RHLE: Automatic Verification of $\forall \exists$-Hyperproperties

Rob Dickerson  
Purdue University  
rc.dickerson@gmail.com

Qianchuan Ye  
Purdue University  
ye202@purdue.edu

Benjamin Delaware  
Purdue University  
bendy@purdue.edu

Abstract
Specifications of program behavior typically consider single executions of a program, usually requiring that every execution never reaches a bad state (a safety property) or that every execution can eventually produce some good state (a liveness property). Many desirable behaviors, however, including refinement and non-interference, range over multiple executions of a program. These sorts of behaviors are instead expressible as a combination of $k$-safety and $k$-liveness hyperproperties. Relational program logics allow for reasoning about the validity of hyperproperties, but, just as Floyd-Hoare logics focus on axiomatic reasoning about safety, existing relational logics have focused on proving $k$-safety properties. Such relational logics are unfortunately not suitable for verifying more general combinations of $k$-safety and $k$-liveness hyperproperties.

This paper presents RHLE, a relational program logic for reasoning about a class of such hyperproperties that we term $\forall \exists$-hyperproperties. RHLE forms the basis for an algorithm capable of automatically verifying this class of hyperproperties. We present an implementation of this algorithm which we have used to automatically verify a number of $\forall \exists$-hyperproperties, including refinement and non-interference properties, on a corpus of representative programs.

Keywords  relational program logic, $k$-liveness, hyperproperties

1 Introduction
The goal of program verification is to prove that a program is correct with respect to some specification of its intended behavior, e.g. that an implementation of quicksort always returns a sorted list or that a program is free of any divide-by-zero errors. The formal methods community has made great strides in the realm of automatic program verification, to the point that it is now possible to certify the correctness of a wide variety of real-world systems [1, 2, 5]. The vast majority of these techniques have focused on specifications of the individual executions of a program.

Many desirable program behaviors fall outside this class, however, requiring specifications that can reference multiple program executions. Examples of such relational properties, or hyperproperties [13], include observational equivalence of two programs, the non-leaking of sensitive information via program outputs, and the refinement of one program by another. Observational equivalence is an example of a $k$-safety hyperproperty, as it requires $k$ executions to never result a “bad” state. Here, the “bad state” is one in which the two programs output different values after starting in the same state. Program refinement, on the other hand, is an example of a $k$-liveness hyperproperty, as it requires $k$ executions to possibly end in some “good” state. In this case, the “good state” is one where the original program outputs the same value as the refined program.

Two broad categories of analyses have been developed to reason about hyperproperties axiomatically: relational program logics [3, 9, 10, 21], which directly reason about the executions of multiple programs and product program-based approaches [6, 7, 17], which attempt to reduce the problem of reasoning about multiple programs to reasoning about a single program which approximates their behaviors. Section 7 compares our proposed program logic to these existing solutions in more detail, but existing approaches have largely focused exclusively on the verification of $k$-safety properties. One exception, by Barthe et al. [8], develops a set of necessary conditions for a product program to capture hyperproperties beyond $k$-safety, but does not discuss a practical means of constructing such product programs.

Several interesting program behaviors fall outside this class of $k$-safety hyperproperties, however. In particular, this paper considers a class of behaviors of the form: for all executions of programs in set $S_1$, there exists a execution for every program in a set $S_2$ such that the final states of both executions satisfies some property. This paper addresses the gap in reasoning about these $\forall \exists$-hyperproperties by introducing RHLE\(^1\), a relational logic capable of proving these kinds of hyperliveness properties. We have realized this logic in an automated tool, called ORHLE, which is capable of automatically verifying $\forall \exists$-hyperproperties, and we have applied ORHLE to analyze a variety of program behaviors encoded as $\forall \exists$-hyperproperties.

Motivating Example  Consider a program that shuffles (randomly permutes) a list of integers and adds three to each element:

Motivating Example  Consider a program that sorts the list of integers before adding three to each element:

Because Sort will always return a list that Shuffle could have returned, when given the same input list, SortPlusThree’s possible outputs are a subset of ShufflePlusThree’s. When one program’s potential behaviors are a subset of another’s,

\(^1\)Pronounced “really”
we will say that the first program refines the second. In this sense, Sort refines Shuffle and SortPlusThree refines ShufflePlusThree.

