Dense Matrix Algorithms

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Topic Overview

- Matrix-Vector Multiplication
- Matrix-Matrix Multiplication
- Solving a System of Linear Equations

Matix Algorithms: Introduction

- Due to their regular structure, parallel computations involving matrices and vectors readily lend themselves to data-decomposition.

- Typical algorithms rely on input, output, or intermediate data decomposition.

- Most algorithms use one- and two-dimensional block, cyclic, and block-cyclic partitionings.

Matrix-Vector Multiplication

- We aim to multiply a dense $n \times n$ matrix A with an $n \times 1$ vector x to yield the $n \times 1$ result vector y.

- The serial algorithm requires n^2 multiplications and additions.

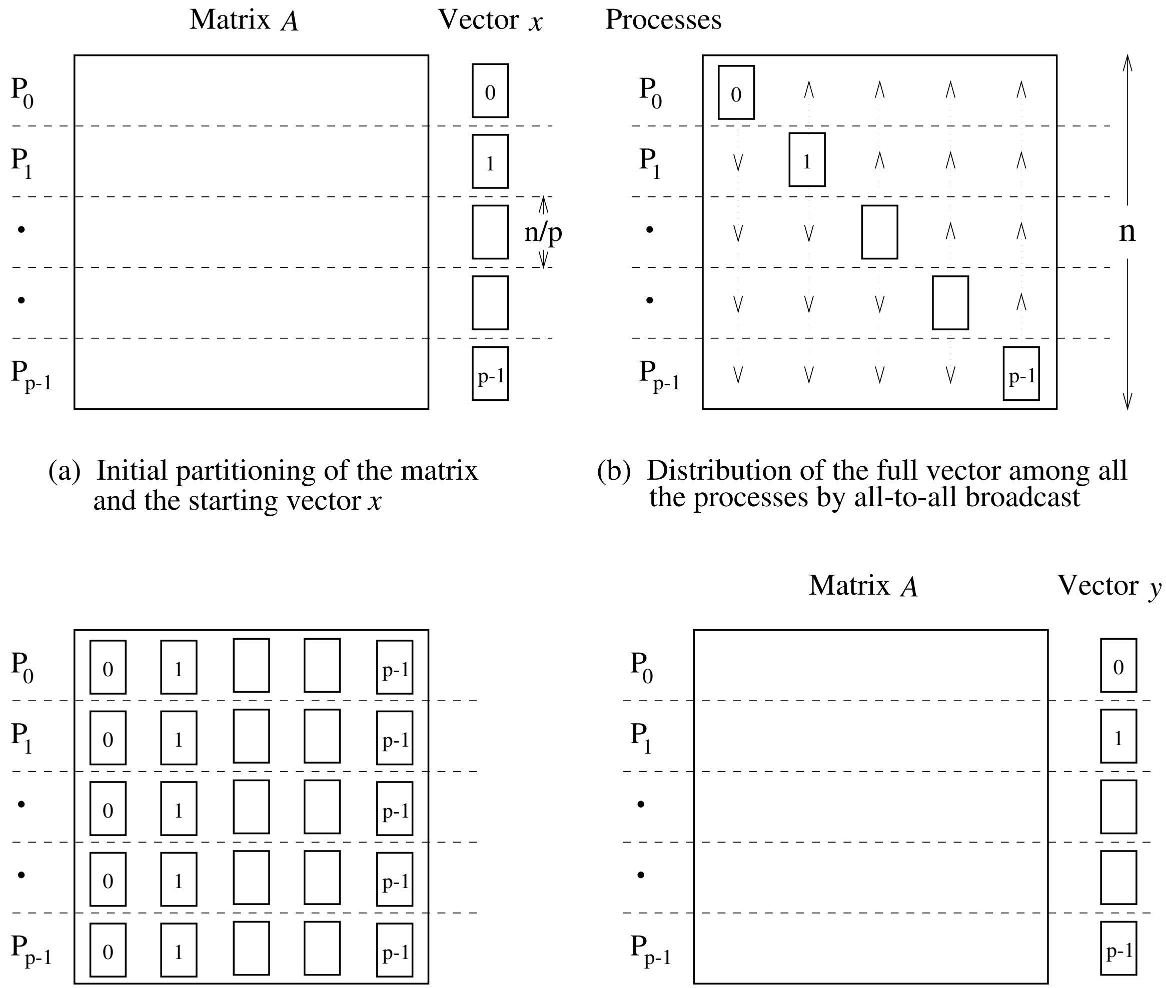
$$W = n^2. \tag{1}$$

Matrix-Vector Multiplication: Rowwise 1-D Partitioning

- The $n \times n$ matrix is partitioned among n processors, with each processor storing complete row of the matrix.

- The $n \times 1$ vector x is distributed such that each process owns one of its elements.

Matrix-Vector Multiplication: Rowwise 1-D Partitioning



(a) Initial partitioning of the matrix and the starting vector x

(b) Distribution of the full vector among all the processes by all-to-all broadcast

(c) Entire vector distributed to each process after the broadcast

(d) Final distribution of the matrix and the result vector y

Multiplication of an $n \times n$ matrix with an $n \times 1$ vector using rowwise block 1-D partitioning. For the one-row-per-process case, $p = n$.

