On Recursive Oblique Projectors

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***Abstract*—This letter proposes a recursive oblique projector. To understand better the recursive oblique projector, we provide a geometrical interpretation of recursive computation and present a brief numerical example.**

***Index Terms*—Adaptive filter, innovation matrix, oblique projection, recursive computation.**

I. Introduction

Since the 1980s, oblique projection has found wide applications in engineering. Especially in recent years, oblique projection has drawn lots of attention in signal processing [1]–[4].

The projection operators (i.e., projection matrices or projectors) can be divided into orthogonal and oblique projectors, and any orthogonal projector is idempotent and Hermitian, while the oblique one is idempotent and not Hermitian. It is well known that the orthogonal projector is a special example of the oblique projector, and the recursive orthogonal projector plays a key role in adaptive signal processing [6]. Unfortunately, there is no recursive oblique projector that severely limits applications of the oblique projector in adaptive signal processing. The aim of this letter is to fill in this gap. To facilitate better understanding, we provide a geometrical interpretation of the recursive oblique projector and present a brief example of its application in blind adaptive multiuser detection in wireless communications.

II. Recursive Oblique Projectors

We use uppercase and lowercase boldfaced letters for matrices with $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^{-1}$, and $(\cdot)^\dagger$ denoting the transpose, Hermitian, inverse, and Moore–Penrose pseudo-inverse operators, respectively. Calligraphic letters denote subspaces, and $\mathcal{C}^n$ represents the n-dimensional complex Euclidean space. For a given matrix $\mathbf{A}$, its row and column spaces are represented by $\mathrm{Row}(\mathbf{A})$ and $\mathrm{Col}(\mathbf{A})$, respectively. In this letter, we mainly consider the case of column spaces and use the calligraphic letter $\mathcal{A}$ for the column space $\mathrm{Col}(\mathbf{A})$ in most cases. For a given space $\mathcal{H}, \mathbf{P}_\mathcal{H} = \mathbf{H}\langle\mathbf{H},\mathbf{H}\rangle^{-1}\mathbf{H}^H$ and $\mathbf{P}_\mathcal{H}^\perp = \mathbf{I}-\mathbf{P}_\mathcal{H}$ denote the corresponding orthogonal projectors on $\mathcal{H}$ and $\mathcal{H}^\perp$, where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$, and $\langle\mathbf{A},\mathbf{B}\rangle = \mathbf{A}^H\mathbf{B}$ is the inner product of $\mathbf{A}$ and $\mathbf{B}$. For two given subspaces $\mathcal{H}$ and $\mathcal{S}$, $\mathbf{E}_{\mathcal{H}|\mathcal{S}}$ denotes the corresponding oblique projector onto $\mathcal{H}$ along $\mathcal{S}$. The oblique projection onto $\mathcal{H}$ along $\mathcal{S}$ of any vector $\mathbf{v} \in \mathbf{C}^n$ is computed by $\mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{v}$. The symbol $\mathcal{H} \oplus \mathcal{S}$ or $(\mathcal{H},\mathcal{S})$ denotes the direct sum of $\mathcal{H}$ and $\mathcal{S}$, and $\mathcal{H} \cap \mathcal{S}$ for their intersection.

The idea of oblique projection and its applications in signal processing is well known [1]. Consider the matrices $\mathbf{H}$ and $\mathbf{S}$ with column spaces $\mathcal{H} = \mathrm{Col}(\mathbf{H}) \subset \mathcal{C}^n$ and $\mathcal{S} = \mathrm{Col}(\mathbf{S}) \subset \mathcal{C}^n$. When $\mathcal{H}$ and $\mathcal{S}$ are disjoint, that is, $\mathcal{H} \cap \mathcal{S} = \{\mathbf{0}\}$, the oblique projector onto $\mathcal{H}$ along $\mathcal{S}$ can be computed as [1]

$$\mathbf{E}_{\mathcal{H}|\mathcal{S}} = \mathbf{H}\left(\mathbf{H}^H\mathbf{P}_\mathcal{S}^\perp\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{P}_\mathcal{S}^\perp \tag{1}$$

where the two subspaces $\mathcal{H}$ and $\mathcal{S}$ are called the *range* and *null* spaces of the projector $\mathbf{E}_{\mathcal{H}|\mathcal{S}}$, respectively.