Refinement is an interesting property because it tells us that we can replace one program or procedure with another without expanding the set of possible end states. A safety property that applies to a program also applies to its refinement. To be a valid refinement, for all executions of the refined program, there must exist an execution of the original program with the same behavior. The prescription of some desirable execution makes this a liveness property, and the fact that we must ensure the existence of at least one satisfying execution of the original program for all executions of the refinement furthermore makes this a \( \exists A \)-hyperproperty.

1.1 Our Contributions

In summary, this paper makes the following contributions:

- We introduce HLE, a sound program logic for verifying liveness properties (Section 3).
- We extend HLE to the relational setting to build a sound program logic for verifying \( \exists A \)-hyperproperties (Section 4).
- We present ORHLE, a tool based on RHLE capable of automatically verifying these \( \exists A \)-hyperproperties (Section 5).
- We apply ORHLE to analyze a variety of relational program behaviors (Section 6).

The anonymized supplementary material includes the implementation of ORHLE, the complete set of benchmarks from Section 6, and a Coq formalization of the logics described in Sections 2-4.

2 Preliminaries: Syntax and Semantics

We begin by defining FunIMP, a core imperative language with function calls, and then consider a program logic for reasoning about 1-safety properties in this language. Figure 1 presents the syntax of FunIMP, a standard IMP language plus syntax for function calls and definitions. Each function has a fixed arity and returns a single value (although ORHLE relaxes the latter restriction.) The semantics of this language is given as a big-step reduction relation, which is parameterized over two contexts used to evaluate function calls. The first, \( \Sigma \in FunName \rightarrow Assert \times Assert \), is a total mapping from function names to assertions, or predicates on states, representing pre- and post-conditions. The second, \( I \in FunName \rightarrow F \), is a partial mapping from function names to function definitions. The semantics for statements other than function calls is the same as standard presentations of IMP. The evaluation of function calls, shown in Figure 2, depends on whether or not an implementation of \( f \) is available in \( I \). If so, the function is evaluated according to its definition per ECALLMPL. If not, the function can return any value consistent with the postcondition given for
$f$ in $\Sigma$, provided the parameters satisfy the corresponding precondition.

**FunIMP Example** As an example, given the specification context $\Sigma \triangleq \text{RandBnd} \mapsto (\top, 0 \leq \text{ret} \leq x)$, the assignment $y := \text{RandBnd}(20)$ can evaluate to any final state where the value of $y$ is non-negative and less than 20. When a specification’s precondition fails to hold or its postcondition is not satisfied, evaluation will fail to produce a result. For example, evaluating $y := \text{RandBnd}(0)$ with the above specification context gets stuck since there is no $\text{ret}$ such that $0 \leq \text{ret} < 0$.

Modifying the specification context to include a precondition that $x$ be sufficiently large, say $\Sigma \triangleq \text{RandBnd}(x) \mapsto (0 < x, 0 \leq \text{ret} < x)$, will also result in a stuck evaluation of $y := \text{RandBnd}(0)$, but this time because the precondition is not met. Finally, if the function definition context contains $\text{RandBnd}(x)$ as $I \triangleq \text{RandBnd}(x) \mapsto \text{return} \ 5$, then $y := \text{RandBnd}(20)$ will always evaluate to final state where $y = 5$.

By equipping function symbols with appropriate pre- and postconditions, FunIMP programs are allowed to nondeterministically evaluate to multiple final values. This semantics also allows us to precisely define what it means to link in “valid” implementations of function specifications. We say that a function definition implements a specification if, whenever its body is run in a starting state satisfying the precondition, every final state it produces satisfies the postcondition:

$$\Sigma, I, \text{def } f(\overline{x}) \{ s; \text{return } e \} \models (P, Q) \equiv$$

$$\forall \overline{v}. \ P(\overline{v}) \land \Sigma, I \triangleright \overline{x} \mapsto \overline{v}, s \parallel \sigma' \land \Sigma, I \triangleright \sigma', e \parallel \overline{v}' \rightarrow Q(\overline{v}', \overline{v})$$

We can straightforwardly lift this property to define valid contexts:

$$I \models \Sigma \equiv \forall f \in I. \Sigma, I, I(f) \models \Sigma(f)$$

**Theorem 2.1.** When executed in a safe initial state (i.e. one in which the preconditions of all function calls are satisfied) under a valid context, a program must produce a subset of the behaviors resulting from running the program with an empty implementation context:

$$\forall I. \Sigma \triangleright \sigma \sigma'. \ I \models \Sigma \land \text{Safe } \sigma \ P \land \Sigma, I \triangleright \sigma, p \parallel \sigma' \rightarrow \Sigma, I \triangleright \sigma, p \parallel \sigma'$$

