Variable-Radius Circles in Cluster Merging
Part II: Rotational Clusters*

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June 29, 2001

Abstract

Variable-radius circles are common constructs in planar constraint solving. We give a complete treatment of variable-radius circles when such circles must be determined simultaneously with placing two groups of geometric entities.

Part I sets up the problem statement and considers clusters where the relative motion is translational. It also reviews past work on the subject. Part II treats rotational clusters motion.

Keywords: geometric constraint solving, variable-radius circles, triangle decomposition, algebraic solver, cyclographic maps.

*Work supported in part by NSF Grant CCR 99-02025 and by ARO Contract 39136-MA, by NSC in Taiwan Grant 39201F, and by the Purdue Visualization Center.

1 Introduction

A geometric constraint problem consists of a (finite) set of geometric elements and a (finite) set of constraints between them. The geometric elements are drawn from a fixed universe such as point, lines, circles and conics in the plane, or points, lines, planes, cylinders and spheres in 3-space. The constraints are logical constraints such as incidence, tangency, perpendicularity, etc., or metric constraints such as distance or angle. The solution of a geometric constraint problem is a coordinate assignment of the geometric elements such that all constraints are satisfied, or a message that such an assignment cannot be found.

We consider planar constraint solving in which a decomposition algorithm has grouped elements recursively into clusters such that three clusters can be combined into a larger one. In the problems we consider, one of the clusters is a variable-radius circle, the other two share a geometric element. The variable-radius circle has four constraints on it, two with elements of one cluster, and two with elements of the other cluster. These types of constraint subproblems arise, for instance, in solvers based on a recursive degree-of-freedom analysis, such as the solvers in [7, 1, 2, 5, 3, 4].

In the first part of this paper, we considered the cases in which the shared geometric element is a line. This constrained the relative motion of the two clusters to a translation. In this paper, we consider that the shared element is a point or a circle of known radius. That is, the relative motion of the two clusters is now a rotation. The rotational case requires solving more complicated systems of equations than the translational case.

Prior work on constraint solving and on variable-radius clusters has been reviewed in Part I. We now give an algebraic solution of the problem of variable-radius circles that are clusters and are determined by cluster merging with rotation. When using triangle decomposition, such circles are merged with two other clusters and have four constraints upon them.

2 Problem Statement and Notation

2.1 Problem Description

We consider two clusters S_1 and S_2 that share as a common element a point or a fixed-radius circle. Thus, S_2 can move relative to S_1 by a rotation about the shared point or the center of the shared circle. The cluster S_1 contains the geometric elements E_1 and E_2, lines or circles. For uniformity, we consider points to be circles of zero-radius. The cluster S_2 contains the geometric elements E_3 and E_4. There is also a variable-radius circle and each of the four elements E_k, $k = 1...4$, constrains that circle, by distance, incidence, or tangency. Note that all those constraints can be reformulated equivalently as incidence/tangency constraints.

2.2 Notation

We use homogeneous coordinates (x, y, z) for points in 2-space; we have $z = 1$ for finite points. Lines have the (homogeneous) coordinates $[a, b, c]$ in the line equation $ax + by + cz = 0$, where (x, y, z) is a point on the line. For finite lines we assume that $a^2 + b^2 = 1$. In the plane, points and lines are dual of each other: If we fix $[a, b, c]$, the equation $ax + by + cz = 0$ represents all points on the line; if we fix (x, y, z), then the equation represents all lines through the point.

We will use mappings from plane geometric objects to geometric objects in 3-space in order to simplify solving the nonlinear equations that arise in the constraint schema. In 3-space, planes have the coordinates $[A, B, C, D]$ and points (X, Y, Z, W), with W the homogenizing coordinate. The duality of points and planes in 3-space is established by the equation $AX + BY + CZ + DW = 0$.

When concentrating on affine (finite) points, we will write $(x, y)_E$ for points in the affine plane and $(X, Y, Z)_E$ for points in affine 3-space. Furthermore, we write $[x, y]_E$ and $[X, Y, Z]_E$ to represent vectors in affine 2- and 3-space. Recall that $(X, Y, Z, W) = (X/W, Y/W, Z/W)_E$ when $W \neq 0$.

We consider oriented geometric elements. This allows us to simplify the algebraic equations and lower the degree of the resulting systems. For example, two circles may have up to four tangents, but two oriented circles have only up to two oriented tangents, because we require that they are tangent with a consistent orientation. We do not lose solutions as long as we consider all relevant orientation combinations. The approach goes back to [6].