It is well known that for the matrix $[\mathbf{U},\mathbf{W}]$, the orthogonal projector $\mathbf{P}_{(\mathcal{U},\mathcal{W})}$ onto the subspace $(\mathcal{U},\mathcal{W}) = \mathrm{Col}([\mathbf{U},\mathbf{W}])$ has the recursion formulae [6]

$$\begin{aligned}\mathbf{P}_{(\mathcal{U},\mathcal{W})} &= \mathbf{P}_\mathcal{U} + \mathbf{P}_\mathcal{U}^\perp\mathbf{W}\left\langle\mathbf{P}_\mathcal{U}^\perp\mathbf{W},\mathbf{P}_\mathcal{U}^\perp\mathbf{W}\right\rangle^{-1}\mathbf{W}^H\mathbf{P}_\mathcal{U}^\perp \\ &= \mathbf{P}_\mathcal{U} + \mathbf{P}_{\mathrm{Col}(\mathbf{P}_\mathcal{U}^\perp\mathbf{W})},\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{P}_{(\mathcal{U},\mathcal{W})}^\perp &= \mathbf{P}_\mathcal{U}^\perp - \mathbf{P}_\mathcal{U}^\perp\mathbf{W}\left\langle\mathbf{P}_\mathcal{U}^\perp\mathbf{W},\mathbf{P}_\mathcal{U}^\perp\mathbf{W}\right\rangle^{-1}\mathbf{W}^H\mathbf{P}_\mathcal{U}^\perp \\ &= \mathbf{P}_\mathcal{U}^\perp - \mathbf{P}_{\mathrm{Col}(\mathbf{P}_\mathcal{U}^\perp\mathbf{W})}.\end{aligned} \tag{3}$$

These formulae play an important role in adaptive signal processing [6], such as adaptive least square lattice filter [5]. The importance of the oblique projectors in signal processing applications compelled us to ask for the big picture: How do you effectively compute the oblique projectors $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}$ and $\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}}$ from $\mathbf{E}_{\mathcal{H}|\mathcal{S}}$ and $\mathbf{E}_{\mathcal{S}|\mathcal{H}}$? The following theorem provides a solution.

Theorem 1: If $\bar{\mathbf{H}} = [\mathbf{H},\mathbf{V}]$ and $\bar{\mathcal{H}} = \mathrm{Col}(\bar{\mathbf{H}})$, and the two column subspaces $\bar{\mathcal{H}}$ and $\mathcal{S} = \mathrm{Col}(\mathbf{S})$ are disjoint, then new oblique projection operators are given by

$$\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}} = \mathbf{E}_{\mathcal{H}|\mathcal{S}} + \mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}} \tag{4}$$

$$\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}} = \mathbf{E}_{\mathcal{S}|\mathcal{H}} - \mathbf{P}_\mathcal{S}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}} \tag{5}$$

where $\bar{\mathcal{V}} = \mathrm{Col}(\bar{\mathbf{V}})$ with $\bar{\mathbf{V}} = \mathbf{V} - \mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{V}$.

Proof: Since the column subspaces $\bar{\mathcal{H}}$ and $\mathcal{S}$ are disjoint, it is known from (1) that the oblique projector $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}$ can be written as

$$\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}} = \bar{\mathbf{H}}\left(\bar{\mathbf{H}}^H\mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}}\right)^{-1}\bar{\mathbf{H}}^H\mathbf{P}_\mathcal{S}^\perp. \tag{6}$$

Since $\mathbf{P}_\mathcal{S}^\perp$ is Hermitian and idempotent, (6) is simplified to

$$\begin{aligned}\mathbf{P}_\mathcal{S}^\perp\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}} &= \mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}}\left(\left(\mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}}\right)^H\mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}}\right)^{-1}\left(\mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}}\right)^H \\ &= \mathbf{P}_{\mathrm{Col}(\mathbf{P}_\mathcal{S}^\perp\bar{\mathbf{H}})}.\end{aligned} \tag{7}$$

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Substituting $\mathbf{U}=\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}$ and $\mathbf{W}=\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}$ in (2), then $[\mathbf{U},\mathbf{W}]=[\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H},\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}]=\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{H}}$. Hence, we have $(\mathcal{U},\mathcal{W})=\mathrm{Col}(\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{H}})$ and can rewrite (2) as

$$\mathbf{P}_{\mathrm{Col}(\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{H}})}=\mathbf{P}_{\mathrm{Col}(\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H})}+\mathbf{P}_{\mathrm{Col}\left(\mathbf{P}^{\perp}_{\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}\right)}. \quad (8)$$