We can straightforwardly construct a sound and relatively complete program logic for FunIMP by augmenting the standard Hoare logic for IMP with the following rule for function calls:\footnote{For convenience, we define the $\text{Pre}_f^x$ and $\text{Proj}_f^x$ projection functions for retrieving the pre- and post-conditions for a function symbol $f$ in the context $\Sigma$.}

$$\Sigma \triangleright \left\{ \text{Pre}_f^x(\overline{x}) \land \forall v. \text{Post}_f^x(v, \overline{x}) \right\} \ y \ := \ f(\overline{x}) \{ \Psi \}$$

The HSpec rule (and thus the logic itself) reasons about function calls strictly via their specifications in $\Sigma$. When combined with Theorem 2.1, however, the logic admits a modular reasoning principle for FunIMP programs:

**Theorem 2.2.** To show that a program $p$ behaves correctly in an environment of function definitions $I$ and specifications $\Sigma$, it suffices to prove that $p$ is functionally correct with respect to $\Sigma$ and that the functions in $I$ meet their specifications:

$$\forall I. \Sigma \ P Q. \ I \models \Sigma \land \Sigma \triangleright \{ P \} p \{ Q \} \land (\forall \sigma. \ P(\sigma) \rightarrow \text{Safe } \sigma \ p) \rightarrow$$

$$\forall \sigma \sigma'. \ P(\sigma) \land \Sigma, I \triangleright \sigma, p \parallel \sigma' \rightarrow Q(\sigma)$$

A direct corollary of this theorem is that function implementations can also be verified using our program logic.

### 3 Existential Reasoning

Before progressing to the relational setting, we begin with a discussion of what a “1-liveness” property looks like. Consider the following program:

```plaintext
hand := card1 + card2;
while hand < 21 do
    card3 := drawCard();
    hand := hand + card3
end
```

This simple program plays a round of blackjack by examining the two cards it is dealt and then continuously drawing cards until it either hits 21 or busts. This rather aggressive strategy is not safe in general, as the player is guaranteed to go bust some of the time. Particularly cavalier players may not care about that however, as long as this strategy is a non-losing one, i.e. it is guaranteed to work at least some of the time, regardless of the hand they were dealt.

We can formally encode this behavior via the following semantic existential Hoare triple:

$$\Sigma \cdot \models [1 \leq \text{card1} \leq 11 \land 1 \leq \text{card2} \leq 10] \ p_1 \ [\text{hand} = 21]$$

where drawCard() is specified to return a card at random:

$$\Sigma \ (\text{drawCard}) = (\top, 1 \leq \text{ret} \leq 11)$$

In other words, for any initial state containing valid values for cards card1 and card2, $p_1$ could yield a winning hand.

More formally, our semantic existential Hoare triples are defined as:

\[ \Sigma, I \models [P] \sigma \models [Q] \equiv \forall \sigma'. \Sigma, I \vdash \sigma, p \models \sigma' \wedge Q(\sigma') \]

These triples encode liveness properties, in that they guarantee a program could enter some desirable final state. Note that this definition forces the existence of a terminating execution satisfying the postcondition. Standard (universal) Hoare triples bound the results of all possible executions, so their semantics also covers non-terminating runs. Our existential triples ensure the existence of a single execution however, so allowing nonterminating executions to satisfy the triple would considerably weaken its guarantees.

**Existential Hoare Logic** We now turn to the question of how to prove the validity of an existential Hoare triple. Given that Section 2 presented a sound and relatively complete logic for universal Hoare triples, it is reasonable to ask if existential triples can be encoded as equivalent universal triples. Consider the following variation on our previous blackjack player, which only draws a single card:

```plaintext
if card1 + card2 < 21 then
    card3 := drawCard();
else card3 := 0;
hand := card1 + card2 + card3;
```

Showing that p₁ can avoid going bust is equivalent to showing that the following universal Hoare triple is not valid:

\[ \Sigma, \cdot \models \{ 1 \leq \text{card1} \leq 11 \land 1 \leq \text{card2} \leq 10 \} \ p₂ \ \text{\{hand > 21\}} \]

The intuition is that if the final hand does not always exceed 21, there must be some execution where it does not bust. This intuition does not hold in general, however:

\[ \Sigma, \cdot \models [P] c [Q] \iff \neg(\Sigma, \cdot \models \{ P \} c \{ \neg Q \}) \]

To see why, consider the following variation on p₂:

```plaintext
if card1 + card2 < 21 then
    card3 := drawCard();
else card3 := 21;
hand := card1 + card2 + card3;
```

Our universal Hoare triple remains invalid for p₃, but the corresponding existential Hoare triple does not hold:

\[ \neg(\Sigma, \cdot \models [1 \leq \text{card1} \leq 11 \land 1 \leq \text{card2} \leq 10] \ p₃ \ [\text{hand} \geq 21]) \]

The intuition is that the universal Hoare triple is invalid as long as there is a single initial state which produces an final state that invalidates the postcondition. The postcondition could hold for initial states which induce a different control flow path, e.g. states leading to the else branch of p₃.