The oriented circle, or *cycle*, in 2D with center $(x, y)_E$ and radius r can be represented as the 3D point $(x, y, z)_E = (x, y, r, 1)$. The sign of r signifies the orientation of the cycle: If $r > 0$, the cycle is oriented counter-clockwise; if $r < 0$, the cycle is oriented clockwise; if $r = 0$, the cycle represents a 2D point and is considered to have both orientations simultaneously.

The oriented line, or *ray*, is defined as the line $[a, b, c]$ with an orientation. That is, the rays $[a, b, c]$ and $[-a, -b, -c]$ have the same underlying line but have opposite orientations. The orientation of a ray is derived from the normal vector $[a, b]_E$ by turning the vector clockwise by $90°$, into the vector $[b, -a]$, so obtaining the direction vector of the ray.

The distance of a point to a ray is measured as a positive quantity if the point is to the left of ray seen in the ray's direction. The radius of a cycle is positive if the cycle oriented counter-clockwise. The angle $\angle(L_i, L_j)$ between the two rays L_i and L_j is measured from the direction of L_i clockwise to the direction of L_j.

We focus only on three elements in each of the two clusters, namely the shared element and the two elements on which the constraints on the variable circle are placed. We denote the elements of first cluster with E_0, E_1, E_2, and those of the second cluster with E_0, E_3, E_4. Ultimately, the form of the algebraic equations that must be solved to merge the three clusters and determine the coordinates of the variable-radius circle depends on the type of the element E_0 and on the type of the other four elements. Therefor, we classify the various instance of the constraint schema with $E_0(E_1E_2, E_3E_4)$.

We write C_i if the i^{th} element is a cycle, and L_i if the i^{th} element is a ray. If the i^{th} element is a point, we write C_i because we can consider the point a cycle with zero radius. $T(E,d)$ denotes the translation of element E along the x-axis by a distance d, and $R(E,\theta)$ denotes the rotation of element E counter-clockwise about the origin by the angle θ.

Recall that the oriented cycle can be mapped to a point in 3-space. The *cyclographic map* of the cycle $(x_0, y_0, z_0, 1)$ is the cone whose apex is $(x_0, y_0, z_0, 1)$, whose axis is parallel to the Z-axis, and whose angle is equal to $\pi/4$; [6]. We call this a *normal cone* and denote it as $C((x_0, y_0, z_0, 1))$. The cyclographic map of a ray $[a, b, c]$ is a plane denoted by $C([a, b, c])$.

2.3 Parameterized Plane Computations

We solve the rotational cluster problems by considering how the intersections of rotating cyclographic maps depend on the rotation angle θ. Recall the circle parameterization, where $t = \tan(\theta/2)$:

$$\begin{aligned} x(t) &= (1-t^2)/(1+t^2) \\ y(t) &= 2t/(1+t^2) \end{aligned}$$

We obtain the following:

Theorem 1 *The cyclographic map for the ray $[a, b, d]$ rotated about the origin by θ has the form*

$$[a(1-t^2) - b(2t), a(2t) + b(1-t^2), c(1+t^2), d(1+t^2)]$$

where $t = tan(\theta/2)$, $-\pi < \theta < \pi$, and $c = -\sqrt{a^2+b^2}$. When $\theta = \pi$, the cyclographic map becomes $[-a, -b, c, d]$.

Theorem 2 *Let $C_1 = (x_1, y_1, z_1, 1)$ and $C_3 = (x_3, y_3, z_3, 1)$ be two normal cones. The intersection plane of $C(C_1)$ and $C(R(C_3, \theta))$ has the form*

$$[a(1-t^2) - b(2t) + e(1+t^2), a(2t) + b(1-t^2) + f(1+t^2), c(1+t^2), d(1+t^2)]$$

where $t = tan(\theta/2)$, $-\pi < \theta < \pi$, $a = x_3, b = y_3, e = -x_1, f = -y_1, c = z_1 - z_3$, and $d = (x_1^2 + y_1^2 - z_1^2 - x_3^2 - y_3^2 + z_3^2)/2$. When $\theta = \pi$, the cyclographic map becomes $[-a+e, -b+f, c, d]$.

