Variable-Radius Circles of Cluster Merging in geometric constraints
Part I: Translational Clusters*

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Abstract

Variable-radius circles are common constructs in planar constraint solving and are usually not handled fully by algebraic constraint solvers. We give a complete treatment of variable-radius circles when such a circle must be determined simultaneously with placing two groups of geometric entities. The problem arises for instance in solvers using triangle decomposition to reduce the complexity of the constraint problem.

Part I sets up the problem statement and considers clusters where the relative motion is translational. Part I also reviews past work on the subject. Part II treats rotational clusters motion.

Keywords: geometric constraint solving, variable-radius circles, triangle decomposition, algebraic solver, cyclographic maps.

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1 Introduction

Informally, a geometric constraint problem consists of a (finite) set of geometric elements and a (finite) set of constraints between them. The geometric elements are drawn from a fixed universe such as point, lines, circles and conics in the plane, or points, lines, planes, cylinders and spheres in 3-space. The constraints are logical constraints such as incidence, tangency, perpendicularity, etc., or metric constraints such as distance or angle.

The solution of a geometric constraint problem is a coordinate assignment to all geometric elements such that all constraints are satisfied, or else a message to the effect that no solution was found. It is understood that a solution is in a particular geometry, for example in the Euclidean plane, the sphere, or in Euclidean 3-space.

When no solution has been found, it means that either the solver could not determine a suitable solution in view of algorithmic limitations, or that there is no solution possible. Since the complexity of geometric constraint solving is doubly exponential, a fact that derives from the ability to express polynomial algebraic equations by geometric constraint configurations, it is accepted that practical constraint solvers are not *complete*, that is, that they solve a subclass of geometric problems.

In this paper we consider solving planar constraint problems. Here, a practically useful class of problems are constraint problems in which the geometric elements are points and lines, and in which the constraint problem can be decomposed using *triangle decomposition*, to be explained later. The geometric elements include fixed-radius circles, circles for which the radius is prescribed, since all such entities can be mapped to points by a suitable transformation of the constraints on them. The purpose of this paper is to add variable-radius circles to the set of geometric elements, circles for which the radius is not explicitly prescribed and must be deduced from other constraints.

1.1 Solver Phases and Decomposition Methods

We consider planar constraint solvers that translate the constraint problem into a graph and decompose the graph to find recursively small subproblems. For planar constraint problems, this is a very practical approach as evidenced by the commercial success of D-Cubed's Dimensional Constraint Manager (DCM) [3, 15]. Owen uses a top-down decomposition approach as discussed in [5, 6]. In our own work, we have used the triangle decomposition approach [1, 5, 12, 13]. It is related to Owen's method in the sense that Owen's algorithm is top-down, whereas ours is bottom-up. The two approaches can be combined [6]. Other approaches include edge orientation techniques; e.g., [14], and the fully general decomposition algorithm of [9, 7, 8].

After decomposition has formulated what could be called a plan for solving all necessary subproblems, a second solver phase must formulate and solve the nonlinear algebraic equations that arise from the decomposition. It is useful that this phase be able to construct a solution using algebraic techniques, because this can make the equation solving highly dependable and is capable of finding all possible solutions. It is important to find solution alternatives when using constraint solving in editing operations, a problem that frequently arises in CAD/CAM practice.

The work presented here best fits the triangle decomposition algorithm, or the method of Owen, and describes how to solve all subproblems that arise when variable-radius circles are allowed. The DCM solver can solve algebraically only those variable-radius problems in which such a circle is constructed as the single unknown entity at that point. Problems that require simultaneously placing other geometric elements have not been so solved before, and in this paper we explain how to do it.

Various extensions to the geometric repertoire of planar constraint solvers can be considered, such as conic arcs and Bezier curves. Of them, variable-radius circles are probably the most immediately useful extension as they permit auxiliary constructions in addition, as explained in [11, 12]. In this paper, we remove the restrictions previously imposed on the use of variable-radius circles. In particular, we consider the use of variable-radius circles as clusters with four constraints on them.

1.2 Variable-Radius Circles

We distinguish between fixed-radius circles and variable-radius circles. In the case of fixed-radius circles, the user stipulates the radius of the circle as a constraint. As shown in [1], this implies that all such circles can be replaced equivalently by points after suitably altering some of the constraints.

A variable-radius circle does not have a prescribed radius, and thus requires generically three constraints to be determined. Once incorporated into a cluster, in the sense of [5, 6], along with other geometric elements, a variable-radius circle has become a fixed-radius circle. However, it is possible that a variable-radius circle forms a cluster by itself and can only be determined when being merged with two other clusters. This is the situation we address in this paper.

1.3 Prior Work on Variable-Radius Circles

When the underlying solver is numerical, and good initial guesses are available for the geometric elements, then variable-radius circles pose no particular problem. For this reason, commercial systems such as DCM [3] include numerical solvers to increase the geometric coverage. In contrast, the algebraic solvers we are aware of permit only *sequential* constructions of variable-radius circles. That is, the circles must be determined from three known geometric elements by three constraints. This includes commercial systems we have studied such as DCM.

The numerical approach to solving constraints has many drawbacks, including poor efficiency, reliance on good starting values, and the inability to explore solution variants [4]. What is needed is an algebraic solution, preferably one in which there is no need to solve high-degree polynomials.

Recall that fixed-radius circles are equivalent to points. There are eight major cases for determining a variable-radius circle in a sequential construction step, according to whether the three known geometric elements constraining the circle are points or lines. Of those, many cases are trivial, for example, constructing a circle of unknown radius that is tangent to three known lines. The hardest sequential case is constructing a circle tangent to three fixed-radius circles, or, equivalently, a circle that is at given distances from three known points.[1] This is the classical Apollonius problem. Apollonius' problem can be solved easily by an algebraic solver requiring only solving univariate quadratic equations. The solution [16] uses *cyclographic maps*, a special case of Laguerre geometry; [10].

Some commercial systems include the sequential steps, but we are unaware of any commercial or research system that allows, in addition, variable-radius circles that must be determined in a nonsequential way, by cluster merging, and does so algebraically.

The purpose of this paper is to give an algebraic solution of the problem of variable-radius circles that are clusters and that must be determined by cluster merging. When using triangle decomposition, such circles are merged with two other clusters and have four constraints upon them.