Matrix-Vector Multiplication: Rowwise 1-D Partitioning

- Since each process starts with only one element of x, an all-to-all broadcast is required to distribute all the elements to all the processes.

- Process P_i now computes $y[i] = \Sigma_{j=0}^{n-1}(A[i,j] \times x[j])$.

- The all-to-all broadcast and the computation of $y[i]$ both take time $\Theta(n)$. Therefore, the parallel time is $\Theta(n)$.

Matrix-Vector Multiplication: Rowwise 1-D Partitioning

- Consider now the case when $p < n$ and we use block 1D partitioning.
- Each process initially stores n/p complete rows of the matrix and a portion of the vector of size n/p.
- The all-to-all broadcast takes place among p processes and involves messages of size n/p.
- This is followed by n/p local dot products.
- Thus, the parallel run time of this procedure is

$$T_P = \frac{n^2}{p} + t_s \log p + t_w n. \tag{2}$$

This is cost-optimal.

Matrix-Vector Multiplication: Rowwise 1-D Partitioning

Scalability Analysis:

- We know that $T_o = pT_P - W$, therefore, we have,

$$T_o = t_s p \log p + t_w np. \quad (3)$$

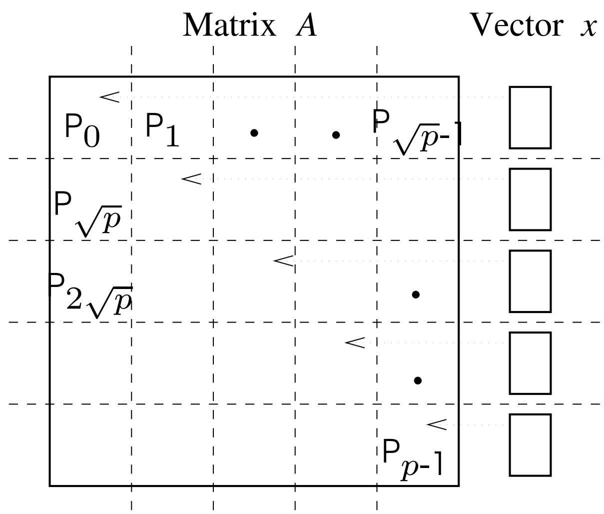
- For isoefficiency, we have $W = KT_o$, where $K = E/(1-E)$ for desired efficiency E.
- From this, we have $W = O(p^2)$ (from the t_w term).
- There is also a bound on isoefficiency because of concurrency. In this case, $p < n$, therefore, $W = n^2 = \Omega(p^2)$.
- Overall isoefficiency is $W = O(p^2)$.

Matrix-Vector Multiplication: 2-D Partitioning

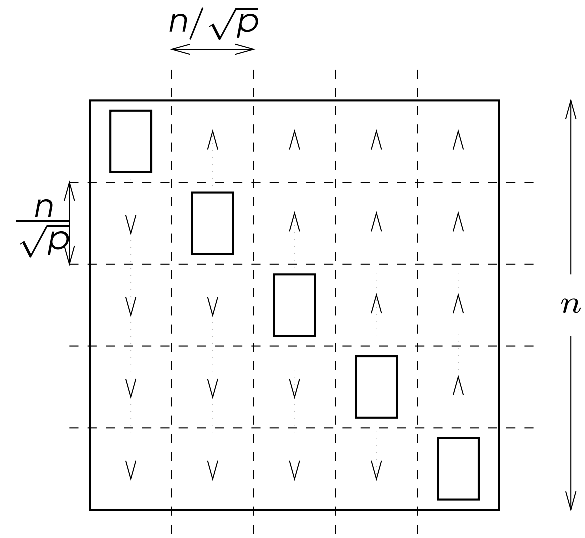
- The $n \times n$ matrix is partitioned among n^2 processors such that each processor owns a single element.

- The $n \times 1$ vector x is distributed only in the last column of n processors.

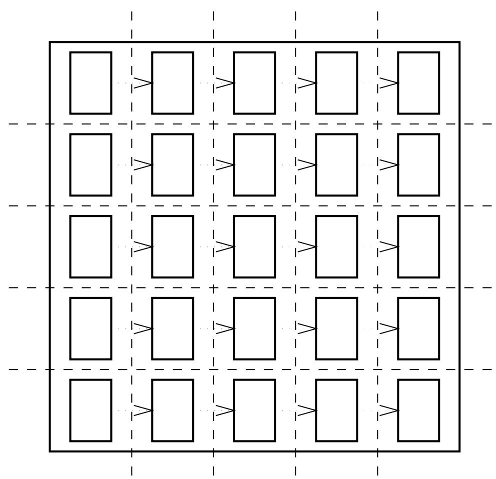
Matrix-Vector Multiplication: 2-D Partitioning



