Sets, Limits, Running time

Today

- Quick review of sets and basic operations
- How to set up and solve limits
- L'Hôpital's Rule for Indeterminate Limits
- Counting work in programs

Sets

- A set is a collection of unique elements
- Basic notation
- $S = \{1,2,3,4,5\}$

Properties of sets

- Contains objects called **elements** or **members**
- Order doesn’t matter
- If order matters, it is called a **sequence**
- There are no repeat elements
- If you have repeat elements it is called a **multiset**

Important sets

- $\mathbb{Z}$ - The integers {...-3, -2, -1, 0, 1, 2, 3, ...}
- $\mathbb{N}$ - Natural numbers {1, 2, 3, 4, 5, ...}
- $\mathbb{Q}$ - Rational numbers (can be written as a fraction of 2 integers)
- $\mathbb{R}$ - Real numbers
- $\mathbb{C}$ - Complex numbers, a+bi, a,b are real numbers
- $\varnothing$ - The empty set. A set with no elements

Set operations

- **Cardinality** (size) - |S| - The number of elements in a set
    - Can be finite e.g. S = {1, 2, 3, 4, 5}. |S| = 5
    - May be infinite e.g. S = integers

- Countably infinite vs uncountably infinite

- Countably infinite: You can write all of the elements in a list
- $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$

- Uncountably infinite: It is impossible to list all of the elements
- $\mathbb{R}$, $\mathbb{C}$

Subsets

- A set A is a subset of another set B if all elements in A are also in B

- A - {1,2,3}
- B = {1,2,3,4,5}
- $A \subseteq B$

- Strict subset $A \subset B$ - A is a subset of B at A != B

- The empty set is a subset of all sets

Membership

- $x \in S$, x is in S
- $x \notin S$, x is not in S

Union, Intersection

- A = {2,3,5,7,11}
- B = {1,2,3,4,5}

- Union: Makes a new set containing all the elements in both united sets
- $A \cup B = \{1,2,3,4,5,7,11\}$

- Intersection: Makes a new set containing only elements in both sets
- $A \cap B = \{2,3,5\}$

XOR / Symmetric Difference, Set difference

- Similar to XOR in logic, same notation. Contains elements in one but not both of the sets

- $A \oplus B = \{1,4,7,11\}$

- Set difference, items in first set with items in second set removed

- $A / B = \{7,11\}$

Set-builder notation

- Shorthand for constructing larger sets
- S = {x | <conditions>}
- All integers greater than 5
- $S = \{x \mid x \in \mathbb{Z}, x > 5\}$ or $S = \{x \in \mathbb{Z} \mid x > 5\}$
- Set of all irrational numbers between e and pi?
- Set of all prime numbers?

Limits

- Used in a few situations
  - Dealing with division by zero
  - Seeing behavior of a function as it approaches infinity

- We know a formula, and need to find some extra information about it

- E.g $$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

- Undefined at x = 1

Ways to work with limits

- By Substitution
- f(x) = 2x + 2, limit approaching 2
- By factoring $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Limits of polynomial fractions

- Consider a limit containing 2 polynomials

$$\lim_{x \to a} \frac{P(x)}{Q(x)}$$

- By cases
  - P(x) is higher degree than Q(x) - infinity (or negative infinity. Copy sign of first coefficient)
  - Q(x) is higher degree than P(x) - zero
  - Same degree - equal to the fraction of the highest degree coefficients

Unsolvable limits

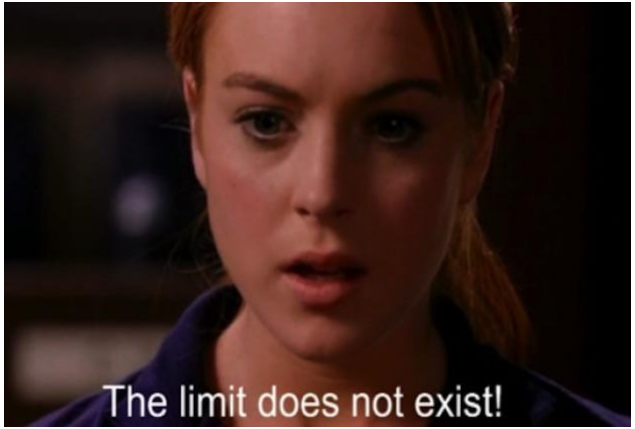
- For a limit to exist, the limit approaching from the left must equal the limit approaching the right

- Notation
  - Right
    $$\lim_{x \to a^+}$$
  - Left
    $$\lim_{x \to a^-}$$

Unsolvable limits

- If the limit from the right and the limit from the left are the same, then the limit exists and is equal to the answers

- If they are not the same…



Example

- Yet more proof you can’t divide by zero

- 1/x

L’Hôpital’s Rule

- Fixes cases where your limit ends up in an indeterminate form
- 0/0, inft/infty

- If the limit ends up in an indeterminate form, you can do the following

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{x \to a} \frac{P'(x)}{Q'(x)}$$

Examples

- $\sin(x) / x$ approaching 0

- e^x / x^2 approaching infinity

Intermission

- Questions?