We have therefore developed a program logic for reasoning about liveness properties. This logic, called HLE, is presented in Figure 3. The proof rules are the same as those for the previous universal logic, with the exception of the rule for function calls, \( \exists \text{Spec} \). To understand this rule, we first consider how the existential behaviors of a function is specified.

To ground this discussion, consider the universal specification for drawCard from Equation 1: \((T, 1 \leq \text{ret} \leq 11)\). There are two immediate interpretations for this specification existentially: a valid implementation of drawCard must have the potential to return every integer meeting the postcondition, or a valid implementation of drawCard must be able to return some integer which meets the postcondition. (Note that neither interpretation gives an exhaustive characterization of the return values of a valid implementation; implementations are free to return other values as well.) Both interpretations have their advantages: the second admits strictly more implementations, while the first provides more guarantees about the set of values an implementation can produce.

HLE attempts to split the difference by extending existential function specifications to include an additional set of arguments, which we call template parameters, which the pre- and postconditions can reference. The underlying intuition is that each instantiation of these parameters corresponds to a single required existential behavior. HLE uses a separate context of existential specifications in
A valid implementation of this specification must be able to return every value less than the argument \( x \). Since our existential specifications permit more behaviors than those that meet the postcondition, even though this implementation could output a value outside the specified range, it could also output the in-bounds value \( 0 \). As long as at least one possible output meets the postcondition, the function is considered to meet its existential specification.

However, we may want to specify that an implementation of randBnd must be able to return every number in the range. To do so, we can utilize a template parameter in the existential specification like so:

\[
\Sigma_\exists \ (\text{randBnd}) = (t, 0 < x \wedge 0 \leq t < x, \text{ret} = t)
\]  

Intuitively, this specification says randBnd must be able to return every \( t \) less than the argument \( x \). Observe that this rules out all the implementations given above; they all have values in this range that can never be returned. Assuming a function rand capable of returning every integer, one possible valid implementation is:

```python
def randBnd(x):
  r := rand();
```
meeting the precondition:

\[ \Sigma, I, \text{ def } f(x); \text{return } e \models_\Sigma (\overline{x}, P, Q) \equiv \forall \overline{x} \overline{\sigma}, P(\overline{x}, \overline{\sigma}) \rightarrow \exists \sigma', \Sigma, I \vdash \overline{x} \mapsto \overline{\sigma}', e \downarrow \sigma' \]

\[ \land \Sigma, I \vdash \sigma', e \downarrow \sigma' \land Q(\sigma', \overline{x}, \overline{\sigma}) \]

The key insight behind how \( \exists \text{Spec} \) uses the existential context is that it allows a proof derivation to freely choose any instantiation of template parameters, \( \overline{x} \), that verify the program. To demonstrate how this works in practice, consider the derivation tree, shown in Figure 4, of the existential triple about the blackjack player \( p_2 \):

\[ \Sigma_3 \vdash [1 \leq \text{card}1 \leq 11 \land 1 \leq \text{card}2 \leq 10] \ p_2 \ [h \leq 21] \_3 \]

The application of \( \exists \text{Spec} \) on the top is particularly noteworthy. By applying \( \exists \text{Spec} \) with the template parameter \( t = 1 \), it is straightforward to verify that the implication in the subsequent application \( \exists \text{Conseq} \) rule holds.