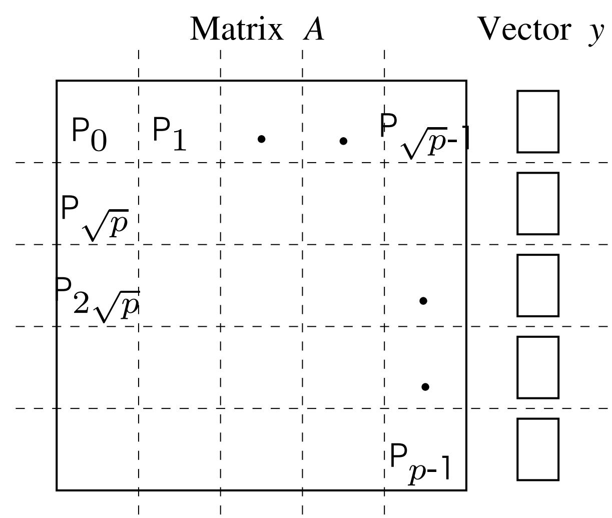
(a) Initial data distribution and communication steps to align the vector along the diagonal



(b) One-to-all broadcast of portions of the vector along process columns



(c) All-to-one reduction of partial results



(d) Final distribution of the result vector

Matrix-vector multiplication with block 2-D partitioning. For the one-element-per-process case, $p = n^2$ if the matrix size is $n \times n$.

Matrix-Vector Multiplication: 2-D Partitioning

- We must first aling the vector with the matrix appropriately.

- The first communication step for the 2-D partitioning aligns the vector x along the principal diagonal of the matrix.

- The second step copies the vector elements from each diagonal process to all the processes in the corresponding column using n simultaneous broadcasts among all processors in the column.

- Finally, the result vector is computed by performing an all-to-one reduction along the columns.

Matrix-Vector Multiplication: 2-D Partitioning

- Three basic communication operations are used in this algorithm: one-to-one communication to align the vector along the main diagonal, one-to-all broadcast of each vector element among the n processes of each column, and all-to-one reduction in each row.

- Each of these operations takes $\Theta(\log n)$ time and the parallel time is $\Theta(\log n)$.

- The cost (process-time product) is $\Theta(n^2 \log n)$; hence, the algorithm is not cost-optimal.

Matrix-Vector Multiplication: 2-D Partitioning

- When using fewer than n^2 processors, each process owns an $(n/\sqrt{p}) \times (n/\sqrt{p})$ block of the matrix.

- The vector is distributed in portions of $n/\sqrt{p}$ elements in the last process-column only.

- In this case, the message sizes for the alignment, broadcast, and reduction are all $(n/\sqrt{p})$.

- The computation is a product of an $(n/\sqrt{p}) \times (n/\sqrt{p})$ submatrix with a vector of length $(n/\sqrt{p})$.

Matrix-Vector Multiplication: 2-D Partitioning

- The first alignment step takes time $t_s + t_w n/\sqrt{p}$.
- The broadcast and reductions take time $(t_s + t_w n/\sqrt{p}) \log(\sqrt{p})$.
- Local matrix-vector products take time $t_c n^2/p$.
- Total time is

$$T_P \approx \frac{n^2}{p} + t_s \log p + t_w \frac{n}{\sqrt{p}} \log p \qquad (4)$$

Matrix-Vector Multiplication: 2-D Partitioning

Scalability Analysis:

- $T_o = pT_p - W = t_s p \log p + t_w n \sqrt{p} \log p$.
- Equating T_o with W, term by term, for isoefficiency, we have, $W = K^2 t_w^2 p \log^2 p$ as the dominant term.
- The isoefficiency due to concurrency is $O(p)$.
- The overall isoefficiency is $O(p \log^2 p)$ (due to the network bandwidth).
- For cost optimality, we have, $W = n^2 = p \log^2 p$. For this, we have, $p = O\left(\frac{n^2}{\log^2 n}\right)$.

Matrix-Matrix Multiplication

- Consider the problem of multiplying two $n \times n$ dense, square matrices A and B to yield the product matrix $C = A \times B$.
- The serial complexity is $O(n^3)$.
- We do not consider better serial algorithms (Strassen's method), although, these can be used as serial kernels in the parallel algorithms.
- A useful concept in this case is called *block* operations. In this view, an $n \times n$ matrix A can be regarded as a $q \times q$ array of blocks $A_{i,j}$ ($0 \leq i, j < q$) such that each block is an $(n/q) \times (n/q)$ submatrix.
- In this view, we perform q^3 matrix multiplications, each involving $(n/q) \times (n/q)$ matrices.

Matrix-Matrix Multiplication

- Consider two $n \times n$ matrices A and B partitioned into p blocks $A_{i,j}$ and $B_{i,j}$ $(0 \leq i, j < \sqrt{p})$ of size $(n/\sqrt{p}) \times (n/\sqrt{p})$ each.
- Process $P_{i,j}$ initially stores $A_{i,j}$ and $B_{i,j}$ and computes block $C_{i,j}$ of the result matrix.
- Computing submatrix $C_{i,j}$ requires all submatrices $A_{i,k}$ and $B_{k,j}$ for $0 \leq k < \sqrt{p}$.
- All-to-all broadcast blocks of A along rows and B along columns.
- Perform local submatrix multiplication.