Calculating $\mathbf{P}^{\perp}_{\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}$, we have

$$\begin{aligned}\mathbf{P}^{\perp}_{\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}&=\left(\mathbf{I}-\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\left(\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H},\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\right)^{-1}\left(\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\right)^{H}\right)\\&\quad\times\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}\\&=\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}-\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\left(\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H},\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\right)^{-1}\mathbf{H}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}\\&=\mathbf{P}_{\mathcal{S}}^{\perp}\left(\mathbf{I}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}\right)\mathbf{V}. \qquad (9)\end{aligned}$$

Combining (7), (8), and (9) and denoting $\bar{\mathbf{V}}=(\mathbf{I}-\mathbf{E}_{\mathcal{H}|\mathcal{S}})\mathbf{V}$, we get

$$\begin{aligned}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}&=\left(\mathbf{P}_{\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}}+\mathbf{P}_{\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{V}}}\right)\\&=\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\left(\mathbf{H}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{H}\right)^{-1}\mathbf{H}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\\&\quad+\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{V}}\left(\bar{\mathbf{V}}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{V}}\right)^{-1}\bar{\mathbf{V}}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\\&=\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{E}_{\mathcal{H}|\mathcal{S}}+\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}} \qquad (10)\end{aligned}$$

namely

$$\mathbf{P}_{\mathcal{S}}^{\perp}(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})=\mathbf{0}. \quad (11)$$

In the following, we use (11) to prove (4). It is known from (11) that any vector $\mathbf{x}\in\mathcal{C}^n$ satisfies $\mathbf{P}_{\mathcal{S}}^{\perp}(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\mathbf{x}=\mathbf{0}$. In other words

$$(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\mathbf{x}\in\mathcal{S}. \quad (12)$$

Since $\mathcal{S}$ and $\mathcal{H}$ are the range and null spaces of the projector $\mathbf{E}_{\mathcal{H}|\mathcal{S}}$, respectively, and the subspace $\bar{\mathcal{H}}=\mathcal{H}\oplus\mathcal{V}=\mathcal{H}\oplus\bar{\mathcal{V}}$ is the direct sum of the two subspaces, we have

$$(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\mathbf{x}\in\bar{\mathcal{H}}. \quad (13)$$

Equations (12) and (13) yield the result

$$(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\mathbf{x}=\bar{\mathcal{H}}\cap\mathcal{S}=\{\mathbf{0}\},\quad\forall\,\mathbf{x}\in\mathcal{C}^n$$

which implies that

$$(\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}-\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\mathbf{x}=\mathbf{0},\quad\forall\,\mathbf{x}\in\mathcal{C}^n$$

or $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}=\mathbf{E}_{\mathcal{H}|\mathcal{S}}+\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}$, i.e., (4) is true.

By the oblique projector property $\mathbf{E}_{\mathcal{S}|\mathcal{H}}=\mathbf{P}_{\mathcal{S}}-\mathbf{P}_{\mathcal{S}}\mathbf{E}_{\mathcal{H}|\mathcal{S}}$ and (4), we immediately have

$$\begin{aligned}\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}}&=\mathbf{P}_{\mathcal{S}}-\mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}=\mathbf{P}_{\mathcal{S}}-\mathbf{P}_{\mathcal{S}}(\mathbf{E}_{\mathcal{H}|\mathcal{S}}+\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}})\\&=\mathbf{E}_{\mathcal{S}|\mathcal{H}}-\mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\end{aligned}$$

which is (5). This completes the proof of Theorem 1.

The three remarks on Theorem 1 are given next.

Remark 1: If the subspace $\bar{\mathcal{H}}$ is orthogonal to $\mathcal{S}$, then the oblique projectors $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}$ and $\mathbf{E}_{\mathcal{H}|\mathcal{S}}$ are equal to $\mathbf{P}_{\bar{\mathcal{H}}}$ and $\mathbf{P}_{\mathcal{H}}$, respectively. Denote $\bar{\mathbf{V}}=(\mathbf{I}-\mathbf{P}_{\mathcal{H}})\mathbf{V}=\mathbf{P}_{\mathcal{H}}^{\perp}\mathbf{V}$; then, $\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}$ is simplified to $\mathbf{P}_{\bar{\mathcal{V}}}$ and $\mathbf{P}_{\mathcal{S}}\mathbf{P}_{\bar{\mathcal{V}}}=\mathbf{P}_{\bar{\mathcal{V}}}$, and thus, (4) and (5) reduce to (2) and (3). In other words, the recursion update of orthogonal projectors is a special case of Theorem 1.

