Euclidean Distance

• Given a (regular) geometric shape $s$, the distance from $s$ defines a map associating each point with its distance from $s$.
  • If $s$ is a point, the iso-lines of the map are concentric circles.
  • The bisector of two points $p$ and $q$ partitions the plane into three regions:
    1. points closer to $p$ than to $q$, an open half plane,
    2. points closer to $q$ than to $p$, also an open half plane,
    3. and points equidistant to $p$ and $q$.

Level Sets
Point / Line Bisector

Voronoi Diagram

Singularities of distance map
Elementary Bisectors

<table>
<thead>
<tr>
<th>Combination</th>
<th>Bisector Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>point/point</td>
<td>perpendicular bisector</td>
</tr>
<tr>
<td>point/line</td>
<td>parabola</td>
</tr>
<tr>
<td>line/line</td>
<td>angle bisector</td>
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<tr>
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<td>parabola</td>
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<tr>
<td>circle/circle</td>
<td>ellipse/hyperbola</td>
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Voronoi Diagrams
Voronoi Cell

- Let $S$ be a finite set of points in the plane, $p$ a point in $S$. Then the Voronoi cell $V(p)$ is the closure of the set of all points in the plane that are closer to $p$ than to any other point in $S$.

- Let $H^*(p,q)$ be all points in the plane closer to $p$ than to $q$. Then
  \[ V(p) = \text{cl}\{ \cap H^*(p,q) \mid q \text{ in } S, p \neq q \} \]

- So $V(p)$ is a convex set.

Voronoi Diagram of a Point Set
Properties

- If \( n \) points \( P \) are collinear, then the Voronoi diagram \( \text{Vor}(P) \) consists of \( n-1 \) parallel lines. Otherwise \( \text{Vor}(P) \) is connected and its edges are either line segments or half lines.

\[ \begin{align*}
\text{Number of Vertices and Edges} \\
\text{- Introduce a fictitious vertex at infinity, incident to all ray edges, then } \text{Vor}(P) \text{ is a planar graph} \\
\text{- Assume } n > 2, \text{ no degenerate configuration, and apply Euler’s formula } \\
\quad (n_v + 1) - n_e + n = 2 \\
\text{- Each vertex has at least valence } 3, \text{ so } \\
\quad 2n_e \geq 3(n_v + 1) \\
\text{- By algebra: } \\
\quad n_v \leq 2n - 5 \\
\quad n_e \leq 3n - 6
\end{align*} \]
Empty Circle Property

Elementary Cyclographics
Voronoi Diagram of a Shape

Level Sets
3D Conceptualization

- Graph of Euclidean distance function
- Level sets from cut with plane $z = d$
Cycles

- Oriented circles; orientation as sign of radius
- Map the cycle \((x_c, y_c, r)\) to the point \((x_c, y_c, r)\) in 3-space; note that \(r\) is a signed quantity
Cyclographic Map

- Orient cycles and curves
- Map curve point to the image of tangent cycles

Medial Axis

- Locus of centers of maximal inscribed circles
- Singularities of Euclidean distance function
- Shocks of propagating wave front
Differences

Cyclographic Map

- Orient cycles and curves
- Map curve point to the image of tangent cycles
Inverse Map of Lines in 3D

\[ \alpha = 0^\circ \quad 0^\circ < \alpha < 45^\circ \quad \alpha = 45^\circ \quad \alpha > 45^\circ \]

Recovery from Ridges
Dimensional Reduction

MAT Uses

- Dimensional reduction
  - Shape recognition and abstraction
  - Stability problem however
- Informationally complete representation
- Conceptual tool for offsets and insets
- Motion planning applications
- Meshing applications
- Constraint solving
Apollonius Problem

• Up to 8 solutions

Topological Types

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
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<td>+</td>
<td>+</td>
</tr>
<tr>
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</tr>
<tr>
<td>cyan</td>
<td>+</td>
<td>+</td>
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</tbody>
</table>
Related Solutions

Obtained by negating all orientations

Computation

• Pick an orientation for the circles
• Let the cones of the cycles be $C_1$, $C_2$, $C_3$
• Intersect the cones for two pairs, they are planes
  • $P_1 = C_1 \cap C_2$
  • $P_2 = C_1 \cap C_3$
• Intersect the planes giving a line in 3-space:
  • $L = P_1 \cap P_3$
• Intersect $L$ with one of the cones
• The two solutions yield two of the cycles
Example

\[ C_1=(0,0;1), \quad C_2=(0,4;2), \quad C_3=(5,1;3). \]
Assume all cycles positively oriented.

\[ C_1 : \quad x^2 + y^2 - (z - 1)^2 = 0 \]
\[ C_2 : \quad x^2 + (y - 3)^2 - (z - 2)^2 = 0 \]
\[ C_3 : \quad (x - 5)^2 + y^2 - (z - 3)^2 = 0 \]

\[ P_{21} : \quad 6y - 2z - 6 = 0 \]
\[ P_{31} : \quad 10x - 4z - 17 = 0 \]

\[ z = 0, \quad x = \frac{17}{10}, \quad y = 1 \]
\[ z = 1, \quad x = \frac{21}{10}, \quad y = \frac{4}{3} \]

\[ L : \left( \frac{17}{10}, 1, 0 \right) + t \left( \frac{4}{10}, \frac{1}{3}, 1 \right) \]
\[ \left( \frac{17}{10} + \frac{4}{10} t \right)^2 + \left( \frac{1}{3} + \frac{1}{3} t \right)^2 - (t - 1)^2 = 0 \]

Another Example

- Constraint problem:

\[ \begin{array}{c}
\alpha_3 \\
A \quad \alpha \quad B \quad C \quad D \quad E
\end{array} \]

\[ \begin{array}{c}
\alpha_1 \\
B \quad \alpha \quad C \quad D \quad E
\end{array} \]
MAT-Based Meshing

- Paving method:
  - Beginning at the boundary, add elements side-by-side, approximately insetting by the thickness of the elements.
  - MA is where the fronts touch or compress

Armstrong’s Method

- Approximate the MA from centers of a Delaunay triangulation of points dense on the boundary of the shape
- Classify triangles into types by how they touch the boundary
- Subdivide highly concave corners with a radius
- Analyze MAT junctions and ends, defining shape molecules with standard subdivision
- Ensure compatibility across boundaries