2 Problem Statement and Notation

2.1 Example

We illustrate the problem we solve with an example. Consider the constraint schema shown in Figure 1: Two clusters S_1 and S_2 have been formed sharing the common geometric element A. The variable-radius circle VC has to be placed subject to constraints between it and the geometric elements B and C, in cluster S_1 and D and E, contained cluster S_2. An example instance of this problem schema is shown in Figure 2. In the example, the clusters are

- S_1 : Points H, F, D, and lines A, G and E, determined by the angle constraints at H and F and the distance constraints (H,F) and (F,D).
- S_2 : Points K and C, and lines A and B, determined by the angle constraint at K and the distance constraint (K,C).
- S_3 : The circle determined by tangency requirements to lines B and E and incidence constraints with points C and D.

The clusters S_1 and S_2 each have the property that they are determined up to position and orientation in the plane, that is, the degrees of freedom, or *dof*, for all geometric elements in a cluster, minus the number of constraints on them, is 3. Note the implicit incidence constraints between points and lines. When combined, clusters S_1 and S_2 have one additional degree of freedom, because

[1] Note that a circle at distance d from a given point p is also tangent to a circle with center p and radius d, and vice-versa.

the common element A results in two constraints. Now the circle VC has three degrees of freedom in isolation. Combined with the two other clusters, therefore, the four constraints between the clusters and the circle result in a combined rigid structure with 3 degrees of freedom, accounting for the undetermined location and orientation in the plane.

In the schema, the complexity of the equations depends on whether the shared geometric element is a line or a point, and whether the other geometric elements, B–E, are lines or points.

2.2 Notation

We use homogeneous coordinates (x, y, z) for points. The homogenizing variable is z. Customarily, one assumes $z = 1$ for finite points. Lines are considered to have the (homogeneous) coordinates $[a, b, c]$ when the line equation is $ax + by + cz = 0$, with (x, y, z) a point on the line. Here it is customary to assume for finite lines that the normal has unit length, i.e., that $a^2 + b^2 = 1$. In the plane, points and lines are dual of each other: If we fix $[a, b, c]$, the equation $ax + by + cz = 0$ represents all points on the line; if we fix (x, y, z), then the equation represents all lines through the point.

We will use mappings from plane geometric objects to geometric objects in 3-space in order to simplify solving the nonlinear equations that arise in the constraint schema. Here, we denote planes by the coordinates $[A, B, C, D]$ and points by (X, Y, Z, W), with W the homogenizing coordinate. The duality of points and planes in 3-space is established by the equation $AX + BY + CZ + DW = 0$.

When concentrating on affine (finite) points, we will write $(x, y)_E$ for points in the affine plane and $(X, Y, Z)_E$ for points in affine 3-space. Furthermore, we write $[x, y]_E$ and $[X, Y, Z]_E$ to represent vectors in affine space. Recall that $(X, Y, Z, W) = (X/W, Y/W, Z/W)_E$ when $W \neq 0$.

We will use oriented geometric elements. Doing so allows us to simplify the algebraic equations and lower the degree of the resulting systems. For example, two circles may have up to four tangents, but two oriented circles have only up to two oriented tangents, because we require that they are tangent with a consistent orientation. We do not lose solutions if we change the unoriented geometric problem into a set of oriented problems as long as we consider all relevant orientation combinations.

The oriented circle, or *cycle*, in 2D with center $(x, y)_E$ and radius r can be represented as the 3D point $(x, y, z)_E = (x, y, r, 1)$. The sign of r signifies the orientation of the cycle: When $r > 0$, the cycle is oriented counter-clockwise; if $r < 0$, the cycle is oriented clockwise; when $r = 0$, the cycle represents a 2D point and is considered to have both orientations simultaneously.

The oriented line, or *ray*, is defined as the line $[a, b, c]$ with an orientation. That is, the rays $[a, b, c]$ and $[-a, -b, -c]$ have the same underlying line but have opposite orientations. The orientation of a ray is derived from the normal vector $[a, b]_E$ as follows: Turn the vector clockwise by $90°$, into the vector $[b, -a]$, to obtain the direction vector of the ray. So, the line $[0, 1, 0]$ is the x-axis oriented from the negative to the positive direction with the normal pointing in the direction of the positive y-axis. We assume that all rays are *normalized*, that is, that $a^2 + b^2 = 1$.

The distance of a point to a ray is measured as a positive quantity if the point is to the left of ray seen in the ray's direction. The radius of a cycle is positive if the cycle oriented counter-clockwise. The angle $\angle(L_i, L_j)$ between the two rays L_i and L_j is measured from the direction of L_i clockwise to the direction of L_j.

To simplify the presentation, we assume that the two clusters each consist of three elements, namely the shared element and the two elements on which the constraints on the variable circle are placed. We denote the elements of first cluster with E_0, E_1, E_2, and the elements of the second cluster with E_0, E_3, E_4. A constraint d_{ij} or θ_{ij} is a distance or angle constraint, respectively, between the i-th and j-th elements. Ultimately, the form of the algebraic equations that must be solved to merge the three clusters and determine the coordinates of the variable-radius circle depends on the type of the element E_0 and on the type of the other four elements. Therefor, we classify the various instance of the constraint schema with $E_0(E_1E_2, E_3E_4)$.

We write C_i if the i^{th} element is a cycle, and L_i if the i^{th} element is a ray. If the i^{th} element is a point, we write C_i because we can consider the point a cycle with zero radius. $T(E, d)$ denotes the translation of element E along the x-axis by a distance d, and $R(E, \theta)$ denotes the rotation of element E counter-clockwise about the origin by the angle θ.

Recall that the oriented cycle can be mapped to a point in 3-space. The *cyclographic map* of the cycle $(x_0, y_0, z_0, 1)$ is the cone whose apex is $(x_0, y_0, z_0, 1)$, whose axis is parallel to the Z-axis, and whose angle is equal to $\pi/4$. We call this kind of cone a *normal cone* and denote it as $C((x_0, y_0, z_0, 1))$. The cyclographic map of a ray $[a, b, c]$ is a plane. We denote this plane as $C([a, b, c])$.

2.3 Basic Properties

The following properties are elementary [2].