Remark 2: Theorem 1 can be understood from the viewpoint of the innovation process, which plays a key role in the Kalman filtering theory. As an extension of the innovation process, in the case of the orthogonal projector, given a data matrix $\mathbf{V}$, its mean-square estimate in the subspace $\mathcal{H}=\mathrm{Col}(\mathbf{H})$ is given by $\mathbf{P}_{\mathcal{H}}\mathbf{V}$, and hence, we can refer to the error matrix $\bar{\mathbf{V}}=\mathbf{V}-\mathbf{P}_{\mathcal{H}}\mathbf{V}$ as the innovation matrix of the original data matrix $\mathbf{V}$ in the subspace $\mathcal{H}$ and the subspace $\bar{\mathcal{V}}=\mathrm{Col}(\bar{\mathbf{V}})$ as the innovation subspace in orthogonal projection. Similarly, $\mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{V}$ is the mean-square estimate of the data matrix $\mathbf{V}$ in the subspace $\mathcal{H}$ along the disjoint subspace $\mathcal{S}$, and we can view $\bar{\mathbf{V}}=\mathbf{V}-\mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{V}$ in Theorem 1 as the innovation matrix of the data matrix $\mathbf{V}$ in $\mathcal{H}$ along $\mathcal{S}$ and $\bar{\mathcal{V}}=\mathrm{Col}(\bar{\mathbf{V}})$ as the innovation subspace in oblique projection.

Remark 3: From a subspace point of view, the vector space $\mathcal{C}^n=\mathcal{H}\oplus\mathcal{S}\oplus(\mathcal{H}\oplus\mathcal{S})^{\perp}$, where $\mathcal{H}$, $\mathcal{S}$, and $(\mathcal{H}\oplus\mathcal{S})^{\perp}$ represent the expected signal (range), structured noise (or interference), and unstructured noise subspaces, respectively. From Theorem 1 and $\mathbf{P}_{(\mathcal{H},\mathcal{S})}=\mathbf{E}_{\mathcal{H}|\mathcal{S}}+\mathbf{E}_{\mathcal{S}|\mathcal{H}}$, we have

$$\bar{\mathbf{V}}=(\mathbf{I}-\mathbf{E}_{\mathcal{H}|\mathcal{S}})\mathbf{V}=\mathbf{E}_{\mathcal{S}|\mathcal{H}}\mathbf{V}+\mathbf{P}^{\perp}_{(\mathcal{H},\mathcal{S})}\mathbf{V}=\mathbf{V}_{S}+\mathbf{V}_{(H,S)^{\perp}}.$$

That is to say, the innovation matrix $\bar{\mathbf{V}}$ of the data matrix $\mathbf{V}$ in $\mathcal{H}$ along $\mathcal{S}$ consists of the structured noise component $\mathbf{V}_{S}=\mathbf{E}_{\mathcal{S}|\mathcal{H}}\mathbf{V}$ in $\mathcal{S}$ and unstructured noise component $\mathbf{V}_{(H,S)^{\perp}}=\mathbf{P}^{\perp}_{(\mathcal{H},\mathcal{S})}\mathbf{V}$ in $(\mathcal{H}\oplus\mathcal{S})^{\perp}$. Clearly, the innovation matrix $\bar{\mathbf{V}}\approx\mathbf{V}_{(H,S)^{\perp}}$ if the contribution of the structured noise component reduces to become negligible as compared to the unstructured noise. In this case, $\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{V}}\approx\mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{V}_{(\mathcal{H},\mathcal{S})^{\perp}}=\mathbf{V}_{(\mathcal{H},\mathcal{S})^{\perp}}$, and the oblique projector reduces to

$$\begin{aligned}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}&=\bar{\mathbf{V}}\left(\bar{\mathbf{V}}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\bar{\mathbf{V}}\right)^{-1}\bar{\mathbf{V}}^{H}\mathbf{P}_{\mathcal{S}}^{\perp}\\&=\mathbf{V}_{(\mathcal{H},\mathcal{S})^{\perp}}\left(\mathbf{V}^{H}_{(H,S)^{\perp}}\mathbf{V}_{(H,S)^{\perp}}\right)^{-1}\mathbf{V}^{H}_{(H,S)^{\perp}} \qquad (14)\end{aligned}$$

which is just the projection matrix of $\mathbf{V}_{(\mathcal{H},\mathcal{S})^{\perp}}$.