Matrix-Matrix Multiplication

- The two broadcasts take time $2(t_s \log(\sqrt{p}) + t_w(n^2/p)(\sqrt{p} - 1))$.
- The computation requires $\sqrt{p}$ multiplications of $(n/\sqrt{p}) \times (n/\sqrt{p})$ sized submatrices.
- The parallel run time is approximately

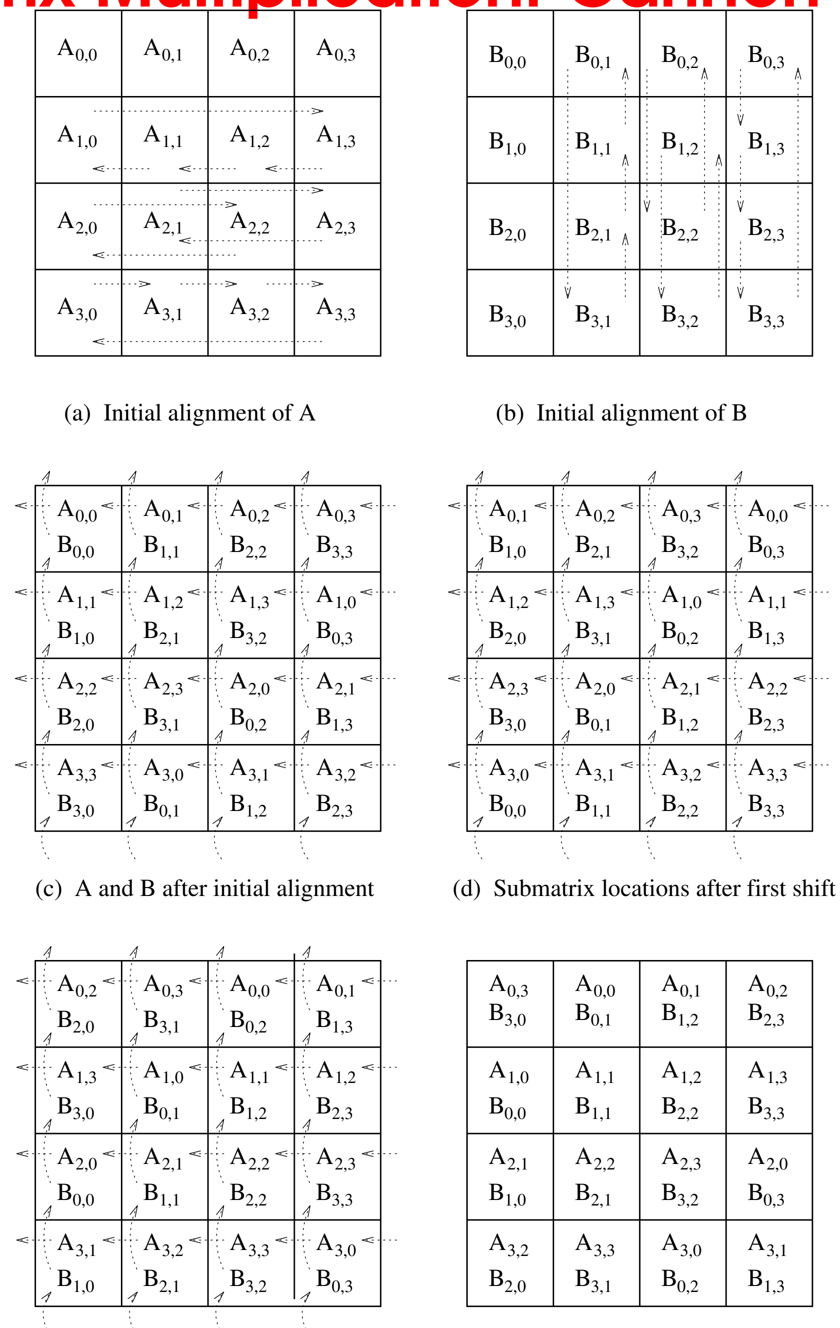
$$T_P = \frac{n^3}{p} + t_s \log p + 2t_w \frac{n^2}{\sqrt{p}}. \qquad (5)$$

- The algorithm is cost optimal and the isoefficiency is $O(p^{1.5})$ due to bandwidth term t_w and concurrency.
- Major drawback of the algorithm is that it is not memory optimal.

Matrix-Matrix Multiplication: Cannon's Algorithm

- In this algorithm, we schedule the computations of the $\sqrt{p}$ processes of the ith row such that, at any given time, each process is using a different block $A_{i,k}$.
- These blocks can be systematically rotated among the processes after every submatrix multiplication so that every process gets a fresh $A_{i,k}$ after each rotation.