Theorem 1 *The cyclographic maps for the cycle* $C{=}(x_0, y_0, z_0, 1)$ *is the normal cone* $C(C)$

$$C(C): \ (X - x_0W)^2 + (Y - y_0W)^2 - (Z - z_0W)^2 = 0$$

The cyclographic map of the ray $l = [a, b, c]$ *is the plane* $[a, b, -1, c]$ *with the equation*

$$C(L): \ aX + bY - Z + cW = 0$$

Theorem 2 *The intersection curve of two normal cones whose apices are* $P_0 = (x_0, y_0, z_0, 1)$ *and* $P_1 = (x_1, y_1, z_1, 1)$ *lies in the plane with normal* $[x_1 - x_0, y_1 - y_0, z_0 - z_1]_E$ *and passing through the point* $(P_0 + P_1)/2$. *That is, the intersection* *of these two cones is equal to the intersection of either one of the cones with this plane:*

$$\begin{aligned} C((x_0, y_0, z_0, 1)) \cap C((x_1, y_1, z_1, 1)) &= C((x_0, y_0, z_0, 1)) \cap \Pi \\ &= C((x_1, y_1, z_1, 1)) \cap \Pi \end{aligned}$$

where $\Pi = [x_1 - x_0, y_1 - y_0, z_0 - z_1, (x_0^2 + y_0^2 - z_0^2 - x_1^2 - y_1^2 + z_1^2)/2]$.

This theorem is reconciled with Bezout's theorem by observing that the intersection of the two cones has a component at infinity that is not present in the intersection with the plane. Since we are interested in finite solutions, the loss of the infinite component is immaterial.

We denote the plane Π containing the intersection curve of the two normal cones $C(C_1)$ and $C(C_2)$ with $\Pi = P(C_1, C_2)$.

Theorem 3 *The cycle* $C = (x, y, z, 1)$ *is tangent to the cycle* $C' = (x', y', z', 1)$ *if the apex of* C' *is on the cone* $C(C)$. *Moreover,* C *is tangent to* C' *if and only if the cycle* $C_r = (x, y, z - r, 1)$ *is tangent to cycle* $C_r' = (x', y', z' - r, 1)$.

The cycle C *is tangent to the ray* $L = [a, b, c]$ *if the plane* $C(L)$ *is tangent to the cone* $C(C)$. *Moreover,* C *is tangent to* L *if and only if the cycle* $C_r = (x, y, z - r, 1)$ *is tangent to the ray* $L_r = [a, b, c - r]$.

Theorem 4 *Let* E_1, E_2 *and* E_3 *be cycles or rays, and consider the cycle* E. *If* $E \in C(E_1) \cap C(E_2) \cap C(E_3)$, *considering* E *as point in 3-space, then the cycle* E *is tangent to the cycles* E_1, E_2 *and* E_3 *with same orientation.*

Suppose we want to find the intersection of the cyclographic maps of two cycles. By theorem 2, we can simplify this cone/cone intersection problem to a cone/plane intersection. That is, $C(C_1) \cap C(C_2) = C(C_1) \cap P(C_1, C_2)$. Similarly, consider the intersection of three cones. Here, we can simplify the intersection problem by first intersecting the two planes, and then intersecting the resulting line with one of the cones. That is, $C(C_1) \cap C(C_2) \cap C(C_3) = C(C_1) \cap P(C_1, C_2) \cap P(C_1, C_3)$. Notice that when these three elements are cycles, the problem becomes the Apollonius problem.

3 The Solving Strategy for 2D Constraint Problems

We consider the instances of the constraint schema by increasing complexity of the equation system. In the following, E_0 is a line. We assume a coordinate system in which E_0 is on the x-axis. So, the first cluster is assumed fixed, whereas the second cluster can only translate along the x-axis.

The 6 major cases for the translational clusters problems are the $L(LL, LL)$, $L(CL, LL)$, $L(CL, CL)$,$L(CC, LL)$, $L(CC, CL)$, and $L(CC, CC)$ problems. The shared element is the line $E_0 = L$, and C and L represent the cycles and rays.

3.1 The L(LL,LL) Problem

We have two clusters sharing the line L_0. In each cluster there are two additional oriented lines, L_1, L_2 in the first, and L_3, L_4 in the second cluster. The second cluster translates along L_0 and must be positioned such that the four rays $L_1, ..., L_4$ are tangent to a common circle in the correct orientation. The coordinate system is such that L_0 is the x-axis and the intersection P_{12} of the lines L_1 and L_2 is on the y-axis.

Conceptually, the cyclographic maps of L_1 and L_2 are two planes that intersect in a line L_{12} in 3-space. By the same reasoning the cyclographic maps of L_3 and L_4 intersect in another line L_{34}. The clusters must be placed such that these two lines in 3-space meet in a common point that maps to the oriented cycle tangent to all four rays L_i. To find this common point, note that the translation of cluster 2 causes the line L_{34} to sweep a plane Π^* whose intersection with the line L_{12} is the point we seek.

Algebraically, we do the following. Let $P_{12} = (0, d_0)_E$, and let $L_k = [a_k, b_k, c_k]$ $k = 1, 2$. Then $c_k = -d_0 b_k$. The cyclographic maps are then

$$\Pi_k = [a_k, b_k, -1, -d_0 b_k], \qquad k = 1, 2$$

The intersection of $\Pi_1 \cap \Pi_2$ is then the line L_{12} with the tangent $[b_2 - b_1, a_1 - a_2, a_1 b_2 - a_2 b_1]_E$ through the intersection point P_{12} of the two rays in the plane.

We assume that the position of lines L_3 and L_4 is a function of the translation distance d, where $d = 0$ means that the intersection point of L_3 and L_4 is on the y-axis. For $d > 0$ the cluster is translating to the right, in the positive x direction. We assume, furthermore, that the lines L_3, L_4 intersect at $(0, d_0')_E$, before the translation. Then the cyclographic maps are

$$\Pi_k(d) = [a_k, b_k, -1, -d_0' b_k - d a_k], \qquad k = 3, 4$$

The intersection of L_{12} with the swept plane Π^* is reached for the value of d at which the determinant vanishes:

$$\begin{vmatrix} a_1 & b_1 & -1 & -d_0 b_1 \\ a_2 & b_2 & -1 & -d_0 b_2 \\ a_3 & b_3 & -1 & -d_0' b_3 - d a_3 \\ a_4 & b_4 & -1 & -d_0' b_4 - d a_4 \end{vmatrix} = 0$$

This is a linear equation in d which can be rewritten as

$$\begin{vmatrix} a_1 & b_1 & 1 & 0 \\ a_2 & b_2 & 1 & 0 \\ a_3 & b_3 & 1 & a_3 \\ a_4 & b_4 & 1 & a_4 \end{vmatrix} d + \begin{vmatrix} a_1 & b_1 & 1 & d_0 b_1 \\ a_2 & b_2 & 1 & d_0 b_2 \\ a_3 & b_3 & 1 & d_0' b_3 \\ a_4 & b_4 & 1 & d_0' b_4 \end{vmatrix} = 0$$

Once d has been computed, the position of the second cluster is fixed, and the coordinates of the variable-radius circle are found from the intersection of the planes Π_k.