Theorem 1 is only available for the column spaces, but its version in the row spaces can be easily derived. In row spaces, the orthogonal projector is given by $\mathbf{P}_{\mathcal{H}}^{(\mathrm{row})}=\mathbf{H}^{H}(\mathbf{H}\mathbf{H}^{H})^{-1}\mathbf{H}$. In a similar way in [1], the oblique projector $\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}$ onto the row spaces $\mathcal{H}=\mathrm{Row}(\mathbf{H})$ along $\mathcal{S}=\mathrm{Row}(\mathbf{S})$ can be defined as

$$\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}=[\mathbf{H}^{H},\mathbf{S}^{H}]\begin{bmatrix}\mathbf{H}\mathbf{H}^{H}&\mathbf{H}\mathbf{S}^{H}\\\mathbf{S}\mathbf{H}^{H}&\mathbf{S}\mathbf{S}^{H}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{H}\\\mathbf{0}\end{bmatrix}, \quad (15)$$

and (15) can be rewritten as

$$\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}=\mathbf{P}_{\mathcal{S}}^{\perp(\mathrm{row})}\mathbf{H}^{H}\left(\mathbf{H}\mathbf{P}_{\mathcal{S}}^{\perp(\mathrm{row})}\mathbf{H}^{H}\right)^{-1}\mathbf{H}. \quad (16)$$

Then, the oblique projection of any vector $\mathbf{v}\in\mathcal{C}^n$ is given by $\mathbf{v}\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}$. The following corollary is a version of Theorem 1 in the case of row spaces.

Corollary 1: If $\bar{\mathbf{H}}=[\mathbf{H},\mathbf{V}]$ and $\bar{\mathcal{H}}=\mathrm{Row}(\bar{\mathbf{H}})$, and the two row subspaces $\bar{\mathcal{H}}$ and $\mathcal{S}=\mathrm{Row}(\mathbf{S})$ are disjoint, then we have the following recursive formulas:

$$\mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{H}}|\mathcal{S}}=\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}+\mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{V}}|\mathcal{S}} \quad (17)$$

$$\mathbf{E}^{(\mathrm{row})}_{\mathcal{S}|\bar{\mathcal{H}}} = \mathbf{E}^{(\mathrm{row})}_{\mathcal{S}|\mathcal{H}} - \mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{V}}|\mathcal{S}} \mathbf{P}^{(\mathrm{row})}_{\mathcal{S}} \tag{18}$$

where $\bar{\mathcal{V}} = \mathrm{Row}(\bar{\mathbf{V}})$ with $\bar{\mathbf{V}} = \mathbf{V} - \mathbf{V}\mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}}$.

Proof: The proof is straightforward. Using $\mathrm{Row}(\mathbf{H}) = \mathrm{Col}(\mathbf{H}^H)$, it is easy to show that

$$\mathbf{E}^{(\mathrm{row})}_{\mathrm{Row}(\mathbf{H})|\mathrm{Row}(\mathbf{S})} = \left(\mathbf{E}_{\mathrm{Col}(\mathbf{H}^H)|\mathrm{Col}(\mathbf{S}^H)}\right)^H. \tag{19}$$

From Theorem 1 and (19), $\mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{H}}|\mathcal{S}}$ can be rewritten as

$$\begin{aligned}\mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{H}}|\mathcal{S}} &= \left(\mathbf{E}_{\mathrm{Col}(\bar{\mathbf{H}}^H)|\mathrm{Col}(\mathbf{S}^H)}\right)^H \\ &= \left(\mathbf{E}_{\mathrm{Col}(\mathbf{H}^H)|\mathrm{Col}(\mathbf{S}^H)} + \mathbf{E}_{\mathrm{Col}(\bar{\mathbf{V}}^H)|\mathrm{Col}(\mathbf{S}^H)}\right)^H \\ &= \mathbf{E}^{(\mathrm{row})}_{\mathcal{H}|\mathcal{S}} + \mathbf{E}^{(\mathrm{row})}_{\bar{\mathcal{V}}|\mathcal{S}}.\end{aligned}$$

Similarly, we can show (18).