Matrix-Matrix Multiplication: Cannon's Algorithm



(a) Initial alignment of A (b) Initial alignment of B

(c) A and B after initial alignment (d) Submatrix locations after first shift

(e) Submatrix locations after second shift (f) Submatrix locations after third shift

communication steps in Cannon's algorithm on 16 processes.

Matrix-Matrix Multiplication: Cannon's Algorithm

- Align the blocks of A and B in such a way that each process multiplies its local submatrices. This is done by shifting all submatrices $A_{i,j}$ to the left (with wraparound) by i steps and all submatrices $B_{i,j}$ up (with wraparound) by j steps.

- Perform local block multiplication.

- Each block of A moves one step left and each block of B moves one step up (again with wraparound).

- Perform next block multiplication, add to partial result, repeat until all $\sqrt{p}$ blocks have been multiplied.

Matrix-Matrix Multiplication: Cannon's Algorithm

- In the alignment step, since the maximum distance over which a block shifts is $\sqrt{p}-1$, the two shift operations require a total of $2(t_s + t_w n^2/p)$ time.

- Each of the $\sqrt{p}$ single-step shifts in the compute-and-shift phase of the algorithm takes $t_s + t_w n^2/p$ time.

- The computation time for multiplying $\sqrt{p}$ matrices of size $(n/\sqrt{p}) \times (n/\sqrt{p})$ is n^3/p.

- The parallel time is approximately:

$$T_P = \frac{n^3}{p} + 2\sqrt{p}t_s + 2t_w \frac{n^2}{\sqrt{p}}. \tag{6}$$

- The cost-efficiency and isoefficiency of the algorithm are identical to the first algorithm, except, this is memory optimal.

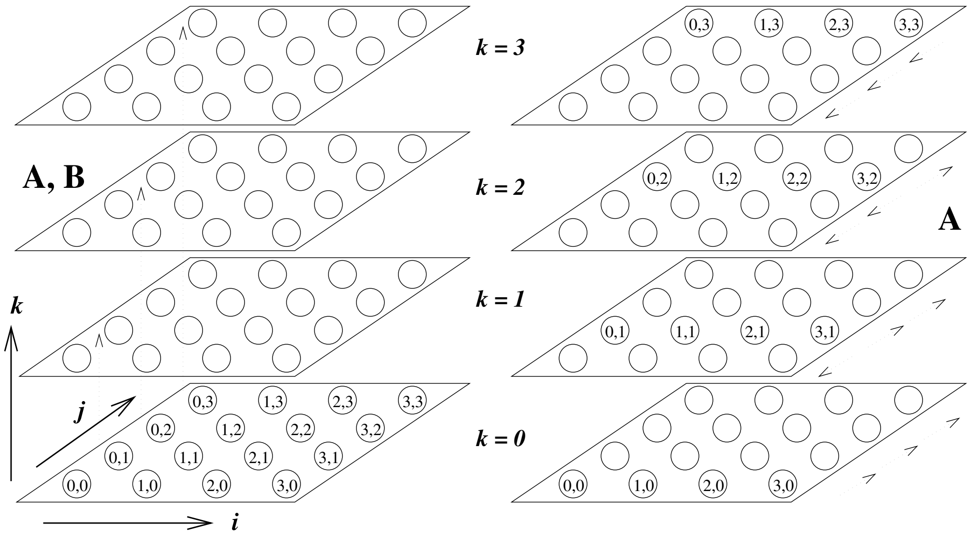
Matrix-Matrix Multiplication: DNS Algorithm

- Uses a 3-D partitioning.
- Visualize the matrix multiplication algorithm as a cube – matrices A and B come in two orthogonal faces and result C comes out the other orthogonal face.
- Each internal node in the cube represents a single add-multiply operation (and thus the complexity).
- DNS algorithm partitions this cube using a 3-D block scheme.

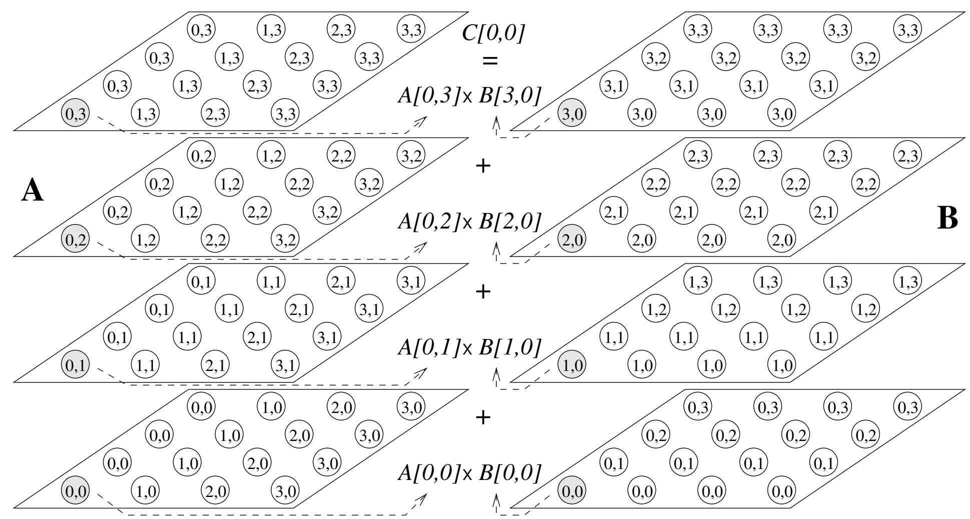
Matrix-Matrix Multiplication: DNS Algorithm