Degeneracies: When both the coefficient of d and the constant term vanish, i.e.,

$$\begin{vmatrix} a_1 & b_1 & 1 & 0 \\ a_2 & b_2 & 1 & 0 \\ a_3 & b_3 & 1 & a_3 \\ a_4 & b_4 & 1 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 & d_0 b_1 \\ a_2 & b_2 & 1 & d_0 b_2 \\ a_3 & b_3 & 1 & d_0' b_3 \\ a_4 & b_4 & 1 & d_0' b_4 \end{vmatrix} = 0$$

then there is a solution for every value of d. This degeneracy arises when the line L_{12} is contained in the plane Π^*. When the lead coefficient vanishes but the constant term does not, then the problem has no solution for this configuration.

Orientation: We should consider all combinations of orienting the rays L_k. Reversing the orientation of all rays and of the cycle simultaneously does not change the geometric solution. Therefore, we only have to consider $2^3 = 8$ combinations, and obtain up to eight solutions.

Extensions: The characteristic of the L(LL,LL) solution is that the intersections of the cyclographic maps of the relevant cluster elements are lines in 3D. Therefore, the solution of L(LL,LL) is adapted easily to those problems and subcases in which lines arise as well in the intersection. They include the problems L(PL,PL) and L(PL,LL) where the variable-radius circle must be incident to the points.

Example 1 *Let the constraints for two clusters be as the following (figure 3):*

$$S_1 : \theta_{01} = 120^0, \theta_{02} = 60^0, d_0 = -10$$

$$S_2 : \theta_{03} = 70^0, \theta_{04} = 40^0, d_0' = -5$$

Find the variable-radius circle tangents to $L_i, i = 1, .., 4$ in every different combination of orientation.

Let $-l$ denote the ray that has the opposite orientation of l. Then we know that $C([a, b, c]) = [a, b, -1, c]$ and $C(-[a, b, c]) = [-a, -b, -1, -c]$. We can then find the variable-radius circles accounting for all different ray orientations. The results are shown in table 1. In the table, the first column shows the orientation of the four rays. For example "$+ + + -$" indicates that the first three elements have the original orientation and the fourth element has the opposite orientation. The second column indicates the translation distance for the second cluster. The third and fourth columns show the center and radius of the variable-radius circle. The associated figures are shown in figure 4. ♡

3.2 The L(CL,LL) and L(PL,LL) Problems

In the L(CL,LL) problem, we find the intersection of the cyclographic maps of a cone and three planes. The same is true for the L(PL,LL) problem except for

	d	center	radius
$++++$	16.86	$(9.72, -10)_E$	8.42
$+++-$	-5.32	$(-1.82, -10)_E$	-1.58
$++-+$	-5.32	$(-1.82, -10)_E$	-1.58
$++--$	-2.58	$(-9.72, -10)_E$	-8.42
$+-++$	-3.02	$(0, -2.88)_E$	-3.56
$+-+-$	-2.15	$(0, -8.06)_E$	-0.97
$+--+$	-9.51	$(0, -18.58)_E$	4.29
$+---$	1.63	$(0, -6.14)_E$	-1.93

Table 1: All solutions of the L(LL,LL) problem of Example 1

the special case of point/circle incidence. Without loss of generality, we fix the cluster containing the fixed-radius circle. Therefore, the cyclographic maps for cluster S_1 are a fixed cone and a fixed plane. Assuming a choice of coordinates in which the center of the cycle C_1 is on the Y-axis, the equations are:

$$\begin{aligned} C(C_1) &= C((0, y_1, z_1, 1)) \\ \Pi_2 &= [a_2, b_2, -1, -d_0 b_2] \end{aligned}$$

Cluster S_2 generates two (moving) planes as cyclographic maps whose equations are, as in the L(LL,LL) case,

$$\begin{aligned} \Pi_3(d) &= [a_3, b_3, c_3, -d_0' b_3 - d a_3] \\ \Pi_4(d) &= [a_4, b_4, c_4, -d_0' b_4 - d a_4] \end{aligned}$$

The three planes $\Pi_2, \Pi_3(d)$ and $\Pi_4(d)$ intersect in a common point whose coordinates are a (linear) function of the translation distance d of the cluster S_2. By placing this point on the cone $C(C_1)$, an intersection of a parametric point with an implicit equation, we obtain two solutions from the resulting quadratic equation in d. Algebraically, the procedure uses Proposition 5 of the appendix to determine the intersection of the planes.

It is advantageous to "lift" the plane in which we solve the problem in the Z-direction by a distance equal to the (signed) radius of the cycle C_1. This has the effect of reducing C_1 to a point and simplifying the cone equation $C(C_1)$. The solution can then be dropped back down, to the original problem plane, by shifting the lines, re-inflating the cycle, and increasing or diminishing the variable radius cycle. The details are routine.

Orientation: We expect up to 16 solutions, counting 16 ways to orient the elements in general, multiplied by two solutions of the quadratic equations arising, and divided by two because of the orientation pairing.

Degeneracies: There are several degenerate cases:

1. The cycle C_1 is a point and the variable-radius circle is to be incident to this point. In this case, the intersection of the cyclographic maps for S_1 becomes a line instead of a conic, and the quadratic equation from before has a double root. Then there are only 8 solutions.

2. The planes Π_2, Π_3 and Π_4 intersect in a common line. Since the line is not fixed in space, an infinite number of solutions ensues for such an orientation configuration, and the constraint problem is underdetermined.

3. The planes Π_2, Π_3 and Π_4 coincide. This means that a parallel plane must contain the x-axis. Since the two lines in S_2 must coincide, the problem is underdetermined.