Theorem 3 *Consider three planes with constant coefficients except for t:*

$$\left\{ \begin{aligned} \Pi_1 &= [a_2,\ b_2,\ c_2,\ -d_2] \\ \Pi_2 &= [a_3(1-t^2) - b_3(2t) + e_3(1+t^2),\ a_3(2t) + b_3(1-t^2) + f_3(1+t^2), \\ &\qquad c_3(1+t^2),\ -d_3(1+t^2)] \\ \Pi_3 &= [a_4(1-t^2) - b_4(2t) + e_4(1+t^2),\ a_4(2t) + b_4(1-t^2) + f_4(1+t^2), \\ &\qquad c_4(1+t^2),\ -d_4(1+t^2)] \end{aligned} \right.$$

Then the coordinates of the intersection point, $(\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta)$, of these planes, are expressions of degree 2 in t.

Proof

By Cramer's rule, we find the intersection point $(\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta) = (1+t^2)(\delta_1,\ \delta_2,\ \delta_3,\ \delta)$ where

$$
\left\{
\begin{array}{rcl}
\delta_1 & = & \begin{vmatrix} d_2 & b_2 & c_2 \\ d_3 & -b_3+f_3 & c_3 \\ d_4 & -b_4+f_4 & c_4 \end{vmatrix} t^2 + \begin{vmatrix} d_2 & 0 & c_2 \\ d_3 & a_3 & c_3 \\ d_4 & a_4 & c_4 \end{vmatrix} (2t) + \begin{vmatrix} d_2 & b_2 & c_2 \\ d_3 & b_3+f_3 & c_3 \\ d_4 & b_4+f_4 & c_4 \end{vmatrix} \\
\delta_2 & = & \begin{vmatrix} a_2 & d_2 & c_2 \\ -a_3+e_3 & d_3 & c_3 \\ -a_4+e_4 & d_4 & c_4 \end{vmatrix} t^2 + \begin{vmatrix} 0 & d_2 & c_2 \\ -b_3 & d_3 & c_3 \\ -b_4 & d_4 & c_4 \end{vmatrix} (2t) + \begin{vmatrix} a_2 & d_2 & c_2 \\ a_3+e_3 & d_3 & c_3 \\ a_4+e_4 & d_4 & c_4 \end{vmatrix} \\
\delta_3 & = & \begin{vmatrix} a_2 & b_2 & d_2 \\ -a_3+e_3 & -b_3+f_3 & d_3 \\ -a_4+e_4 & -b_4+f_4 & d_4 \end{vmatrix} t^2 \\
& & + \left[\begin{vmatrix} a_2 & b_2 & 0 \\ -b_3 & a_3 & d_3 \\ -b_4 & a_4 & d_4 \end{vmatrix} + d_2 \left(\begin{vmatrix} e_3 & a_3 \\ e_4 & a_4 \end{vmatrix} - \begin{vmatrix} b_3 & f_3 \\ b_4 & f_4 \end{vmatrix} \right) \right] (2t) \\
& & + \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3+e_3 & b_3+f_3 & d_3 \\ a_4+e_4 & b_4+f_4 & d_4 \end{vmatrix} \\
\delta & = & \begin{vmatrix} a_2 & b_2 & c_2 \\ -a_3+e_3 & -b_3+f_3 & c_3 \\ -a_4+e_4 & -b_4+f_4 & c_4 \end{vmatrix} t^2 \\
& & + \left[\begin{vmatrix} a_2 & b_2 & 0 \\ -b_3 & a_3 & c_3 \\ -b_4 & a_4 & c_4 \end{vmatrix} + c_2 \left(\begin{vmatrix} e_3 & a_3 \\ e_4 & a_4 \end{vmatrix} - \begin{vmatrix} b_3 & f_3 \\ b_4 & f_4 \end{vmatrix} \right) \right] (2t) \\
& & + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3+e_3 & b_3+f_3 & c_3 \\ a_4+e_4 & b_4+f_4 & c_4 \end{vmatrix}
\end{array}
\right.
$$

Since $(\Delta_1, \Delta_2, \Delta_3, \Delta)$ are homogeneous coordinates and $1+t^2 \neq 0$, the intersection point coordinates are equivalently $(\delta_1, \delta_2, \delta_3, \delta)$. Each coordinate is a quadratic expression in t.

3 The Solving Strategy for 2d Constraint Problems

As in Part I, we explain each constraint problem in turn, in increasing order of complexity. We assume that the relevant elements in the two clusters are the shared element and the two elements on which the constraints on the variable-radius circle are given. Let E_0, E_1, E_2 be those elements of first cluster and E_0, E_3, E_4 those elements of the second cluster.

We denote the constraint problems as $E_0(E_1E_2, E_3E_4)$. In the following, E_0 is a point. We assume a coordinate system in which E_0 is at the origin. The cluster S_1 is assumed fixed, cluster S_2 can rotate about the origin.