- Assume an $n \times n \times n$ mesh of processors.
- Move the columns of A and rows of B and perform broadcast.
- Each processor computes a single add-multiply.
- This is followed by an accumulation along the C dimension.
- Since each add-multiply takes constant time and accumulation and broadcast takes $\log n$ time, the total runtime is $\log n$.
- This is not cost optimal. It can be made cost optimal by using $n / \log n$ processors along the direction of accumulation.

Matrix-Matrix Multiplication: DNS Algorithm



(a) Initial distribution of A and B (b) After moving $A[i,j]$ from $P_{i,j,0}$ to $P_{i,j,j}$



(c) After broadcasting $A[i,j]$ along j axis (d) Corresponding distribution of B

The communication steps in the DNS algorithm while multiplying 4×4 matrices A and B on 64 processes.

Matrix-Matrix Multiplication: DNS Algorithm

Using fewer than n^3 processors.

- Assume that the number of processes p is equal to q^3 for some $q < n$.
- The two matrices are partitioned into blocks of size $(n/q) \times (n/q)$. Each matrix can thus be regarded as a $q \times q$ two-dimensional square array of blocks.
- The algorithm follows from the previous one, except, in this case, we operate on blocks rather than on individual elements.

Matrix-Matrix Multiplication: DNS Algorithm

Using fewer than n^3 processors.

- The first one-to-one communication step is performed for both A and B, and takes $t_s + t_w(n/q)^2$ time for each matrix.
- The two one-to-all broadcasts take $2(t_s \log q + t_w(n/q)^2 \log q)$ time for each matrix.
- The reduction takes time $t_s \log q + t_w(n/q)^2 \log q$.
- Multiplication of $(n/q) \times (n/q)$ submatrices takes $(n/q)^3$ time.
- The parallel time is approximated by:

$$T_P = \frac{n^3}{p} + t_s \log p + t_w \frac{n^2}{p^{2/3}} \log p. \tag{7}$$

The isoefficiency function is $\Theta(p(\log p)^3)$.

Solving a System of Linear Equations

Consider the problem of solving linear equations of the kind:

$$\begin{array}{ccccccccc}
a_{0,0}x_0 & + & a_{0,1}x_1 & + & \cdots & + & a_{0,n-1}x_{n-1} & = & b_0, \\
a_{1,0}x_0 & + & a_{1,1}x_1 & + & \cdots & + & a_{1,n-1}x_{n-1} & = & b_1, \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n-1,0}x_0 & + & a_{n-1,1}x_1 & + & \cdots & + & a_{n-1,n-1}x_{n-1} & = & b_{n-1}.
\end{array}$$

This is written as $Ax = b$, where A is an $n \times n$ matrix with $A[i,j] = a_{i,j}$, b is an $n \times 1$ vector $[b_0, b_1, \ldots, b_{n-1}]^T$, and x is the solution.

Solving a System of Linear Equations

Two steps in solution are: reduction to triangular form, and back-substitution. The triangular form is as:

$$\begin{array}{ccccccccccc}
x_0 & + & u_{0,1}x_1 & + & u_{0,2}x_2 & + & \cdots & + & u_{0,n-1}x_{n-1} & = & y_0, \\
 & & x_1 & + & u_{1,2}x_2 & + & \cdots & + & u_{1,n-1}x_{n-1} & = & y_1, \\
 & & & & & & & & \vdots & & \vdots \\
 & & & & & & & & x_{n-1} & = & y_{n-1}.
\end{array}$$

We write this as: $Ux = y$.

A commonly used method for transforming a given matrix into an upper-triangular matrix is Gaussian Elimination.