3.3 The L(CL,CL) problem

In this problem, both the fixed and the moving cluster include a cycle, and hence a cone as cyclographic map. With the enumeration of the elements as before and d the distance parameter, one plane and one cone are fixed, the others translate along the x-axis where we have placed L_0. We want to find the intersection of the cyclographic maps for C_1, L_2, $T(c_3, d)$ and $T(L_4, d)$. From Theorem 1 and Proposition 2, we have:

$$\begin{cases} C(C_1) & = & C((x_1, y_1, z_1, 1)) \\ \Pi_2 = C(L_2) & = & [a_2, b_2, -1, c_2] \\ C(T(c_3, d)) & = & C((x_3 + d, y_3, z_3, 1)) \\ \Pi_4(d) = C(T(L_4, d)) & = & [a_4, b_4, -1, c_4 - a_4 d] \end{cases}$$

A straightforward equation formulation raises the degree of the system to be solved unnecessarily. Instead, we simplify the intersection of the two cones and two planes to intersecting one cone and three planes. The cone is $C(C_1)$. Two of the three planes are Π_2 and $\Pi_4(d)$. The other plane will be the plane that contains the intersection of the cones $C(C_1)$ and $C(T(C_3, d))$. From theorem 2, we write this plane as

$$\begin{aligned} \Pi_3(d) &= P(C_1, (T(C_3, d)) \\ &= [x_3 + d - x_1, y_3 - y_1, z_1 - z_3, (x_1^2 + y_1^2 - z_1^2 - (x_3 + d)^2 - y_3^2 + z_3^2)/2] \end{aligned}$$

So, we obtain

$$\begin{cases} Eq & : & (X - Wx_1)^2 + (Y - Wy_1)^2 - (Z - Wz_1)^2 = 0 \\ \Pi_2 & : & [a_2, b_2, -1, c_2] \\ \Pi_3 & : & [x_3 + d - x_1, y_3 - y_1, z_1 - z_3, (x_1^2 + y_1^2 - z_1^2 - (x_3 + d)^2 - y_3^2 + z_3^2)/2] \\ \Pi_4 & : & [a_4, b_4, -1, c_4 - da_4] \end{cases}$$

In general, the three planes intersect in one point, and this point should also lie on the cone $C(C_1)$. We use Proposition 5 to find the intersection point $(\Delta_1, \Delta_2, \Delta_3, \Delta)$ of the planes. Here, the degrees of the polynomials $\Delta_1, \Delta_2, \Delta_3$ are two, and the degree of Δ is one. We substitute the intersection point $(X, Y, Z, W) = (\Delta_1, \Delta_2, \Delta_3, \Delta)$ into the cone equation. We obtain an equation of degree 4, in the variable d, so there are 4 solutions for this problem. After solving the equation, we know the distance that the second cluster should be translated by, and also the cycle tangent to the four elements in the two clusters. The cycle has the center $(X/W, Y/W)_E$ and the (signed) radius Z/W.

Orientation:

The eight basic orientations allow up to 4 solutions each, so that the maximum number of geometric solutions is 32.

Degeneracies: The three planes may meet in a common line or coincide. The intersection $(\Delta_1, \Delta_2, \Delta_3, \Delta)$ of the three planes is at infinity when $\Delta = 0$ and at least one of the $\Delta_k \neq 0, k = 1, .., 3$. The planes intersect in a line or coincide when $\Delta_1 = \Delta_2 = \Delta_3 = \Delta = 0$. Since $[a, b, c, d]$ and $[ra, rb, rc, rd], r \neq 0$ represent the same plane, we can check easily whether three planes coincide. If they are not coincident, but the coordinates Δ and Δ_k all vanish, the intersection is a line.

When the three planes intersect in a line, we use Proposition 6 to obtain a parametric representation of the line, $(x(s), y(s), z(s), w)$. The line coordinates must be linear in d if Π_2 and Π_4 are not parallel. Substituting into the equation of the cone, we get the degree 2 equation

$$(x(s) - x_0 w)^2 + (y(s) - y_0 w)^2 - (z(s) - z_0 w)^2 = 0$$

We solve the equation for s. If s is real, then we find the intersection of a line and a cone, obtaining two points in general. If s is complex, the line and cone do not intersect. If the equation vanishes, the line is tangent to the cone.

Example 2 *Consider the problem of figure 5. The first cluster contains the circle C_1 and one of its tangent ray L_2 that has the angle $120°$ with the x-axis. The second cluster contains the circle C_3 and one of its tangents L_4 at an angle of $135°$ with the x-axis. Find a tangent circle of C_1 and C_4 at distances 30 and 25, respectively, to the lines L_2 and L_4.*

We translate the rays L_2 and L_4 away from their normals by 25 and 30 units, obtaining the rays L'_2 and L'_4. Now the above example is transformed to finding a circle tangent to C_1, L'_2, C_3 and L'_4, suitably translating the cluster with C_3 and L'_4 along x-axis by d. In this case, we have the initial configuration for the two clusters:

$$\begin{cases} C_1 &= (0, \frac{15}{2} - 40\sqrt{3}, 15, 1) \\ L'_2 &= [\frac{\sqrt{3}}{2}, \frac{1}{2}, -1, \frac{15}{4} + \frac{45}{2}\sqrt{3}] \\ C_3 &= (0, \frac{45}{2}\sqrt{2}, 20, 1) \\ L'_4 &= [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1, 55] \end{cases}$$

We get four solutions, as shown in table 2. Figure 5 shows two solutions. Intuitively, the solutions differ by whether the variable-radius circle achieves tangency to L'_2 and L'_4 inside or outside the contour of figure 5. ♡

d	center	radius
3.16	$(-3.88, -72.15)_E$	8.96
-39.33	$(-29.17, -119.17)_E$	-36.45
-177.73	$(-88, -93.52)_E$	-74.58
-1523.80	$(-763.62, -629.63)_E$	-927.74

Table 2: All solutions of the L(CL,CL) problem

3.4 The L(CC,LL) Problem

The cluster S_1 contains the elements L_0, C_1, C_2 and the cluster S_2 contains L_0, L_3, L_4. As before, S_1 is fixed and the shared line L_0 is on the x-axis. We follow the same strategy as before, considering three planes and intersecting the resulting point with one of the cones.

The first plane contains the intersection of the two cones $C(C_1)$ and $C(C_2)$. Since S_1 is fixed, this plane $\Pi_1 = P(C_1, C_2)$ has constant coefficients. The second and third planes are the cyclographic maps of L_3 and L_4, accounting for translation. That is, $\Pi_2(d) = C(T(L_3, d))$ and $\Pi_3(d) = C(T(L_4, d))$. Note that the second and third plane have coordinates that are linear in d. Therefore, Δ_1, Δ_2, Δ_3, have degree 1 and Δ is constant. After we substitute the point $(\Delta_1, \Delta_2, \Delta_3, \Delta)$ into the equation of the first cone, we get a quadratic equation in d. The two solutions determine the position of S_2 and the variable-radius circle. There is a maximum of 16 solutions.