3.1 The Rotational Clusters

There are 6 major cases for the rotational clusters problems. They are, in order of increasing complexity, the $C(LL,LL)$, $C(CL,LL)$, $C(CL,CL)$,$C(CC,LL)$, $C(CC,CL)$, and $C(CC,CC)$ problems. The shared element C is a point (or the center of a shared circle).

3.1.1 The C(LL,LL) Problem

Consider four rays L_1, L_2 in the first cluster and L_3, L_4 in the second cluster. We want to rotate the second cluster, so that there exists a variable-radius cycle that is tangent to the four rays.

The cyclographic maps of the four rays are planes $C(L_i) = [a_i, b_i, c_i, -d_i]$, $i = 1,..,4$. We will find the intersection point of three planes $C(L_2)$, $C(R(L_3,\theta))$, $C(R(L_4,\theta))$ and then put this point onto the plane $C(L_1)$. By Theorem 1, the third and the fourth plane have the equation

$$R(C(L_i),\theta) = [a_i(1-t^2)-b_i(2t),\ a_i(2t)+b_i(1-t^2),\ c_i(1+t^2),\ -d_i(1+t^2)],\ \ i = 3,4,$$

where $t = \tan(\frac{\theta}{2})$. By Theorem 3 with $e_3 = e_4 = f_3 = f_4 = 0$, we compute the intersection point $(\delta_1,\ \delta_2,\ \delta_3,\ \delta_4)$, where

$$\begin{cases} \delta_1 = \left(\begin{vmatrix} d_2 & b_2 & c_2 \\ d_3 & -b_3 & c_3 \\ d_4 & -b_4 & c_4 \end{vmatrix} t^2 + \begin{vmatrix} d_2 & 0 & c_2 \\ d_3 & a_3 & c_3 \\ d_4 & a_4 & c_4 \end{vmatrix} (2t) + \begin{vmatrix} d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \\ d_4 & b_4 & c_4 \end{vmatrix} \right) \\ \delta_2 = \left(\begin{vmatrix} a_2 & d_2 & c_2 \\ -a_3 & d_3 & c_3 \\ -a_4 & d_4 & c_4 \end{vmatrix} t^2 + \begin{vmatrix} 0 & d_2 & c_2 \\ -b_3 & d_3 & c_3 \\ -b_4 & d_4 & c_4 \end{vmatrix} (2t) + \begin{vmatrix} a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \\ a_4 & d_4 & c_4 \end{vmatrix} \right) \\ \delta_3 = \left(\begin{vmatrix} a_2 & b_2 & d_2 \\ -a_3 & -b_3 & d_3 \\ -a_4 & -b_4 & d_4 \end{vmatrix} t^2 + \begin{vmatrix} a_2 & b_2 & 0 \\ -b_3 & a_3 & d_3 \\ -b_4 & a_4 & d_4 \end{vmatrix} (2t) + \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} \right) \\ \delta = \left(\begin{vmatrix} a_2 & b_2 & c_2 \\ -a_3 & -b_3 & c_3 \\ -a_4 & -b_4 & c_4 \end{vmatrix} t^2 + \begin{vmatrix} a_2 & b_2 & 0 \\ -b_3 & a_3 & c_3 \\ -b_4 & a_4 & c_4 \end{vmatrix} (2t) + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} \right) \end{cases}$$

To put the intersection point $(\delta_1,\ \delta_2,\ \delta_3,\ \delta)$ onto the plane $C(L_1)$ we require $[a_1,\ b_1,\ c_1,\ -d_1]\cdot(\delta_1,\ \delta_2,\ \delta_3,\ \delta) = 0$. Therefore, we have the quadratic equation

$At^2 + Bt + C = 0$, where

$$A = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ -a_3 & -b_3 & c_3 & d_3 \\ -a_4 & -b_4 & c_4 & d_4 \end{vmatrix}$$

$$B = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & c_2 & d_2 \\ -b_3 & a_3 & c_3 & d_3 \\ -b_4 & a_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ -b_3 & a_3 & c_3 & d_3 \\ -b_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

$$C = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

After solving the equation we find the rotation angle from $\theta = 2\tan^{-1}(t)$, and determine the center and radius of the circle as $(\delta_1/\delta, \delta_2/\delta)_E$ and δ_3/δ.