HLE admits a modular reasoning theorem similar to Theorem 2.2:

**Theorem 3.1.** To show that a program \( p \) can produce a desired state from a given set of initial states when run in an environment of function definitions \( I \) and specifications \( \Sigma \), it suffices to prove that \( p \) can result in such a state in existential context \( \Sigma_3 \), as long \( I \) is consistent with \( \Sigma_3 \), and \( \Sigma \) and \( \Sigma_3 \) are consistent:

\[ \forall \Sigma, \Sigma_3 \ p \ P \ Q. I \models_3 \Sigma_3 \]

\[ \land \Sigma_3 \vdash [P] p \ [Q]_3 \]

\[ \land \text{Consistent} \Sigma, \Sigma_3 \rightarrow \Sigma, I \vdash [P] \ e \ c \ [Q]_3 \]

In order to connect the existential context used in HLE with the universal context used in the semantics of FunIMP, this theorem requires that they are consistent, i.e., every behavior required by \( \Sigma_3 \) meets the corresponding safety condition in \( \Sigma \):

**Consistent** \( \Sigma, \Sigma_3 \equiv \forall f \ P \ Q \overline{\sigma} P_3 \ Q_3. \)

\[ \Sigma(f) = (P, Q) \land \Sigma_3(f) = (\overline{\sigma}, P_3, Q_3) \rightarrow \]

\[ (\forall \overline{x} \overline{\sigma}, P_3(\overline{x}, \overline{\sigma}) \rightarrow P(\overline{\sigma})) \land \forall \overline{x} \overline{\sigma}, Q_3(\overline{x}, \overline{\sigma}) \rightarrow Q(\overline{\sigma}, \overline{\sigma}) \]

### 4 RHLE: A program logic for \( \forall \exists \)-hyperproperties

When \( k = 1, k \)-liveness properties are for the most part an intellectual curiosity. They become much more useful in the relational setting (where \( k > 2 \)) or when considered in the context of universally quantified program executions. As a simple example of the latter, we might want to verify that

our blackjack player \( p_1 \) has a winning strategy when playing against \( p_2 \). This can be expressed as the following (semantic) relational Hoare triple:

\[ \Sigma, \Sigma_3, I \models [f] P \vdash_\Sigma P_2 \ \land [f] \ p_1 \ (\exists \text{hand}_2 > 21) \land [f] \ p_2 \ (\exists \text{hand}_2 < 21) \land [f] \ p_2 \ \land [f] \ p_1 \ \land [f] \ p_2 \ (\exists \text{hand}_1 \leq 21) \]

Following Sousa and Dillig [21], the pre- and post-conditions of this triple can reference specific variables via a numeric index: e.g., \( \text{hand}_2 \) in the postcondition refers to the value of hand in the second program (\( p_2 \) in this case). Executions of every program to the left of \( \sim_3 \) are universally quantified, while those to the right are existentially quantified. Formally, the definition of this triple is:

\[ \Sigma, \Sigma_3, I \models [f] P \vdash_\Sigma P_2 \ \land [f] \ p_1 \ (\exists \text{hand}_2 > 21) \land [f] \ p_2 \ (\exists \text{hand}_2 < 21) \land [f] \ p_2 \ (\exists \text{hand}_1 \leq 21) \]

Figure 5 gives examples of other interesting hyperproperties encoded using these relational Hoare triples, including a generalization of our winning strategy property and the refinement property discussed in Section 1. We also give an encoding of a Generalized Non-Interference property based on [18] which says that given any two program executions with potentially different High security inputs, there exists a third execution whose High security inputs are the same as the first program, but whose Low security outputs are the same as the second. (This guarantees that no unprivileged output is ever serving as a way to distinguish between two sets of privileged state.) Lastly, we encode a property which says every input parameter to a program is meaningful in the sense that the program can always reach at least two different end states when all other parameters are fixed.

RHLE is a relational program logic for reasoning about the sorts of hyperproperties encoded by these semantic relational Hoare triples\(^3\). Figure 6 gives a complete listing of the proof rules of the logic. The first three rules are used to end derivations: the \texttt{SkpIntro} and \texttt{SkpIntro}\_3 rules are used to append a final skip statement to the end of universal and existential programs, respectively. These final \texttt{skips} act as flags that allow the \texttt{Skp} axiom to be applied to end a derivation after the proof has fully considered the behavior of each program. The \texttt{Step} and \texttt{Step}_3 rules both mimic the standard sequence rule, first leveraging the logic for their respective (single) program logics to prove the topmost command of one of the programs meets some postcondition \( \Psi' \), which is then used as the precondition for further relational reasoning.

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\(^3\)To avoid clashing with FunIMP’s sequence operator, the rules use \( \oplus \) to separate programs on either side of \( \sim_3 \).
Theorem 4.1. This section presents an algorithm based on RHLE for automatically verifying \( \forall \exists \)-hyperproperties. This algorithm is parameterized over the underlying program logic as well as a context of universal and existential function specifications. The general approach is to generate verification conditions by reasoning backward from the relational postcondition by applying \( \text{Step}_\forall \) and \( \text{Step}_\exists \) until all program statements have been processed. A final formula is then constructed which ensures that the accumulated verification conditions are implied by the initial relational precondition. This formula can then be checked using a solver for the underlying assertion logic.

