Computational Geometry
CS 531

IMPLICIT ALGEBRAICS
• Bezout’s theorem
• Algebraic curves & surfaces
• Conics
• Continuation eval

2D Curves

• A polynomial $f(x,y)=0$ defines an affine algebraic curve. Coefficients usually are integer.
  - $x^2+y^2 - 1 = 0$
  - $x^2 - xy+y^2 - 3x+2y - 10 = 0$
  - $xy+15x - 10 = 0$
  - $x^3 - yx + x^2 = 0$
Formally

- A bivariate polynomial $f(x,y)$ with coefficients in $\mathbb{C}$ defines an implicit algebraic curve as:
  $$\sum_k a_k x^{p_k} y^{q_k} = 0$$
- The degree of the term $ax^p y^q$ is $p+q$. The degree of the polynomial is the highest degree of its terms.

Singular Cubics

- $y - x^3 = 0$
- $y^2 - x^3 = 0$
- $y^2 - x^2 - x^3 = 0$
Homogeneous

By homogenizing, we can extend algebraic curves and study their behavior at infinity

\[ x^2 + y^2 - 1 = 0 \quad x^2 + y^2 - w^2 = 0 \]
\[ x^3 - yxw + x^2w = 0 \quad x^3 - yxw + x^2w = 0 \]

Two ways to homogenize,

1. Raise all terms to the degree of the polynomial by multiplying with a suitable power of \( w \).
2. Use \( (x/z, y/z) \) and clear the denominator.

Example

\[ 3xy^2 - 3x^2y + 4x^2 - 2xy + 5y - 2 \]
\[ 3xy^2 - 3x^2y + 4x^2w - 2xyw + 5yw^2 - 2w^3 \]

Set \( w \) to zero to obtain the degree form, which governs the behavior at infinity. Here we get

\[ 3xy^2 - 3x^2y = 3xy(y - x) \]

So, 3 solutions (cf. Bezout!):

\( (1,0,0), (0,1,0), (1,1,0) \)
Example (2)

\[3xy^2 - 3x^2y + 4x^2 - 2xy + 5y - 2\]

At infinity: \((1,0,0), (0,1,0), (1,1,0)\)

Basics of Algebraic Curves

- Curve degree is degree of highest term(s)
  - \(x^4, xy^3, x^2y^2\) all are degree 4 terms
- Bezout: A line intersects a curve of degree \(n\) in exactly \(n\) points, properly counted
  - Point multiplicity
  - Points at infinity
  - Complex points
- Bezout: A curve of degree \(n\) intersects a curve of degree \(m\) in \(n \cdot m\) points, properly counted
Means What?

• Bezout’s theorem means that the algebraic notion of degree is also anchored in a geometric degree notion.
• Roughly, degree \( n \) means \( n \) “wiggles.”

How to Count for a Circle

By Bezout’s theorem, the unit circle intersects any line in exactly two points:
• Obvious for the line \( y = 0 \).
• The line \( y = 1 \) is tangent, so the intersection counts as a double point. Justifiable from geometry with lines \( y = 1 - \epsilon \).
Complex Intersections

• The line \( y = 2 \) intersects the unit circle in two complex points:
  - \( x^2 + y^2 - 1 = 0 \) and \( y = 2 \), so:
  - \( x^2 + 4 - 1 = 0 \), i.e., \( x^2 = -3 \)
  - \( x = \pm i\sqrt{3} \) giving
    \((i\sqrt{3}, 2, 1)\) and \((-i\sqrt{3}, 2, 1)\)

Bezout vs. Circle

Intersection of two circles should be 4 points.
So, where are the other two?

\[
(x - 1)^2 + y^2 - 4 = 0
\]
\[
(x + 1)^2 + y^2 - 4 = 0
\]

Subtract, then substitute:

\[
4x = 0, \quad y^2 = 3
\]

So \((0, \pm \sqrt{3})\) are the two intersections.
Infinite Intersections

- The line at infinity intersects every circle at the absolute circle points.
  - \((x-c_x)^2+(y-c_y)^2 - r^2 = 0\)
  - \((x-c_xw)^2+(y-c_yw)^2 - r^2 w^2 = 0\); substitute \(w=0\)
  - \(x^2+y^2 = 0\)
  - Solutions: \((1, i, 0)\) and \((1, -i, 0)\)

Algebraic Surfaces

- A polynomial \(f(x,y,z)=0\) defines an algebraic surface
  - Quadric surfaces
  - Homogeneous form to deal with surface in projective space
  - Formalism exactly as with algebraic curves.
Bezout’s Theorem

• A plane intersects a surface of degree $n$ in an algebraic plane curve of degree $n$
• Two algebraic surfaces, of degree $m$ and $n$, intersect in an algebraic space curve of degree $m \cdot n$
• An algebraic space curve of degree $d$ intersects a plane in $d$ points.
• So, again there is a geometric interpretation of algebraic degree.