Degeneracies: When $A = B = C = 0$, there is a solution for every value of t. This degeneracy arises, for example, when L_1,L_2,L_3 and L_4 are four sides of a square centered at the origin, and their directions are oriented counter-clockwise. When $A = B = 0$ and $C \neq 0$, then there is no solution among $-\infty < t < \infty$ or $\pi < \theta < \pi$ for this configuration.

Orientation: Recall that the orientation of lines induces an orientation on the planes $C(L_k)$, and that we can pair solutions when all orientations are reversed. Thus, there are 2^3 significant orientations to be chosen, yielding a total of eight solutions in the generic case.

Special Case ($t = \infty, \theta = \pi$): When $A = 0$ and $B \neq 0$, there is only one solution. In that case, it is possible that π is also a solution of this problem, and we check this separately.

Special Case (t exists and $\delta = 0$): In this case the three planes $C(L_2)$, $C(R(L_3, \theta))$, $C(R(L_4, \theta))$, are in degenerate position. The degenerate case is treated in the next section.

Example 1 *Consider the problem of figure 1. The first cluster contains the ray L_1 underlying a segment of length 85 at an angle of $45°$ with the x-axis, and the ray L_2 at angle $85°$ with L_1, through the segment end point. The second cluster contains the ray L_3 supporting a segment of length 75, at angle $45°$ with the x-axis, and the ray L_4 at an angle of $100°$ with the L_3. Find a circle at distance 15 from each of these four rays.*

We transform the example problem to finding a circle tangent to the four rays. Concentric with the result, and with a radius smaller by 15, will be the circle of the untransformed problem. The two clusters have the following initial configuration:

$$\begin{cases} L_1 &=& [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1, 0] \\ L_2 &=& [\cos(50), \sin(50), -1, \frac{85}{\sqrt{2}}(\cos(50)+\sin(50))] \\ L_3 &=& [-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1, 0] \\ L_4 &=& [-\cos(35), \sin(35), -1, \frac{75}{\sqrt{2}}(\sin(50)-\cos(50))] \end{cases}$$

We get two solutions, namely $-2.93°$ and $90°$. Figure 1 shows the solution for rotation by $-2.93°$ degree. The center and the radius are $(-1.47, -57.54)_E$ and 39.65. The other solution rotates the second cluster by $90°$ degree, which makes the rays L_1 and L_3 coincident. In this case we find a circle whose center and radius are $(62.46, -54.75)_E$ and 82.88 respectively.

♡

3.1.2 The C(CE,EE) Problem

We can combine the five cases C(CL,LL), C(CL,CL), C(CC,LL), C(CC,CL) and C(CC,CC), naming them $C(C_1E_2, E_3E_4)$. The three planes of the C(LL,LL) problem are replaced with three planes that arise as follows. If E_2 is a ray, then Π_1 is the cyclographic map of the ray. If E_2 is a cycle, then Π_1 is the plane containing the intersection of $C(C_1)$ and $C(E_2)$. To abbreviate:

$$\text{If } E_2 = L_2 \text{ then } \Pi_2 = C(L_2) \text{ else } \Pi_2 = P(C_1, C_2).$$

Using the same notation, we consider the other planes:

$$\text{If } E_3 = L_3 \text{ then } \Pi_3 = C(R(L_3,\theta)) \text{ else } \Pi_3 = P(C_1, R(C_3,\theta)).$$
$$\text{If } E_4 = L_3 \text{ then } \Pi_4 = C(R(L_2,\theta)) \text{ else } \Pi_4 = P(C_1, R(C_4,\theta)).$$

With this construction, we know the plane Π_2 is fixed, of the form $[a_2, b_2, c_2, -d_2]$. Furthermore, from Theorem 1 and Theorem 2, the planes Π_i, where $i = 3, 4$, have the form

$$\Pi_i = [a_i(1-t^2)-b_i(2t)+e_i(1+t^2), a_i(2t)+b_i(1-t^2)+f_i(1+t^2), c_i(1+t^2), d_i(1+t^2)]$$

From Theorem 3, we know that the coordinates of the intersection point $(\delta_1, \delta_2, \delta_3, \delta)$ are quadratic in t. After we substitute the intersection point into the equation of the first cone, we have the degree 4 equation

$$(\delta_1 - x_1\delta)^2 + (\delta_2 - y_1\delta)^2 - (\delta_3 - z_1\delta)^2 = 0$$

from which we determine the rotation angle.