III. Geometrical Interpretation

For a geometrical interpretation of the recursive construction of oblique projectors described by (4) and (5), consider the recursive quantities of oblique projectors in Fig. 1, where subspaces are represented by directed lines or planes, whereas matrices or data are represented by vectors with specific lengths. For example, the directed lines $\overrightarrow{OH}$, $\overrightarrow{OS}$ and plane $\overline{HOS}$ denote the subspaces $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{H} \oplus \mathcal{S}$, and the vectors $\overrightarrow{OV}$ and $\overrightarrow{OY}$ denote the added data matrix $\mathbf{V}$ and arbitrary data $\mathbf{y} \in \mathcal{C}^n$, respectively. Assume without loss of generality that $\mathbf{y}$ lies in the subspace $\mathcal{H} \oplus \mathcal{S} \oplus \mathcal{V}$.[1]

As illustrated in Fig. 1(a), where $\overrightarrow{O_1Y}$ are perpendicular to plane $\overline{HOS}(O_1 \in \overline{HOS})$, and $\overrightarrow{OB}$ and $\overrightarrow{OA}$ are, respectively, the parallel decomposition components of $\overrightarrow{OO_1}$ along the directed lines $\overrightarrow{OH}$ and $\overrightarrow{OS}$, the vectors $\overrightarrow{OO_1}$, $\overrightarrow{OB}$, and $\overrightarrow{OA}$ represent the projections $\mathbf{P}_{(\mathcal{H},\mathcal{S})}\mathbf{y}$, $\mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{y}$, and $\mathbf{E}_{\mathcal{S}|\mathcal{H}}\mathbf{y}$, respectively. Similarly, the oblique projections $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}\mathbf{y}$ and $\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}}\mathbf{y}$ can be represented by $\overrightarrow{OO_3}$ and $\overrightarrow{OD}$ in parallelogram $\overline{OO_3YD}$, respectively. That is, when the range subspace expands to $\bar{\mathcal{H}} = \mathcal{H} \oplus \mathcal{V}$ from $\mathcal{H}$, the updated components of oblique projections are denoted by $\overrightarrow{BO_3}$ and $\overrightarrow{DA}$. In Figs. 1(b) and (c), we show that $\overrightarrow{BO_3} = \mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$ and $\overrightarrow{DA} = \mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$.

In Fig. 1(b), the vectors $\overrightarrow{OW}$ and $\overrightarrow{OU}$ in the parallelogram $\overline{OWO_2U}$ represent the projections $\mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{V}$ and $\mathbf{E}_{\mathcal{S}|\mathcal{H}}\mathbf{V}$, respectively, and thus, $\overrightarrow{WV} = \overrightarrow{OV} - \overrightarrow{OW}$ represents the matrix $\bar{\mathbf{V}} = \mathbf{V} - \mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{V}$. With $\overrightarrow{O_2W}//\overrightarrow{O_1B}$ and $\overrightarrow{O_2V}//\overrightarrow{O_1Y}$, the plane $\overline{BO_1YO_3}//\overline{WVO_2}$ represents the subspace $\mathcal{S} \oplus \bar{\mathcal{V}}$. To obtain the oblique projection $\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$, let $\overrightarrow{BY_1} = \overrightarrow{OY}$ and $\overrightarrow{Y_1O_4} \perp \overline{YO_3}$; then, $\overrightarrow{YY_1}//\overrightarrow{BO}$, i.e.,

$$\overline{Y_1YO_3} \,//\, \overline{HOS} \quad \Rightarrow \quad \overrightarrow{YO_1} \perp \overrightarrow{Y_1O_4}. \tag{20}$$

[1]Because the component of $\mathbf{y}$ in the complementary subspace $(\mathcal{H} \oplus \mathcal{S} \oplus \mathcal{V})^{\perp}$ does not affect the projections in $\mathcal{H} \oplus \mathcal{S} \oplus \mathcal{V}$, we can obtain the orthogonal projection of $\mathbf{y}$ onto the subspace $\mathcal{H} \oplus \mathcal{S} \oplus \mathcal{V}$ and then calculate the corresponding oblique projections using $\mathbf{P}_{(\mathcal{H},\mathcal{S},\mathcal{V})}\mathbf{y}$, if $\mathbf{y} \notin (\mathcal{H} \oplus \mathcal{S} \oplus \mathcal{V})$.

