance of the center from both surfaces to be blended. Hence we add the equation

\[
S_r : (x - u_1)^2 + (y - u_2)^2 + (z - u_3)^2 - r^2 = 0
\]

to describe the family of spheres whose envelope is the desired blending surface.

In order to define the envelope of the family, we must add an equation that is the derivative of \( S_r \) in the tangent direction. At the point \( (u_1, u_2, u_3) \), the tangent direction to the spine is given by the cross product of the surface normals of \( \text{Vor}(f, g) \) and of \( h \). The normal to \( h \) is easily specified as the cross product

\[
N_h = \left( \frac{\partial h_1}{\partial s_h}, \frac{\partial h_2}{\partial s_h}, \frac{\partial h_3}{\partial s_h} \right) \times \left( \frac{\partial h_1}{\partial t_h}, \frac{\partial h_2}{\partial t_h}, \frac{\partial h_3}{\partial t_h} \right).
\]

The normal to \( \text{Vor}(f, g) \) is obtained as follows. Consider the lines connecting the point \( p = (u_1, u_2, u_3) \) on the Voronoi surface with its projections \( p_f = (f_1, f_2, f_3) \) and \( p_g = (g_1, g_2, g_3) \), where the coordinate functions \( f_j \) and \( g_j \) are evaluated at \( (s_f, t_f) \) and at \( (s_g, t_g) \) respectively. Since the lines \( (p, p_f) \) and \( (p, p_g) \) are perpendicular to \( f \) and \( g \), respectively, it follows that the tangent plane to the Voronoi surface at \( p \) is the bisecting plane of the two lines...
[Wolter '85], as illustrated in Fig. 7. Since the vectors \((\overrightarrow{p}, \overrightarrow{p_y})\) and \((\overrightarrow{p}, \overrightarrow{p_x})\) have equal length, the normal to the Voronoi surface at \(p\) is therefore the vector
\[ N_v = (\overrightarrow{p}, \overrightarrow{p_y}) - (\overrightarrow{p}, \overrightarrow{p_x}) = (f_1 - g_1, f_2 - g_2, f_3 - g_3). \]

We compute the cross product of the two normals and obtain the tangent vector to the intersection curve at \(p\) as
\[ T = N_y \times N_v. \]

So, the directional derivative of \(S_h\), to be adjoined to the system, is
\[ D = \left( \frac{\partial S_y}{\partial u_1}, \frac{\partial S_y}{\partial u_2}, \frac{\partial S_y}{\partial u_3} \right) \cdot T = 0. \]

This gives us as final system describing the variable radius blend
\[ S_f: \quad (u_1 - f_1)^2 + (u_2 - f_2)^2 + (u_3 - f_3)^2 - r^2 = 0, \]
\[ \frac{\partial S_f}{\partial u_1} = 0, \quad \frac{\partial S_f}{\partial u_2} = 0, \quad \frac{\partial S_f}{\partial u_3} = 0, \]
\[ S_g: \quad (u_1 - g_1)^2 + (u_2 - g_2)^2 + (u_3 - g_3)^2 - r^2 = 0, \]
\[ \frac{\partial S_g}{\partial u_1} = 0, \quad \frac{\partial S_g}{\partial u_2} = 0, \quad \frac{\partial S_g}{\partial u_3} = 0, \]
\[ u_1 = h_1(s_h, t_h), \quad u_2 = h_2(s_h, t_h), \quad u_3 = h_3(s_h, t_h). \]
\[ S_h: \quad (x - u_1)^2 + (y - u_2)^2 + (z - u_3)^2 - r^2 = 0, \]
\[ D = 0. \]

It is important to note that the radius \(r\) varies as a function of the point \((u_1, u_2, u_3)\). In consequence, the partial derivative of \(S_h\) by \(u_1\), is
\[ -2(x - u_1) - 2r \frac{\partial r}{\partial u_1}. \]