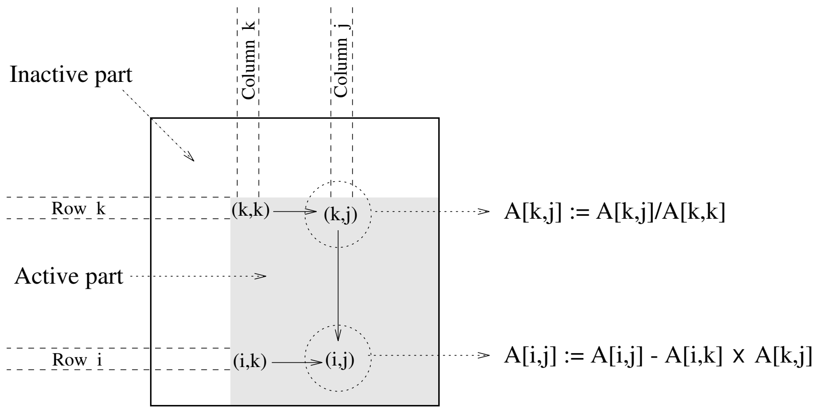
Gaussian Elimination

```
1.  procedure GAUSSIAN_ELIMINATION (A, b, y)
2.  begin
3.     for k := 0 to n − 1 do                  /* Outer loop */
4.     begin
5.        for j := k + 1 to n − 1 do
6.           A[k, j] := A[k, j]/A[k, k];    /* Division step */
7.        y[k] := b[k]/A[k, k];
8.        A[k, k] := 1;
9.        for i := k + 1 to n − 1 do
10.       begin
11.          for j := k + 1 to n − 1 do
12.             A[i, j] := A[i, j] − A[i, k] × A[k, j]; /* Elimination step */
13.          b[i] := b[i] − A[i, k] × y[k];
14.          A[i, k] := 0;
15.       endfor;             /* Line 9 */
16.    endfor;                /* Line 3 */
17. end GAUSSIAN_ELIMINATION
```

Serial Gaussian Elimination

Gaussian Elimination

- The computation has three nested loops – in the kth iteration of the outer loop, the algorithm performs $(n-k)^2$ computations. Summing from $k = 1..n$, we have roughly $(n^3/3)$ multiplications-subtractions.



A typical computation in Gaussian elimination.

Parallel Gaussian Elimination

- Assume $p = n$ with each row assigned to a processor.
- The first step of the algorithm normalizes the row. This is a serial operation and takes time $(n-k)$ in the kth iteration.
- In the second step, the normalized row is broadcast to all the processors. This takes time $(t_s + t_w(n-k-1)) \log n$.
- Each processor can independently eliminate this row from its own. This requires $(n-k-1)$ multiplications and subtractions.
- The total parallel time can be computed by summing from $k = 1..n-1$ as

$$T_P = \frac{3}{2}n(n-1) + t_s n \log n + \frac{1}{2}t_w n(n-1) \log n. \quad (8)$$

- The formulation is not cost optimal because of the t_w term.

Parallel Gaussian Elimination

P_0	1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
P_1	0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)
P_2	0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)
P_3	0	0	0	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)
P_4	0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)
P_5	0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)
P_6	0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)
P_7	0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)

(a) Computation:

(i) A[k,j] := A[k,j]/A[k,k] for k < j < n

(ii) A[k,k] := 1

P_0	1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
P_1	0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)
P_2	0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)
P_3	0	0	0	1	(3,4)	(3,5)	(3,6)	(3,7)
P_4	0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)
P_5	0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)
P_6	0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)
P_7	0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)

(b) Communication:

One-to-all brodcast of row A[k,*]

P_0	1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
P_1	0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)
P_2	0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)
P_3	0	0	0	1	(3,4)	(3,5)	(3,6)	(3,7)
P_4	0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)
P_5	0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)
P_6	0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)
P_7	0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)

(c) Computation:

(i) A[i,j] := A[i,j] - A[i,k] × A[k,j]
for k < i < n and k < j < n

(ii) A[i,k] := 0 for k < i < n

Gaussian elimination steps during the iteration corresponding to $k = 3$ for an 8×8 matrix partitioned rowwise among eight processes.