The procedure for generating verification conditions has two components; \( \text{StmtVC} \), presented in Algorithm 3, for reasoning about non-relational Hoare triples and \( \text{RhleVC} \), presented in Algorithm 4, for stepping through relational triples. Note that \( \text{StmtVC} \) handles both universal and existential Hoare triples by taking a quant parameter that indicates whether the universal or existential logic should be used. (This parameter is only used for function calls and loops.) \( \text{RhleVC} \) operates by applying the \( \text{Step} \) rules to single out program statements that can be given to \( \text{StmtVC} \), which will generate the appropriate verification conditions for the chosen statement. The recursion terminates when all statements have been considered, at which point the output formula can be discharged by a solver for the underlying theory.

For a \( \forall \exists \)-hyperproperty over \( k \) programs, each application of \( \text{RhleVC} \) can choose \( k \) possible targets for \( \text{Step} \). Different selection strategies can affect the performance of the verification algorithm, with sophisticated strategies potentially being able to discard irrelevant subprograms altogether. Rather than hardcode this strategy, \( \text{RhleVC} \) is parameterized over a \( \text{ChooseStep} \) function. This function accepts a collection of quantified programs and returns the next statement to step over, which quantifier group the statement came from, and the collection of programs in their post-step form. It is important to note that, because verification conditions for existential executions may include existential quantifiers, in the worst case it is necessary to completely step through all the existential programs before targeting universal programs. This ensures existential quantifiers occur inside the scope of the quantifiers from the universal executions, preserving the “forall-exists” semantics of RHLE triples. This, however, does not preclude \( \text{ChooseStep} \) from selecting a universal statement before an existential one in all cases, e.g. when the existential statement is deterministic.

5 Verification

This section presents an algorithm based on RHLE for automatically verifying \( \forall \exists \)-hyperproperties. This algorithm is parameterized over the underlying program logic as well as a context of universal and existential function specifications. The general approach is to generate verification conditions by reasoning backward from the relational postcondition by applying \( \text{Step}_\forall \) and \( \text{Step}_\exists \) until all program statements have been processed. A final formula is then constructed which
Algorithm 3: StmtVC(quant, stmt, ψ)

**Inputs**: quant, quantifier in which stmt appears
stmt, a FunNMP program statement
ψ, a relational postcondition

**Output**: verification condition for stmt to meet ψ in the given quant context

begin

match stmt:

  case skip do
    return ψ
  case x := a do
    return ψ[x → a]
  case y := f(x) do
    if quant = ∀ then
      ψpre ← Pre_f(ψ)
      ψpost ← Post_f(y, ψ) ⇒ ψ
    return ψpre ∧ ψpost
    else
      ψpre ← Pre_f(ψ)
      ψpost ← Post_f(y, ψ)
      ψend ← ψpre ∧ ψpost
      return ψpre ∧ ψend ∧ ψend
  case s₁; s₂ do
    ψ ← StmtVC(quant, s₁, ψ)
    return StmtVC(quant, s₂, ψ)
  case if c then s₁ else s₂ do
    ψ₁ ← StmtVC(quant, s₁, ψ)
    ψ₁ ← StmtVC(quant, s₂, ψ)
    return c ⇒ ψ₁ ∧ ¬c ⇒ ψ₂
  case while c do s end, I, M do
    v ← FreeVars(stmt)
    v’ ← Fresh(v)
    if quant = ∀ then
      ψbody ← StmtVC(quant, s, I)
    else
      ψmeas ← M < M[v → v’]
      ψbody ← StmtVC(quant, s, I ∧ ψmeas)
      ψloop ← ∀v’.c ∧ I ⇒ ψbody[v → v’]
      ψend ← ∀v.¬c ∧ I ⇒ ψ)
    return I ∧ ψloop ∧ ψend

end

To see how this algorithm works in practice, consider the variation of the refinement example from Section 1 presented in Figure 7. To formally verify that SortPlusThree is a refinement of ShufflePlusThree, we need to check the following ∀∃-hyperproperty:

Σ, ∃₃ ⊢ (list_sort = list_shuf)
SortPlusThree ∼₃ ShufflePlusThree
〈return_sort = return_shuf〉

where list_sort and list_shuf are the inputs to SortPlusThree and ShufflePlusThree respectively, and return_sort and return_shuf are the respective outputs.