Usually, \(r\) is not known explicitly as a function of the \(u_i\). Therefore, we proceed as follows.
From a geometric point of view, equations \( S_p \) and \( D = 0 \) express that the surface points of the variable radius blend are on a curve that is the limit of the intersection of the sphere centered at \( p = (u_1, u_2, u_3) \) with the sphere at a point \( q \) on the spine as \( q \) approaches \( p \). Since the intersection of two spheres is a circle, the limit curve is also a circle. Moreover, since the tangent is a first-order approximation of the spine at \( p \), the limit circle must lie in a plane \( P \) that is perpendicular to the tangent at \( p \). Finally, the points \( p_1 = (f_1, f_2, f_3) \) and \( p_2 = (g_1, g_2, g_3) \) at which the sphere centered at \( p \) touches \( f \) and \( g \), respectively, must lie on this circle. Therefore, the equation \( D = 0 \) can be replaced with equations that describe the plane \( P \) and state that \( P \) contains \( p_1, p_2, \) and the point \( (x, y, z) \). See also Fig. 8. Note that the components \((t_1, t_2, t_3)\) of the tangent vector can be used as the coefficients of the \( x, y, \) and \( z \) term in the plane equation. The constant term \( d \) of the plane equation is an additional variable whose value is proportional to the distance of the center of the sphere, \((u_1, u_2, u_3)\), to \( P \). In summary, the equation \( D = 0 \) in (8) is replaced with the equations

\[
P: \quad t_1x + t_2y + t_3z + d = 0, \\
t_1f_1 + t_2f_2 + t_3f_3 + d = 0, \\
t_1g_1 + t_2g_2 + t_3g_3 + d = 0.
\]

Since both \( p_1 \) and \( p_2 \) lie in \( P \), and since we know the plane normal \((t_1, t_2, t_3)\), it follows that the last two equations are algebraically dependent. One or the other could therefore be omitted from the final system.

**Example 4.** We construct a variable radius blend to the surfaces \( f \) and \( g \) of Example 3. We use as reference surface the surface \( h \) of that example. Then the equations describing the blending surface are

\[
S_f: \quad (u_1 - f_1)^2 + (u_2 - f_2)^2 + (u_3 - f_3)^2 - r^2 = 0, \\
\frac{\partial S_f}{\partial s} = 0, \quad \frac{\partial S_f}{\partial t} = 0, \\
S_g: \quad (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 - r^2 = 0, \\
g: \quad 4(2v_1 - 3)^2 + 9(2v_2 - 3)^2 + 36(v_3 - 1)^2 - 36 = 0, \\
9(2v_2 - 3)(u_1 - v_1) - 4(2v_1 - 3)(u_2 - v_2) = 0, \\
36(v_3 - 1)(u_1 - v_1) - 8(2v_1 - 3)(u_2 - v_2) = 0, \\
h: \quad 1250u_1^2 - 3750u_1 + 4050u_2^2 - 10530u_3 + 5607 = 0, \\
S_h: \quad (x - u_1)^2 + (y - u_2)^2 + (z - u_3)^2 - r^2 = 0, \\
\left( \frac{\partial S_h}{\partial u_1}, \frac{\partial S_h}{\partial u_2}, \frac{\partial S_h}{\partial u_3} \right) = 0, \\
T = 0
\]

where \( T \) is

\[
T = (250u_1 - 375, 810u_2 - 1053, 0) \times (v_1 - f_1, v_2 - f_2, v_3 - f_3).
\]

The last equation is replaced with

\[
P: \quad t_1x + t_2y + t_3z + d = 0, \\
t_1v_1 + t_2v_2 + t_3v_3 + d = 0
\]

where

\[
t_1: \quad (810u_2 - 1053)(v_3 - f_3), \\
t_2: \quad -(250u_1 - 375)(v_3 - f_3), \\
t_3: \quad (250u_1 - 375)(v_2 - f_2) - (810u_2 - 1053)(v_1 - f_1).
\]
The first equation states that \( P \) contains the point \((x, y, z)\) of the blending surface. The second equation states that \( p_\xi = (v_1, v_2, v_3) \) is also on the plane.