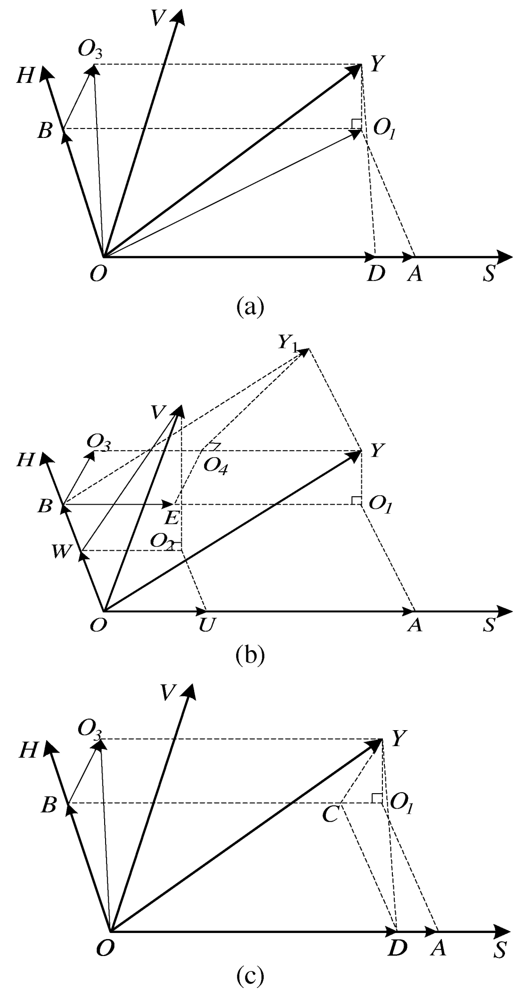

Fig. 1. Geometrical representation of recursive oblique projectors. (a) Illustration of oblique projections. (b) Illustration of $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}\mathbf{y}$ in (4). (c) Illustration of $\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}}\mathbf{y}$ in (5).

Hence, $\overrightarrow{Y_1O_4} \perp \overline{BO_3YO_1}$, which shows that $\overrightarrow{BO_4}$ equals the orthogonal projection $\mathbf{P}_{(\mathcal{S},\bar{\mathcal{V}})}\mathbf{y}$. Then, the parallel decomposition component $\overrightarrow{BO_3}$ obviously equals the oblique projection $\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$, i.e., $\overrightarrow{BO_3} = \mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$. Now, we have $\mathbf{E}_{\bar{\mathcal{H}}|\mathcal{S}}\mathbf{y} = \mathbf{E}_{\mathcal{H}|\mathcal{S}}\mathbf{y} + \mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$, which is a geometrical demonstration of the recursive relation of (4) in Theorem 1.

In a similar way, $\overrightarrow{DA} = \mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$ can be easily found in the oblique prism $OBO_3 - DCY$ in Fig. 1(c). Since $\overrightarrow{CY} \overset{//}{=} \overrightarrow{BO_3}$ and $\overrightarrow{DA} \overset{//}{=} \overrightarrow{CO_1}$, then $\overrightarrow{DA}$ illustrates the orthogonal projection $\mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$, i.e., $\mathbf{E}_{\mathcal{S}|\bar{\mathcal{H}}}\mathbf{y} = \mathbf{E}_{\mathcal{S}|\mathcal{H}}\mathbf{y} - \mathbf{P}_{\mathcal{S}}\mathbf{E}_{\bar{\mathcal{V}}|\mathcal{S}}\mathbf{y}$, which is a geometrical interpretation of (5) in Theorem 1.