It is advantageous to "lift" the plane in which we solve the problem in the Z-direction by a distance equal to the (signed) radius of the cycle C_1. This has the effect of reducing C_1 to a point and simplifying the cone equation $C(C_1)$. The solution can then be dropped down, to the original problem plane, by shifting the lines, re-inflating the cycle, and increasing or diminishing the variable radius cycle. The details are routine.

Degeneracies:

In the degenerate case, three planes may meet in a common line or have no finite intersection. The symbolic intersection $(\delta_1,\ \delta_2,\ \delta_3,\ \delta)$ yields no finite intersection or no common intersection at all when $\delta = 0$ and at least one of the $\delta_i \neq 0, i = 1, .., 3$. The planes intersect in a common line or coincide when $\delta_1 = \delta_2 = \delta_3 = \delta = 0$. From the fact that $[a,\ b,\ c,\ -d]$ and $[ra,\ rb,\ rc,\ -rd], r \neq 0$, represent the same plane, we can check that whether three planes coincide.

When the planes meet in a common line, the three planes are linearly dependent. So, we can select two of them to define the line, and parameterize the line as $(x(s),\ y(s),\ z(s),\ w)$. Substitute the parametric equation into the implicit equation of the cone:

$$(x(s) - x_1 w)^2 + (y(s) - y_1 w)^2 - (z(s) - z_1 w)^2 = 0$$

The solve the equation for variable s. If s is a real number, then we fine the intersection points of a line and a cone at two points. Otherwise, the line and cone do not intersect. If the equation vanishes, the line is on the cone, and there is an infinity of solutions.

We give two examples in this section, for the C(CL,CL) and the C(CC,CC) problem.

Example 2 *Consider the problem of figure 2. The first cluster contains the cycle C_1 and a tangent ray L_2 that is coincident with the x-axis. The second cluster contains the cycle C_3 and a tangent ray L_4 that is coincident with the y-axis. We find a tangent circle of C_1 and C_4 at distances 18 and 17, respectively, from the rays L_2 and L_4.*

We translate the rays L_2 and L_4 by 18 and 17, respectively, obtaining the rays L'_2 and L'_4. This transforms the problem into finding a circle tangent to C_1, L'_2, C_3 and L'_4, allowing a suitable rotation of the second cluster about the origin by an angle θ. We have the initial configuration for the two clusters:

$$\begin{cases} C_1 &= (-65, -10, 10, 1) \\ L'_2 &= [0, 1, -1, 18] \\ C_3 &= (-10, -70, 10, 1) \\ L'_4 &= [1, 0, -1, 17] \end{cases}$$

There are four solutions for θ, namely $-73.19°, 21.48°, 90°, 90°$. For $\theta = -73.19$, we find the cycle centered at $(-67.43, -10.48)_E$ with radius 7.52; this cycle is inside C_1 and $R(C_3, \theta)$. For $\theta = 21.48$, we find the cycle centered at $(-48.95, -73.44)_E$ with radius -55.44; this cycle is the solution shown in Figure 2. For $\theta = 90°$, the lines L_2 and L_4 are both coincident with the y-axis; this is a degenerate case that has no solution. ♡

Example 3 *Consider the problem of figure 3. The first cluster contains the cycles C_1 and C_2 at distance 5 from the x-axis with radii 7 and 5, respectively. The second cluster contains the cycles C_3 and C_4 with distance 5 to the y-axis, and with radii 7 and 5, respectively. We find a cycle that has distance 5 from the four circles, as shown in the figure.*

We translate the above problem by finding a circle tangent to these four cycles, allowing rotation of the second cluster by θ. The initial configuration for the two clusters is:

$$\begin{cases} C_1 &= (-23, -12, 7, 1) \\ C_2 &= (-65, 10, 5, 1) \\ C_3 &= (-12, -23, 7, 1) \\ C_4 &= (10, -75, 5, 1) \end{cases}$$

There are four solutions for the rotation angle θ, namely $-34.89°, -34.89°, 62.21°$, and $90°$. For $\theta = -34.89°$, the cycles C_1 and $R(C_3, \theta)$ are coincident, so we have a degenerate case where $(\delta = \delta_1 = \delta_2 = \delta_3 = 0)$. Because the two cycles coincide, this problem becomes the Apollonius problem. Here, we find the line generated by two planes $\Pi_1 = P(C_1, C_2)$ and $\Pi_2 = (C_1, R(C_4, \theta))$, and intersect this line with $C(C_1)$. There are two intersection points, namely $(-43.53, -28.49, 33.33)_E$ and $(-46.20, -29.30, -21.94)_E$, representing two cycles: The one with a negative radius contains all four cycles, and the one with positive radius contains none of the cycles. For $\theta = 62.21$, we find the circle centered at $(-70, -282.98)_E$ with radius -268.00. This cycle is tangent to the four cycles, so that reducing the radius by 5 yields the solution; see figure 3. For $\theta = 90°$, we have another degenerate case with $\delta = \delta_1 = 0$ and $\delta_2 = -\delta_3 \neq 0$. In this case, $\Pi_1 = [-21, 1, 1, 919], \Pi_2 = [1, 0, 0, 0]$, and $\Pi_3 = [\frac{75}{2}, 1, 1, 1269]$. The three planes have no (finite) intersection. So, there is no solution when the rotation angle is 90 degree.