Sample questions

$$\lim_{x \to 5} \frac{x^2 - 25}{x^2 + x - 30}$$

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$$

Counting work

- Leading into tomorrow's topic, asymptotics
- Need to get some function for running times of code segments
- Need to count basic operations

Random Access Machine

- Model most often used for counting work

- All operations take the same amount of time
  - 4 basic ops, boolean evaluation, assignment, etc

- Not necessarily a perfect representation of real life

- Division takes ~20 times longer than multiplication, for example

Counting steps

- Not super important to get the constant exact. Big O will take care of that soon enough
- More important to get the sums right
- Most of the sums will come out as arithmetic or geometric sequences

Example

Pseudo code of maxOfArray

```
Algorithm maxOfArray(A, n)
   Input: an array A storing n integers
   Output: the maximum of in A
max ← A[1]
for i ← 2 to n do
   if max < A[i] then
      max ← A[i]
return max
```

x varies between 0 and $n-1$ depending on the input

$x=0$: $n, n-1, n-2, \ldots\ldots, 3,2,1$

$x=n-1$: $1,2,3,\ldots\ldots,n-2,n-1,n$

indexing × 1 + assignment ×1

\+

assignment ×1 + increment × $(n-1)$ + comparison × n

\+

Indexing × $(n-1)$ + comparison × $(n-1)$

\+

indexing × x + assignment × x

\+

return × 1

=

$4 \times n + 2 \times x$

Arithmetic sequence

- Definition
  - $Given\ a\ sequence\ of\ numbers\ a_1, a_2, \ldots, a_n, if$

$$a_i = a_1 + d \times (i - 1) \qquad (i = 1,2,3, \ldots, n)$$

$satisfies, it\ is\ called\ an\ arithmetic\ sequence, d\ is$
$called\ the$ $common\ difference$

- Example
  - $a_1 = 1, a_2 = 4, a_3 = 7, a_4 = 10, a_5 = 13$ is an arithmetic sequence
    - $a_i = 1 + 3 \times (i - 1)$ satisfies for all $i = 1, 2,3,4,5;$
  - 3 is the common difference

Summation of arithmetic sequences

- Let s be the summation of an arithmetic sequence $a_1, a_2, \dots, a_n$;

$$s = \frac{n \times (a_1 + a_n)}{2}$$

Pf: $s = a_1 + a_2 + \cdots + a_n$ ...... (1)

$s = a_n + a_{n-1} + \cdots + a_1$ ...... (2)

(1) + (2) → $2 \times s = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots + (a_n + a_1)$

$$s = \frac{a_1 + a_n}{2} + \frac{a_2 + a_{n-1}}{2} + \cdots + \frac{a_n + a_1}{2}$$

$$\because a_1 + a_n = a_2 + a_{n-1} = \cdots = a_n + a_1 = 2 \times a_1 + n \times d$$

$$\therefore s = \frac{a_1 + a_n}{2} + \frac{a_1 + a_n}{2} + \cdots + \frac{a_1 + a_n}{2} = \frac{n \times (a_1 + a_n)}{2}$$

Geometric sequence

- Definition
  - $Given\ a\ sequence\ of\ numbers\ a_1, a_2, \ldots, a_n, if$
    $$a_i = a_1 \times q^{i-1} \qquad (i = 1,2,3,\ldots,n)$$
    $satisfies, a_i\ is\ called\ a\ geometric\ sequence, q\ is\ called\ the\ common\ ratio$
- Example
  - $a_1 = 1\ a_2 = 2, a_3 = 4, a_4 = 8, a_5 = 16$ is a geometric sequence
    - Because $a_i = 1 \times 2^{i-1}$ satisfies for $i = 1, 2,3,4,5$
  - The common ratio is 2

Summation of geometric sequences

- Let s be the summation of a geometric sequence $a_1, a_2, \dots, a_n$ with common ratio q

$$s = \frac{a_1 - a_n \times q}{1 - q} = \frac{a_1 \times (1 - q^n)}{1 - q}$$

Pf: $s = a_1 + a_2 + a_3 + \cdots + a_n$ .... (1)

$q \times s = a_1 \times q + a_2 \times q + \cdots + a_n \times q = a_2 + a_3 + \cdots + a_n + a_n \times q$... (2)

(1) - (2) → $(1 - q) \times s = a_1 - a_n \times q$

$$\therefore s = \frac{a_1 - a_n \times q}{1 - q} = \frac{a_1 \times (1 - q^n)}{1 - q}$$

Examples with series

```
int temp = 4;
int sum = 0;
for(int i = 1; i < n; i++)
{
    for(int j = 0; j < temp; j++)
        sum++;
    temp *= 4;
}
```

Amortization

- Sometimes the cost of a function is not constant, and varies over time
- May run in O(n) some times, O(1) other times
- Amortization is the “in the long run” average cost per call
- Sum of costs of *k* calls divided by *k*
- More detail if time, otherwise discuss tomorrow