3.5 The L(CC,CL) Problem

The clusters S_1 and S_2 contain the elements L_0, C_1, C_2 and L_0, C_3, L_4, respectively. To keep the solution complexity low, we again identify three planes and intersect them. The resulting point, parameterized by the translation variable d of the cluster S_2, is then substituted into one of the cone equations yielding the solution.

We construct the three planes as $\Pi_1 = P(C_1, C_2)$, $\Pi_2 = P(C_1, T(C_3, d))$, and $\Pi_3 = C(T(L_4, d))$. The first plane has constant coefficients. The coefficients of the other two planes are a function of d. The four coefficients of Π_2 have degree (1,0,0,2), those of Π_3 have degree (0,0,0,1). So, Δ_1, Δ_2, Δ_3, and Δ have degree $2, 2, 2$ and 1 respectively. After we substitute the point $(\Delta_1, \Delta_2, \Delta_3, \Delta)$ into the equation of the first cone, we get a quartic equation in d. The solutions determine the cycle tangent to the four elements C_1, C_2, C_3, and L_4. Notice that the solution procedure of this problem is the same as of L(CL,CL) except for the determination of the first plane.

3.6 The L(CC,CC) problem

The most complex configuration for translational cluster motion is the L(CC,CC) problem. Enumerating the cycles as C_1, C_2, C_3, C_4, we consider the planes

$$\begin{aligned} \Pi_1 &= P(C_1, C_2) \\ \Pi_2(d) &= P(C_1, T(C_3, d)) \\ \Pi_3(d) &= P(C_1, T(C_4, d)) \end{aligned}$$

Plane Π_1 is constant; Π_2 and Π_3 have a linear first coordinate and a quadratic fourth coordinate in d, the translation distance. Let

$$p = (\Delta_1, \Delta_2, \Delta_3, \Delta)$$

be the intersection of the planes. Then the coordinates of p are polynomials in d of degree $(2, 3, 3, 1)$. After we substitute the coordinates into the equation of the cone $C(C_1)$, we get a degree 6 equation in d whose solutions determine the position of S_2 and the variable-radius circle.

There is a better choice of three planes for this problem: Instead of Π_3, we can use the planes

$$\Pi_3'(d) = T(P(C_3, C_4), d))$$

The difference is that the third plane is generated by the intersection of the cones $C(C_3)$ and $C(C_4)$ *before* a translation. Clearly

$$\Pi_1 \cap \Pi_2 \cap P(C_1, T(C_4, d)) = \Pi_1 \cap \Pi_2 \cap T(P(C_3, C_4), d)$$

Here, the third plane has constant coefficients before a translation, so the coefficients of Π_3' are linear instead of quadratic. Therefore, the intersection P now has coefficients of degree $(2, 2, 2, 1)$. Substitution into $C(C_1)$ yields an algebraic equation of degree 4 in d. The degree reduction simplifies solving the problem and reduces the estimate for the a-priori number of distinct solutions from 48 to 32. The degenerate case is treated the same as before.

Example 3 *Consider the problem of figure 6. The first cluster contains the top two circles C_1, C_2 and the second cluster contains the bottom two circles C_3, C_4. The radius for the circles C_1 and C_3 are 10, and the radius for the other two circles are 15. Find the circle which has distance 12 to C_1,C_3, and 15 to C_2, C_4.*

We enlarge the circles C_2 and C_4 by 3, the difference between the required distances, obtaining the circles C_2' and C_4'. Now the above example is transformed to finding a circle tangent to C_1, C_2', C_3 and C_4', suitably translating the cluster with C_3 and C_4' along x-axis by d. The resulting circle has a radius 12 units larger than the one we need for the sketch. The centers of C_1 and C_3 coincide with the y-axis initially, so we have the initial configuration for the two clusters:

$$\left\{ \begin{aligned} C_1 &= (0, 110, 10, 1) \\ C_2' &= (-8, 75, 18, 1) \\ C_3 &= (0, 22, 10, 1) \\ C_4' &= (35, 30, 18, 1) \end{aligned} \right.$$

	d	center	radius
1	18.78	$(30.02, 71.90)_E$	63.12
2	85.88	$(70.57, 92.96)_E$	-62.60
3	-88	$(-69.98, 91.98)_E$	82.26
4		$(-132.91, 154.99)_E$	-130.39

Table 3: All solutions of the L(CC,CC) problem for example 3

We get four solutions for d, namely 18.78, 85.88, -88, -88. The case $d = -88$ is degenerate and gives two solutions as shown in table 3. Figure 6 shows one of them, solution 4. Solution 3 is similar, but the variable-radius circle contains all four smaller circles. The other two solutions differ in that the second cluster is on the other side of the first cluster. ♡

4 Conclusion

The solution strategy for solving variable-radius-circle clusters has the following pattern.

1. Fix cluster S_1 and place the coordinate system so that the x-axis coincides with the line L_0 shared by cluster S_2.
2. Construct the cyclographic maps of all elements, accounting for the translation of cluster S_2. Where possible, replace the cone/cone intersection with a cone/plane intersection, thus lowering the algebraic degree.
3. Derive a univariate polynomial whose solution determines the position of S_2 and the variable-radius circle of the third cluster.

The choice of which of the two clusters to fix is based on which cluster has the more complicated elements. Constraints on circles (and nonzero distance constraints on points) are algebraically more complex than distance constraints from lines. Hence, the cluster with more circle elements constraining the variable-radius circle is chosen as S_1.

By working with planes that contain the intersection of two cones, we consistently achieve the following solution method:

1. Construct three planes and their common intersection.
2. Substitute the intersection into one of the cone equations, or, in the case L(LL,LL), into the plane equation, so deriving a univariate polynomial in the translation distance d.
3. Solve the polynomial.

	Eqn	Π_1	Π_2	Π_3	Degree
L(LL,LL)	$Eqn(L_1)$	$C(L_2)$	$C(L_3)$	$C(L_4)$	1
L(CL,LL)	$Eqn(C_1)$	$C(L_2)$	$C(L_3)$	$C(L_4)$	2
L(CL,CL)	$Eqn(C_1)$	$C(L_2)$	$P(C(C_1), C(T(C_3, d)))$	$C(T(L_4, d))$	4
L(CC,LL)	$Eqn(C_1)$	$P(C(C_1, C_2)$	$C(T(L_3, d))$	$C(T(L_4, d))$	2
L(CC,CL)	$Eqn(C_1)$	$P(C(C_1, C_2)$	$P(C(C_1), C(T(C_3, d)))$	$C(T(C_4, d)))$	4
L(CC,CC)	$Eqn(C_1)$	$P(C(C_1, C_2)$	$P(C(C_1), C(T(C_3, d)))$	$T(P(C_3, C_4), d))$	4

Table 4: Plane construction table

Depending on the planes that must be constructed, the polynomial in d has degree up to 4. The number of solutions must be multiplied with 8, the number of essentially distinct orientations of lines and cycles, leading up to 32 solutions in the worst case. The plane constructions and the degree of the polynomial are summarized in Table 4.