Implicit Vs. Parametric

• Every (rational) parametric curve/surface is also an (implicit) algebraic curve/surface.
• Not every algebraic curve/surface has a rational parametric form.
• Characterizations are not simple.
• Conversions are not simple.
Examples

- **Unit circle** $x^2 + y^2 - 1 = 0$
  \[ x(s) = \frac{1 - s^2}{1 + s^2}, \quad y(s) = \frac{2s}{1 + s^2} \]

- **Right hyperbola** $x^2 - y^2 - 1 = 0$
  \[ x(s) = \frac{1 + s^2}{1 - s^2}, \quad y(s) = \frac{2s}{1 - s^2} \]

- **Parabola** $y - x^2 = 0$
  \[ x(s) = s, \quad y(s) = s^2 \]

Conics

- Every line has a parametric form, without denominator, that is easy to find.
- Every conic has a rational parametric form that is sort of easy to find.
- The parabola has a parametric form that requires no denominator.
- Every singular cubic has a rational parametric form.
- Nonsingular cubics are not rational.
Why Implicit or Parametric Form?

- Implicit form allows simple membership test.
- Parametric form allows simple point generation.
- Ideally, we want both, but conversions need not be straightforward.

Conics

- How to tell the type of a conic?
  - \( x^2 + xy + y^2 + 4x + 3y - 12 = 0 \)
  - \( x^2 + 2xy + y^2 + 4x + 3y - 12 = 0 \)
  - \( x^2 + 3xy + y^2 + 4x + 3y - 12 = 0 \)
- Need to know what happens at infinity:
  - Two real points – hyperbola (nondegenerate…)
  - One double point – parabola
  - Two complex points – ellipse
Liming’s Conics

• Implicit equation is
  \[(1 - s) L_1L_2 + s L_3 L_4 = 0\]

The 5th Point

• Implicit equation is
  \[(1 - s) L_1L_2 + s L_3 L_4 = 0\]
Special Case $L_3 = L_4$

- Extra point $P$ determines $s$:

![Diagram of a conic section with points and lines labeled $L_1$, $L_2$, and $L_3 = L_4$]

How to Evaluate a Conic

- Keep Liming’s form, unless we want to do more than only evaluate.
- Evaluation of arbitrary points:

$$f(x, y) > 0$$

$$f(x, y) < 0$$

$$f(x, y) = 0$$
Evaluate on a Grid

- Color inside points red, outside points blue
- Infer curve from the dots
- Better to interpolate curve, e.g:

16 Cases

- Eight plus symmetry by opposite signs
Interpolation Method

• Linear interpolation:

\[
\frac{f_1}{a} = \frac{-f_2}{b}
\]

\[
a = \frac{f_1}{f_1 - f_2} d, \quad \text{where} \quad d = a + b
\]

Marching Squares/Cubes

• Evaluate only where the curve is…
Marching Squares/Cubes

• Evaluate only where the curve is…

[Diagram]

Marching Squares/Cubes

• Evaluate only where the curve is…

[Diagram]
Marching Squares/Cubes

- Evaluate only where the curve is…

Marching Squares/Cubes

- Evaluate only where the curve is…
Ambiguity

- Why one or the other?

Arbitrary Resolution

- Work with triangle grid…
  - Triangle orientation defines a preferred direction
  - Which one?
Regular Case

• If there is no self-intersection, then refining the grid eventually separates the branches and arrives at the correct topology.
• If there is a self-intersection, then no amount of refinement can determine what should happen

Complexity and Marching

• Evaluation at every grid point too costly
• Find a point on the curve, then march depending on the square and the sign distribution
• How to determine all branches?
Adaptive Variants

- At low curvature, grid can be widely spaced.
- At high curvature we need a dense grid.
- How do we make the grid adaptive?

Gap Fixing

- Gap due to error of linear interpolation
- Newton iteration as possible fix-up
  - May be too expensive
- Simple hack
  - Choose gap middle as curve point
Implicit Surfaces

- Same evaluation principle, but on a cubic grid. For instance:

How Many Cases?

Without accounting for symmetry get $2^8$ cases. Need a map that informs case and orientation:

1 positive: 8 positions

2 positive where x positions?

2 positive 

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Face Matching

Surface Issues

- Ambiguous cases as before, resolvable by working with tetrahedra.
  - How many tetra in a cube?
- Gap problem as before, but more complicated to fix up.
  - Examine all unmatched lines on the plane of refinement.
3-D Tetrahedral Decomposition

Five

Six

Five

Six