4 Conclusions

Our solution strategy for solving variable-radius circle clusters has the following pattern.

1. Fix cluster S_1 and place the coordinate system so that the axis of the normal cone C_0, shared by cluster S_2, passes through the origin.

2. Construct the cyclographic maps of all elements, accounting for the rotation of cluster S_2. Where possible, replace the cone/cone intersection with a cone/plane intersection, thus lowering the algebraic degree.

3. Derive a univariate polynomial whose solution determines the position of S_2 and the variable-radius circle of the third cluster.

We fix the cluster that has the more complicated elements. Constraints on circles (and nonzero distance constraints on points) are algebraically more complicated than distance constraints on lines. Hence, the cluster with more circle elements constraining the variable-radius circle is designated as S_1.

By working with planes that contain the intersection of two cones, we were consistently able to achieve the following solution method:

1. Construct three planes and their common intersection.

	Eqn_1	Π_1	Π_2	Π_3	Degree
C(LL,LL)	$Eqn(L_1)$	$C(L_2)$	$C(L_3)$	$C(L_4)$	2
C(CL,LL)	$Eqn(C_1)$	$C(L_2)$	$C(L_3)$	$C(L_4)$	4
C(CL,CL)	$Eqn(C_1)$	$C(L_2)$	$P(C_1, R(C_3, \theta))$	$C(R(L_4, \theta))$	4
C(CC,LL)	$Eqn(C_1)$	$P(C_1, C_2)$	$C(R(L_3, \theta))$	$C(R(L_4, \theta))$	4
C(CC,CL)	$Eqn(C_1)$	$P(C_1, C_2)$	$P(C_1, R(C_3, \theta))$	$C(R(L_4, \theta))$	4
C(CC,CC)	$Eqn(C_1)$	$P(C_1, C_2)$	$P(C_1, R(C_3, \theta))$	$P(C_3, R(C_4, \theta))$	4

Table 1: Plane construction table

2. Substitute the intersection into one of the cone equations, or, in the case C(LL,LL), into the plane equation, deriving a univariate polynomial in a variable $t = \tan(\frac{\theta}{2})$ that yields the required angles of rotation.

3. Solve the polynomial.

Depending on the planes that must be constructed, the polynomial in d has degree up to 4. The number of solutions must be multiplied with 8, the number of essentially distinct orientations of lines and cycles, leading to up to 32 solutions in the worst case. The plane constructions and the degree of the polynomial are summarized in Table 1. Note that no equation has degree higher than 4, an accomplishment due to working with rotating planes instead of rotating cones.

There are several ways in which the problem can become degenerate. It is possible that the three planes intersect in a common line or even coincide. In the common line case, there is the possibility of obtaining an infinite number of solutions, i.e., of dealing with an underconstrained instance. This case is approached by deriving the parametric line equation and substituting it into the cone or plane equation representing the cyclographic map of element 1 in S_1. If the parameter vanishes, no intersection is also a possibility.