There are several ways in which these problems can become degenerate. It is possible, that the three planes intersect in a common line, or even coincide. In the line case, there is the possibility of obtaining an infinite number of solutions, i.e., of dealing with an underconstrained instance. This case is approached by deriving the parametric line equation and substituting it into the cone or plane equation representing the cyclographic map of element 1 in S_1. If the parameter vanishes, no intersection is also a possibility.

When two of the three planes coincide, there is a possibility that an intersection line can be found with the other plane. In the cases we have studied, however, usually no solution exists.

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A Appendix

Proposition 1 *The line or ray from point* $P_1 = (x_1, y_1)$ *to* $P_2 = (x_2, y_2)$ *is*

$$\Lambda = [y_1 - y_2, x_2 - x_1, x_1 y_2 - x_2 y_2].$$

Proposition 2 *The line or ray*

$$\Lambda = [a, b, c],$$

when translated along the X-axis by the distance d, becomes the line or ray

$$\Lambda' = T(\Lambda, d) = [a, b, c - ad].$$

Moreover, when rotated counter-clockwise about the origin by the angle θ, it becomes

$$\Lambda'' = R(\Lambda, \theta) = [a\cos(\theta) - b\sin(\theta), a\sin(\theta) + b\cos(\theta), c].$$

We write $T([a, b, c], d) = [a, b, c - ad]$ for the translation of a line or ray by the distance d, and $R([a, b, c], \theta) = [a\cos(\theta) - b\sin(\theta), a\sin(\theta) + bcos(\theta), c]$ for the rotation of a line or ray by the angle θ.

Proposition 3 *The plane*

$$\Pi = [a, b, c, d],$$

when translated along the X-axis by d, becomes the plane

$$\Pi' = T(\Pi, d) = [a, b, c, d - ad].$$

Moreover, when rotated counter-clockwise about the Z-axis by the angle θ, it becomes the plane

$$\Pi'' = R(\Pi, \theta) = [a\cos(\theta) - b\sin(\theta), a\sin(\theta) + b\cos(\theta), c, d].$$

Proposition 4 *The cycle*

$$C = (x, y, r, 1),$$

translated along the X-axis by d, becomes

$$T(C, d) = (x + d, y, r, 1).$$

Rotated counter-clockwise about the Z-axis by θ, it becomes

$$R(C, \theta) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), r, 1).$$

Proposition 5 *The plane spanned by the three points*

$$P_k = (x_k, y_k, z_k, -w_k), \qquad k = 1, 2, 3,$$

is

$$\Pi = [\Delta_1, \Delta_2, \Delta_3, \Delta]$$

where

$$\Delta_1 = \begin{vmatrix} w_1 & y_1 & z_1 \\ w_2 & y_2 & z_2 \\ w_3 & y_3 & z_3 \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} x_1 & w_1 & z_1 \\ x_2 & w_2 & z_2 \\ x_3 & w_3 & z_3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \\ x_3 & y_3 & w_3 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The intersection point of the three planes

$$\Pi_k = [a_k, b_k, c_k, -d_k], \qquad k = 1, 2, 3,$$

is the point

$$P = (\Delta_1, \Delta_2, \Delta_3, \Delta)$$

where

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proposition 6 *Let $\Pi_1 = [a_1, b_1, c_1, d_1]$ and $\Pi_2 = [a_2, b_2, c_2, d_2]$ be two planes. Define $q_{mn} = m_1 n_2 - m_2 n_1$, where $m, n \in \{a, b, c, d\}$. If these two planes intersect, the intersection line has direction $[q_{bc}, q_{ca}, q_{ab}]$. If $q_{bc} \neq 0$, this line passes through the point $(0, -q_{dc}, q_{db}, q_{bc})$. If $q_{ca} \neq 0$, this line passes through the point $(q_{dc}, 0, -q_{da}, q_{ca})$. If $q_{ab} \neq 0$, this line passes through the point $(-q_{db}, q_{da}, 0, q_{ab})$.*

Proposition 7 *A line or ray that has distance d to a point $(x_0, y_0, 1)$ has the coordinates*

$$[a, b, d\sqrt{a^2 + b^2} - (ax_0 + by_0)].$$

Proposition 8 *The ray that intersects the X-axis at the angle θ, measuring the angle clockwise from X-axis to ray, is $[-sin(\theta), -cos(\theta), d]$. Furthermore, when the ray contains the point $(x_0, y_0, 1)$, then $d = x_0 \sin(\theta) + y_0 \cos(\theta)$.*

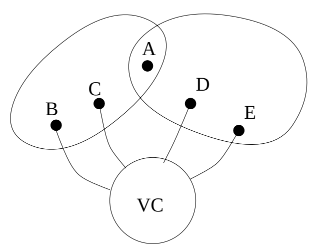


Figure 1: Variable-radius circle as cluster

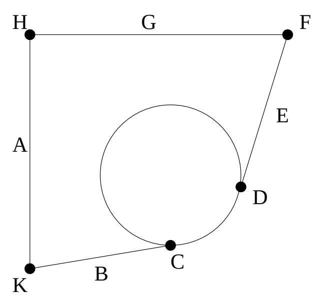


Figure 2: Example problem with distance constraints between points (H,F), (F,D) and (K,C); angle constraints between lines (A,G), (G,E), and (A,B). The circle is to be tangent to line B at C and to line E at D.