IV. Brief Numerical Example

As a brief numerical example of the recursive oblique projectors, let us consider the blind adaptive multiuser detection. In a code-division multiple access (CDMA) channel, K users transmit simultaneously over a shared channel, with different signature vector $\mathbf{s}_k$. For the desired user 1 with signature vector $\mathbf{s}_1$, its linear detector has two canonical representations [7]

$$\text{Type I}: \quad \mathbf{c}_1(n) = \mathbf{s}_1 + \mathbf{x}_1(n), \quad \text{subject to } \langle \mathbf{s}_1, \mathbf{x}_1 \rangle = 0 \tag{21}$$

and [8]

$$\text{Type II}: \quad \mathbf{c}_1(n) = \mathbf{s}_1 - \mathbf{C}_{1,\text{null}}\mathbf{w}_1(n) \tag{22}$$

where the columns of the matrix $\mathbf{C}_{1,\text{null}}$ span the null space of $\mathbf{s}_1$, i.e., $\langle \mathbf{C}_{1,\text{null}}, \mathbf{s}_1 \rangle = \mathbf{0}$.

By [3], the blind multiuser detector can be expressed in the form of oblique projectors as follows:

$$\mathbf{c}_{\text{opt}} = \left(\mathbf{s}_1^{\dagger}\mathbf{E}_{s_1|\bar{S}}\right)^H \tag{23}$$

where $\bar{\mathbf{S}}$ is other users' signature matrix.

Let $S = s_1 = \text{span}(\mathbf{s}_1)$, $H = \text{span}(\mathbf{0})$, and $V = \bar{S} = \text{span}(\bar{\mathbf{S}})$ in (5); then, $\mathbf{E}_{H|S} = \mathbf{0}$, $\bar{\mathbf{V}} = \mathbf{V} - \mathbf{E}_{H|S}\mathbf{V} = \mathbf{V}$, and then, we have

$$\begin{aligned} \mathbf{c}_{\text{opt}} &= \left[\mathbf{s}_1^{\dagger}\left(\mathbf{E}_{s_1|0} - \mathbf{P}_{s_1}\mathbf{E}_{\tilde{S}|s_1}\right)\right]^H \\ &= \mathbf{s}_1 - \mathbf{P}_{s_1}^{\perp}\bar{\mathbf{S}}\left(\bar{\mathbf{S}}^H\mathbf{P}_{s_1}^{\perp}\bar{\mathbf{S}}\right)^{-1}\bar{\mathbf{S}}^H\mathbf{s}_1 \\ &= \mathbf{s}_1 - \mathbf{E}_{\tilde{S}|s_1}^H\mathbf{s}_1 \end{aligned} \tag{24}$$

which unifies Type I and Type II canonical representations for blind multiuser detection. From (24), we can get the optimal adaptive filters associated with Type I and Type II canonical blind detectors, respectively, given by

$$\mathbf{x}_{1,\text{opt}} = -\mathbf{E}_{\bar{S}|s_1}^H\mathbf{s}_1 = -\mathbf{P}_{s_1}^{\perp}\bar{\mathbf{S}}\left(\bar{\mathbf{S}}^H\mathbf{P}_{s_1}^{\perp}\bar{\mathbf{S}}\right)^{-1}\bar{\mathbf{S}}^H\mathbf{s}_1 \tag{25}$$

and

$$\mathbf{w}_{1,\text{opt}} = \left(\mathbf{C}_{1,\text{null}}^H\mathbf{C}_{1,\text{null}}\right)^{-1}\mathbf{C}_{1,\text{null}}^H\bar{\mathbf{S}}\left(\bar{\mathbf{S}}^H\mathbf{P}_{s_1}^{\perp}\bar{\mathbf{S}}\right)^{-1}\bar{\mathbf{S}}^H\mathbf{s}_1. \tag{26}$$

Since $\mathbf{P}_{s_1}^{\perp} = \mathbf{C}_{1,\text{null}}(\mathbf{C}_{1,\text{null}}^H\mathbf{C}_{1,\text{null}})^{-1}\mathbf{C}_{1,\text{null}}^H$, we have $-\mathbf{C}_{1,\text{null}}\mathbf{w}_{1,\text{opt}}(n) = \mathbf{x}_{1,\text{opt}}$, which means the two types of canonical representations for blind multiuser detection are in complete agreement.

V. Conclusion

This letter has proposed a recursive oblique projector that contains the well-known recursive orthogonal projector as a special example. The recursive oblique projector has a clear geometrical interpretation. As a brief numerical example, we have applied the recursive oblique projector to derive a unified blind adaptive multiuser detector that gives, respectively, the optimal adaptive filters of the two well-known canonical representations for a linear detector. The principal angle associated with the recursive oblique projector is an interesting problem and remains to be studied.

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