These equations differ somewhat from system (8) because \( g \) and \( h \) are given implicitly rather than parametrically. To obtain a visual impression of the surface, we intersect the blend with planes through the axis of symmetry of \( h \), given by

\[
\cos(\omega)(10x - 13) + \sin(\omega)(10y - 15) = 0
\]

for several angles \( \omega \). The resulting curves are approximately circular but are not circles in general. All intersection curves are shown in Fig. 9, along with the two curves at which the blend is tangent to the bicubic surface and to the ellipsoid. \qed

6. Conclusions

We have demonstrated that certain surface operations can be formulated straightforwardly in higher-dimensional spaces, and that such formulations can be used directly in surface
interrogation. One advantage of this paradigm is its success in coping with curves and surfaces of very high algebraic degree.

The Table 1 shows in summary the number of variables for each example, the highest actual degree of the equations in the higher-dimensional problem formulation, and a very conservative estimate of the degrees of the intersection curves. The curve degrees were estimated as follows: By Bezout's theorem, the intersection degree equals the product of the degrees of the intersecting surfaces. Implicitization shows that the ellipsoid offset has degree 8. The bicubic patch has degree 18. An offset and a Voronoi surface constructed from a surface of degree n should have a degree at least 4n. Likewise, the variable-radius blend should have a degree at least four times the spine degree.

A number of approaches to surface intersection may be conceptually classified as follows. One extreme is the mapping approach in which the surface intersection curve is mapped birationally to a single, bivariate equation; e.g., [Garrity & Warren '89]. It has the minimum number of variables possible, and the highest algebraic degree. Another extreme is our approach that trades algebraic degree of the equations against the number of variables. Intermediate approaches would correspond to problem formulations with fewer equations and fewer variables. Algorithmic conversion of the higher-dimensional formulation towards the mapping approach is the process of elimination. Conversion is the other direction would be an algorithm that replaces a single equation by several equivalent equations, in more variables, but of strictly lower degrees. There are only very few examples of this process, and they are not algorithmic. For instance, certain surfaces of degree 4 can be represented by linear and quadratic equations in a five-dimensional space; e.g., [Blutel 1890, Degen '82]. We feel that an algorithm that systematically achieves such a conversion would be of value, for it is our experience that it is numerically easier to work with lower-degree equations, in higher dimensions, than with a single bivariate algebraic equation of very high degree. In this paper, we have demonstrated that the geometric structure of offsets, blends, and Voronoi surfaces, can be exploited to derive a low-degree, higher-dimensional formulation.

When analyzing surface intersections, one of the problems that must be solved is to find initial starting points on each branch of the intersection curve. In joint work with C.-S. Chiang, we have begun to explore this problem. Briefly, by converting each equation to a Bezier representation, standard subdivision and domain reduction techniques can be generalized to find initial points. However, care has to be taken that the number of needed control points is kept manageable. In other, unpublished work, joint with J.-H. Chuang, we give techniques for analyzing surfaces defined by a set of algebraic equations in the vicinity of a given point. Methods for deriving local parameterizations have been found that do not depend on the structure of the system of equations, and therefore could be used in a much broader context than the setting of this paper.

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4 See, e.g., [Sederberg '88] for a method to intersect two plane algebraic curves in this manner.
In Section 3 we mentioned that the equations defining the offset surface may degenerate at certain special points at which one or several of the equations vanish, or become algebraically dependent. In the context of rational plane curves, Farouki and Neff [Farouki & Neff '89] have analyzed these phenomena. They give methods for excluding degeneracies of this kind completely in the case of polynomially rational curves, and partially for general rational plane curves. Excluding degeneracies completely is an aspect of the problem of faithfulness, discussed in [Hoffmann '89b], and we are working on a systematic solution for it.

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