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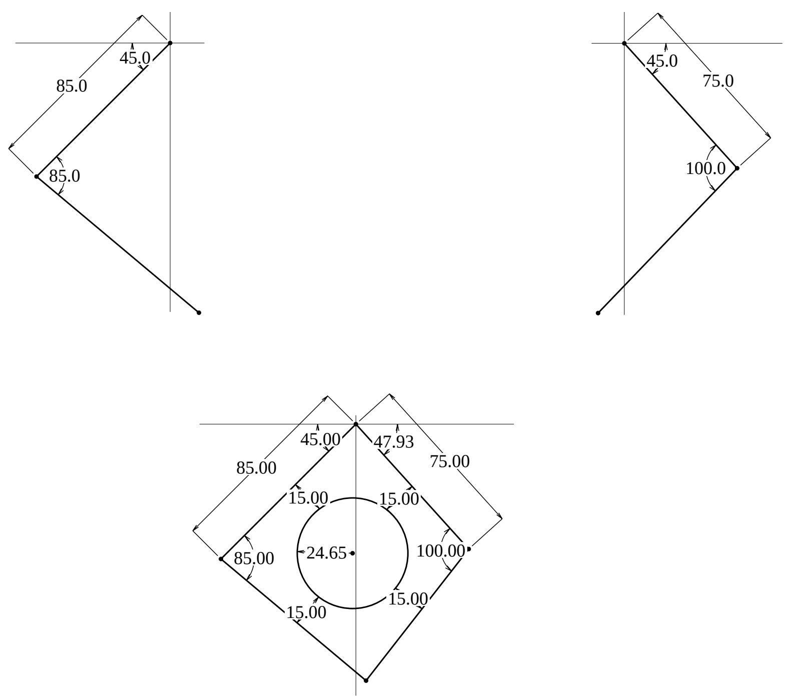

Figure 1: Top – the two clusters of the C(LL,LL) problem of example 1; Bottom – one solution with $\theta = -2.93$.

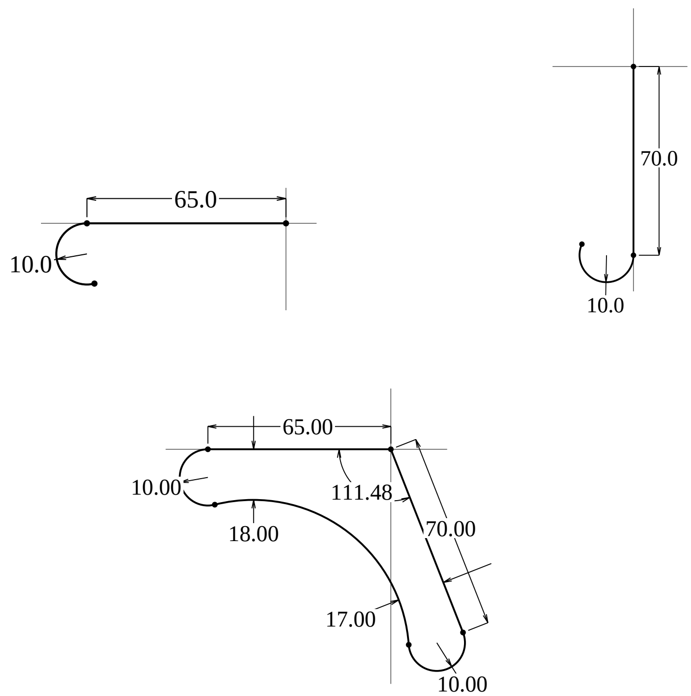

Figure 2: Top – the two clusters of the L(CL,CL) problem of example 2; Bottom – one solution with $\theta = 21.48$.

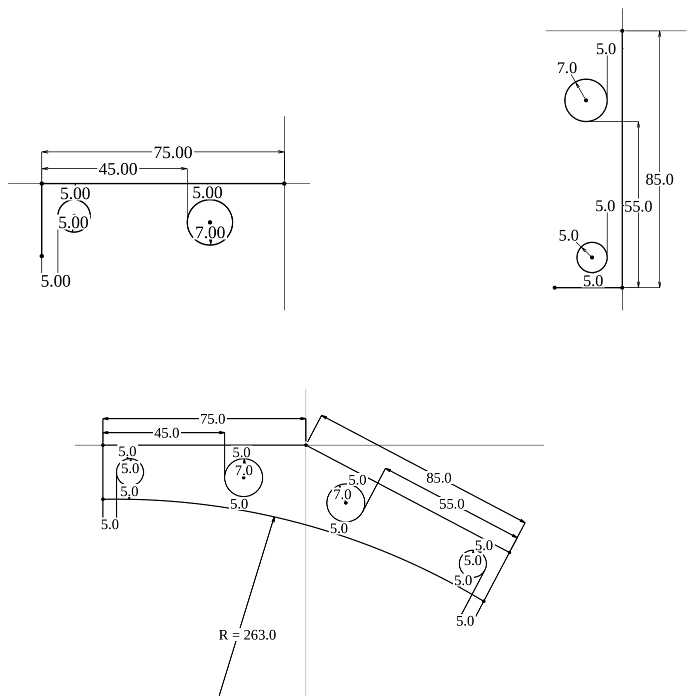

Figure 3: Top – the two clusters of the L(CC,CC) problem of example 3; Bottom – one solution with $\theta = 62.21$.

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