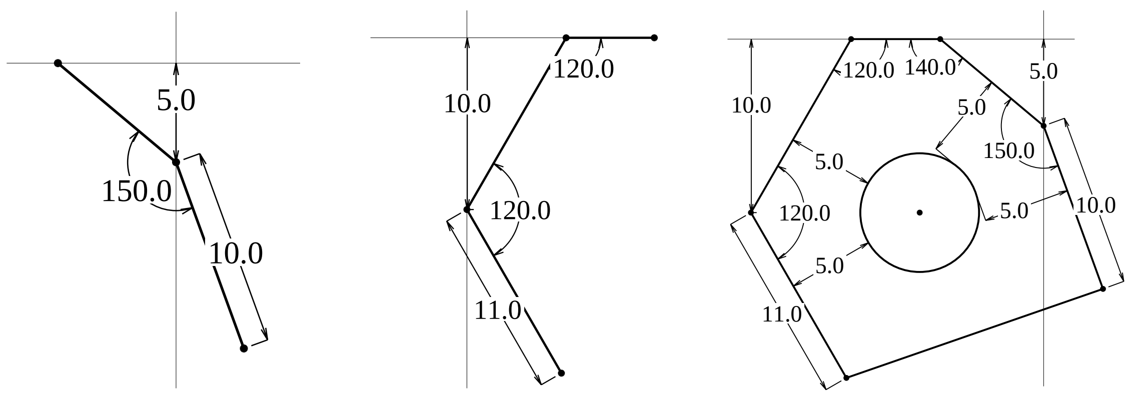
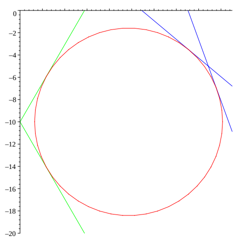
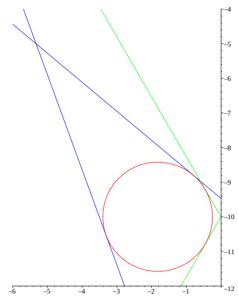
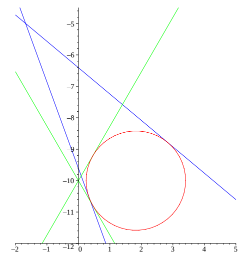


Figure 3: L(LL,LL) example with two clusters determining the circle. The two clusters have been reduced to the relevant geometric entities.

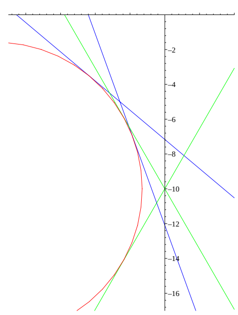


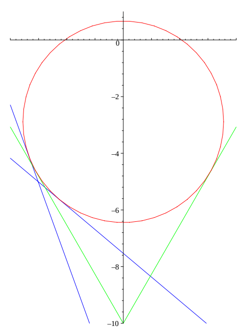


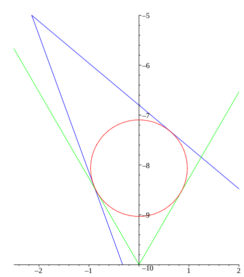


(b) $(+++-), (---+)$ (c) $(++-+), (--+-)$

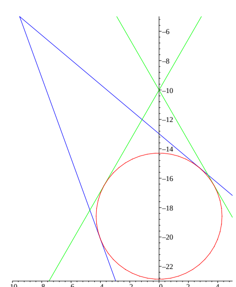
(a) $(++++), (----)$

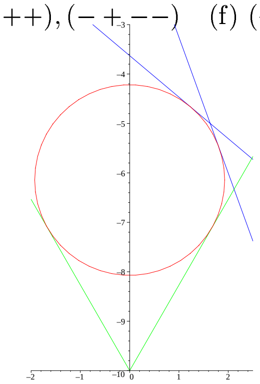






(d) $(++--), (--++)$ (e) $(+-++), (-+--)$ (f) $(+-+-), (-+-+)$





(g) $(+--+), (-++-)$ (h) $(+---), (-+++)$

Figure 4: All solutions of the L(LL,LL) problem of example 1

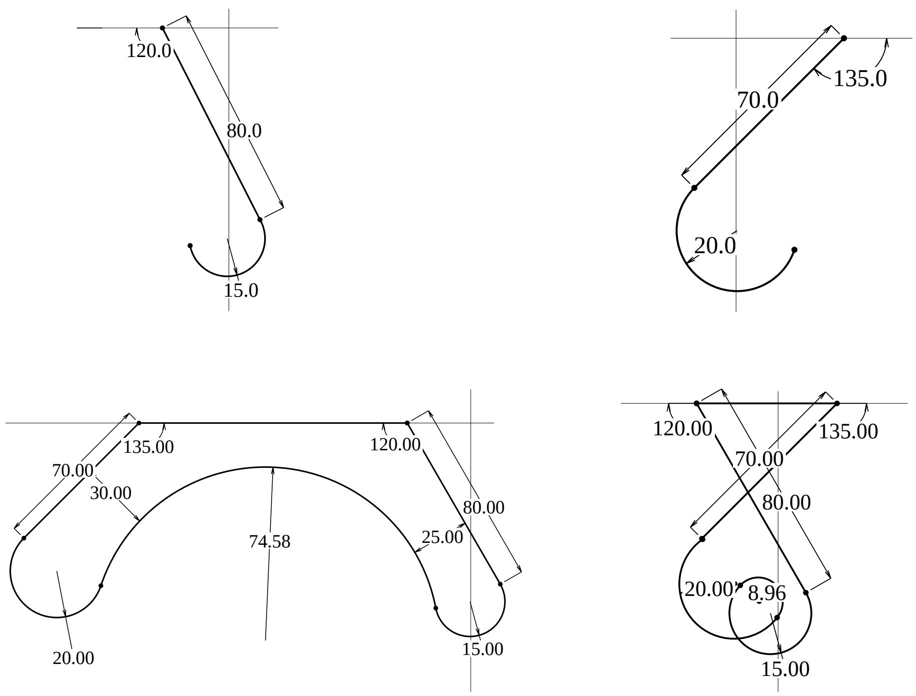


Figure 5: Top – the two clusters of the L(CL,CL) problem of example 2; Bottom – two solutions, one with $r = -74.58$, the other with $r = 8.96$.

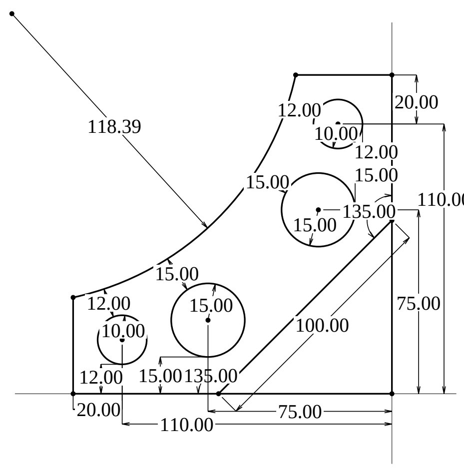


Figure 6: Solution 4 of the L(CC,CC) problem of example 3

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