The specification for Sort requires the output to be a permutation in ascending order:

Pre_sort(l_in, l_out) → τ
Post_sort(l_in, l_out) → permutation(l_in, l_out)
∧ ∀i ∈ {1...size(l_in)}.l_out[i − 1] ≤ l_out[i]

Algorithm 4: RhleVC

**Inputs**: Φ, a relational precondition
χ₁, universally quantified programs
χ₂, existentially quantified programs
Ψ, a relational postcondition

**Output**: verification condition for (Φ) χ₁ ∼₃ χ₂ (Ψ)

begin
  if empty(χ₁) ∧ empty(χ₂) then
    return Φ ⇒ Ψ
  else
    (quant, stmt, χ₁, χ₂) ← ChooseStep(χ₁, χ₂)
    ψ’ ← StmtVC(quant, stmt, ψ)
  return RhleVC(Φ, χ₁, χ₂, ψ’)

end
while the specification for Shuffle simply requires the output to be any permutation:

\[ \forall l_t. \text{Pre}_{\text{shuffle}}(l_{in}, l_{out}) \leftrightarrow \text{permutation}(l_{in}, l_t) \]
\[ \text{Post}_{\text{shuffle}}(l_{in}, l_{out}) \leftrightarrow l_{out} = l_t \]

Note that since Shuffle appears in an existentially quantified program, its specification is given in terms of a universally quantified template variable \( l_t \). This specification should be read as, “for any list \( l_t \), if \( l_t \) is a permutation of \( l_{in} \), then the output \( l_{out} \) of Shuffle can be \( l_t \).” A goal of the verification algorithm will be to ensure the existence of at least one such permutation \( l_t \) that meets whatever other constraints are imposed on it by the rest of the program analysis.

If we implement ChooseStep as a simple sequential step through the existing program, RHLeVC will proceed by first stepping backward through ShufflePlusThree. This will dispatch two sets of calls to StmtVC with \( quant = \exists \); one for the addition statements and one for the call to Shuffle. The full verification condition is omitted here for brevity, but note that the verification condition at this point is of the form \( \exists l_t \ldots \), indicating verification will succeed only when the template variable \( l_t \) can be suitably instantiated. RHLeVC then proceeds to step back through SortPlusThree, again triggering two sets of calls to StmtVC to handle the additions followed by the function call. After these steps, no program statements are left, and we are ready to predicate the resulting verification condition on the overall precondition:

\[ \text{list}_{\text{sort}} = \text{list}_{\text{shuf}}. \] This gives the final verification condition the form \( \text{list}_{\text{sort}} = \text{list}_{\text{shuf}} \implies \ldots \implies (\exists l_t, \ldots) \). With free variables implicitly universally quantified, this final verification condition has the expected \( \forall \ldots \exists \ldots \) form, and is ready to be validated by a solver.

6 Implementation and Evaluation

ORHLE is an automated verifier for \( \exists \)-hyperproperty based on Algorithm 4. It is implemented in Haskell and uses Z3 as its backend solver. ORHLE specifications are written in SMT-LIB2, and the tool operates over FunMP programs. Input programs must be annotated with loop invariants and function specifications. When a set of programs fails to verify, ORHLE outputs a falsifying model for the verification conditions. ORHLE implementation for ChooseStep currently steps through each existential execution in order followed by each universal execution, a strategy that has proved sufficient for our benchmark programs.

We evaluated ORHLE on a set of representative programs that exhibit or fail to exhibit certain \( \exists \)-hyperproperties drawn from Figure 5. (See Figure 8 for an example ORHLE input listing.) Our experimental results are summarized in Figure 9. Timings were taken using an 8th Generation Intel Core i7-8750H CPU with 6 2.2GHz cores. The examples included a mix of programs that were expected to verify or fail to verify, and ORHLE returned the expected result in all cases. Verification across this example set was reasonably quick, with most verification tasks completing in under 300ms.

7 Related Work

This section discusses related work.

Hyperproperties The concept of a hyperproperty was originally introduced by Clarkson and Schneider [13], building off of earlier work by Terauchi and Aiken [22]. Clarkson and Schneider define hyperproperties as finite properties over a
of CHL for reasoning about a larger class of hyperproperties via an existentially quantified Hoare logic.

**Product Programs** Product programs are another common technique for verification of relational properties. The basic technique is to transform multiple programs with a relational property into an equivalent single program with a non-relational property[7]. Standard verification tools may then be used to verify the resulting product program. The advantage to this approach is it can leverage existing non-relational verification tools and techniques, but large state space of product programs makes verification difficult in practice. Product programs have been used to verify k-safety properties and reason about non-interference and secure information flow[6, 17]. The most closely related work in product programs was that of Barthe et al. [8], which develops a set of necessary conditions for “left-product programs”. These product programs can be used to verify hyperproperties outside of k-safety, including the V3-hyperproperties RHLE deals with, but the work does not address how to construct left-product programs.

**Temporal Logics** Temporal logics like LTL and CTL are popular systems for verifying liveness properties, and variants like HyperLTL and HyperCTL can be used to verify liveness hyperproperties. (See [11, 12] for surveys.) A recent paper by Coenen et al.[14] examines verification and synthesis of computational models using HyperLTL formulas with alternating quantifiers, leading to analysis of the kinds of V3-hyperproperties considered here. While these temporal logics are capable of reasoning about these hyperliveness properties, they operate over finite state models of software systems instead of reasoning directly over program syntax in the way RHLE does.

**Dynamic Logic** First-order dynamic logic[19] is a reinterpretation of Hoare logic in first order, multi-modal logic. Dynamic logic programs are constructed by regular expression operators extended by other operators like testing and assignment. The more conventional programming constructs, such as the ones presented in this work, can be encoded in terms of the regular operators. For each program S, there are modal operators [S] and ∃S. A modal formula [S]P means that after executing S, all end states satisfy P, and the dual ∃S P means that after executing S, there exists an end state satisfying P. The axiomatic proof rules for dynamic logic closely correspond to the weakest precondition transformer, in that [S]P ↔ wlp(S, P), where wlp is the weakest liberal precondition, and (∃S)P ↔ wp₂(S, P), where wp₂ is the weakest precondition for existential Hoare logic. Therefore, the universal Hoare triple {P}S[Q] corresponds to P → [S]Q, and the existential Hoare triple {P}S[Q] → corresponds to P → (∃S)Q. In contrast to dynamic logic, we present a more conventional imperative language, and allow function calls, potentially without implementations.
The verification algorithm presented in this work also allow the users to specify loop invariants, making the system more practical. In addition, we focus on the verification of hyperproperties.

8 Future Work and Conclusion

Efficiency Improvements Currently ORHLE uses a simple implementation of the ChooseStep subroutine Algorithm 4 which steps sequentially through all existential programs before their universal counterparts. This approach builds a formula where each an existential trace needs to be exhibited for each universal trace, resulting in a potentially expensive call to Z3. Although this was not an issue in our experimental evaluation, we believe that a smarter selection strategy could result in a more efficient (and thus scalable) verification algorithm. More sophisticated stepping may also be able to elide verification of irrelevant program segments altogether using strategies similar to those employed in CHL[21]. Opportunities to prune existentially quantified subprograms could be particularly fruitful, as the verifier only needs to establish the existence of one satisfying execution. Once a satisfying execution is established, the behavior of different execution paths becomes irrelevant and do not need to be considered.

Existential Loop Invariant Inference ORHLE requires the user to specify all loop invariants. There is a wide variety of existing work on loop invariant inference (see section 5 of [16] for a survey) that ORHLE should be able to apply to its universally quantified programs, but to our knowledge there is no work on inferring loop invariants in an existentially quantified context. We believe this is an interesting problem to investigate: since only one suitable execution path in the loop body needs to preserve the invariant, inferring a valid invariant for an existentially quantified loop may be substantially easier than doing the same in a universally quantified context.

Existential Specification Inference ORHLE requires the user to give full specifications for each function call, but for some applications it may be useful to require only universal functional specifications from users and have the verification tool report the most permissive satisfying existential specification. If ORHLE were capable of proving that, e.g., any function implementation that can return 2 and 6 results in the desired program behavior, it should become much easier to prove function implementations meet their existential specifications. Albarghouthi et al.[4] have described a technique for abducting maximal function specifications that may be applicable here; by replacing existential functions with abducible, the verification conditions generated by ORHLE could form the basis of a useful multiabduction query.

In summary, this paper described RHLE, a verification algorithm for reasoning axiomatically about $\forall \exists$-hyperproperties, as well as an algorithm for automatically verifying this class of hyperproperties. This verification algorithm has been used to build ORHLE, a verification tool that operates over an arbitrary number of $\forall \exists$-quantified program executions. We have used ORHLE to successfully verify or falsify a variety of representative $\forall \exists$-hyperproperties.

References