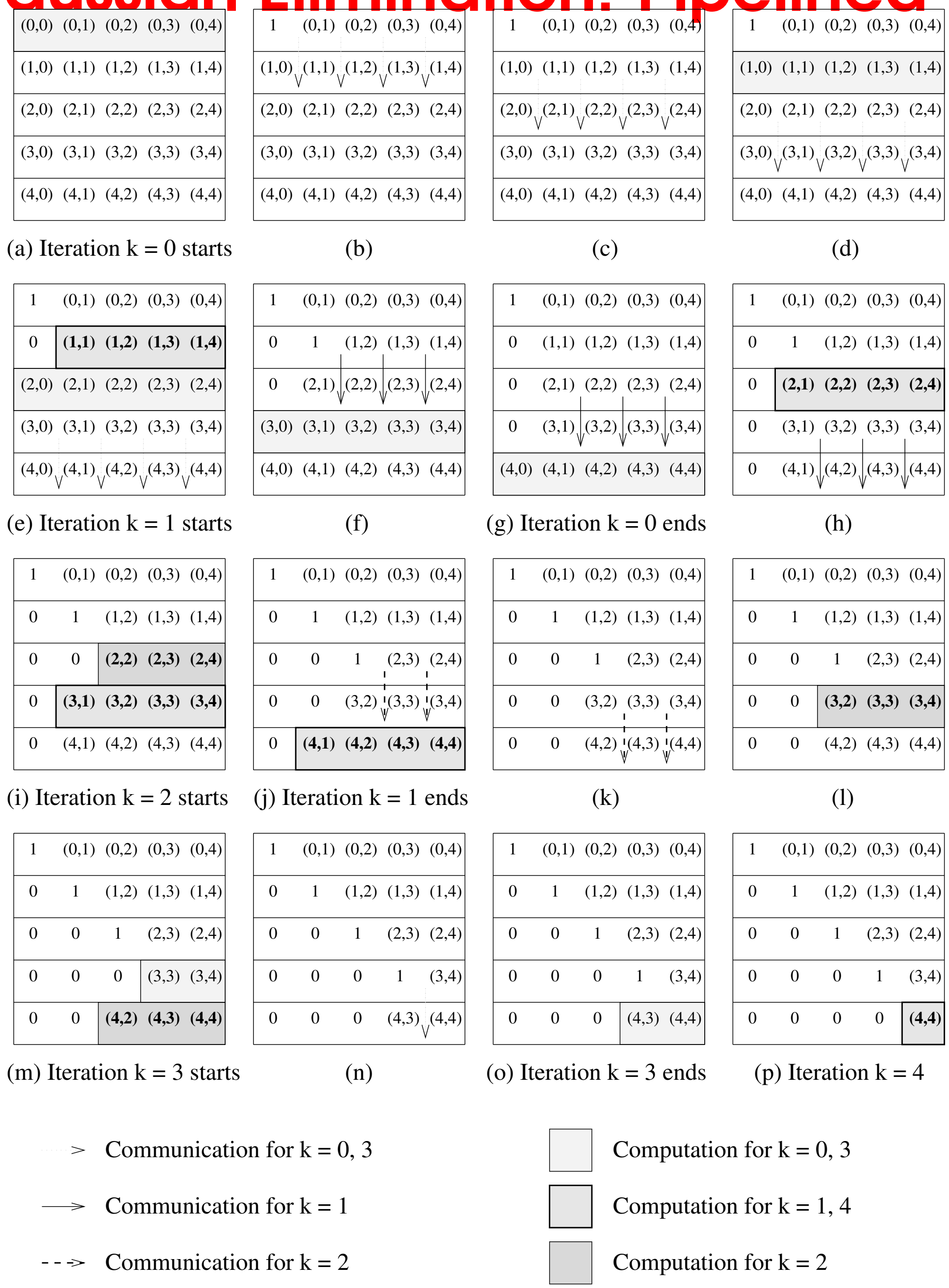
Parallel Gaussian Elimination: Pipelined Execution

- In the previous formulation, the $(k+1)$st iteration starts only after all the computation and communication for the kth iteration is complete.

- In the pipelined version, there are three steps – normalization of a row, communication, and elimination. These steps are performed in an asynchronous fashion.

- A processor P_k waits to receive and eliminate all rows prior to k. Once it has done this, it forwards its own row to processor P_{k+1}.

Parallel Gaussian Elimination: Pipelined Execution



Pipelined Gaussian elimination on a 5×5 matrix partitioned with one row per process.

Parallel Gaussian Elimination: Pipelined Execution

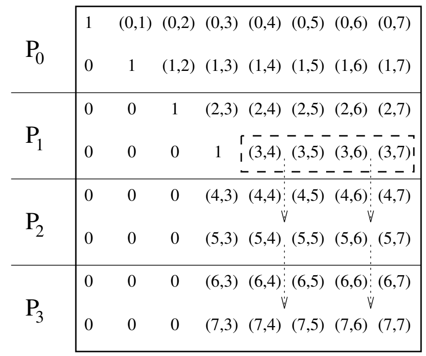
- The total number of steps in the entire pipelined procedure is $\Theta(n)$.

- In any step, either $O(n)$ elements are communicated between directly-connected processes, or a division step is performed on $O(n)$ elements of a row, or an elimination step is performed on $O(n)$ elements of a row.

- The parallel time is therefore $O(n^2)$.

- This is cost optimal.

Parallel Gaussian Elimination: Pipelined Execution



The communication in the Gaussian elimination iteration corresponding to $k = 3$ for an 8×8 matrix distributed among four processes using block 1-D partitioning.

Parallel Gaussian Elimination: Block 1D with $p < n$

- The above algorithm can be easily adapted to the case when $p < n$.

- In the kth iteration, a processor with all rows belonging to the active part of the matrix performs $(n - k - 1)n/p$ multiplications and subtractions.

- In the pipelined version, for $n > p$, computation dominates communication.

- The parallel time is given by: $2(n/p)\Sigma_{k=0}^{n-1}(n - k - 1)$, or approximately, n^3/p.

- While the algorithm is cost optimal, the cost of the parallel algorithm is higher than the sequential run time by a factor of 3/2.

Parallel Gaussian Elimination: Block 1D with $p < n$

P_0	1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
	0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)
P_1	0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)
	0	0	0	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)
P_2	0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)
	0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)
P_3	0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)
	0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)

(a) Block 1-D mapping

1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	P_0
0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	
0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	P_1
0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	
0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	P_2
0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	
0	0	0	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	P_3
0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	

(b) Cyclic 1-D mapping

Computation load on different processes in block and cyclic 1-D partitioning of an 8 $\times$ 8 matrix on four processes during the Gaussian elimination iteration corresponding to k = 3.

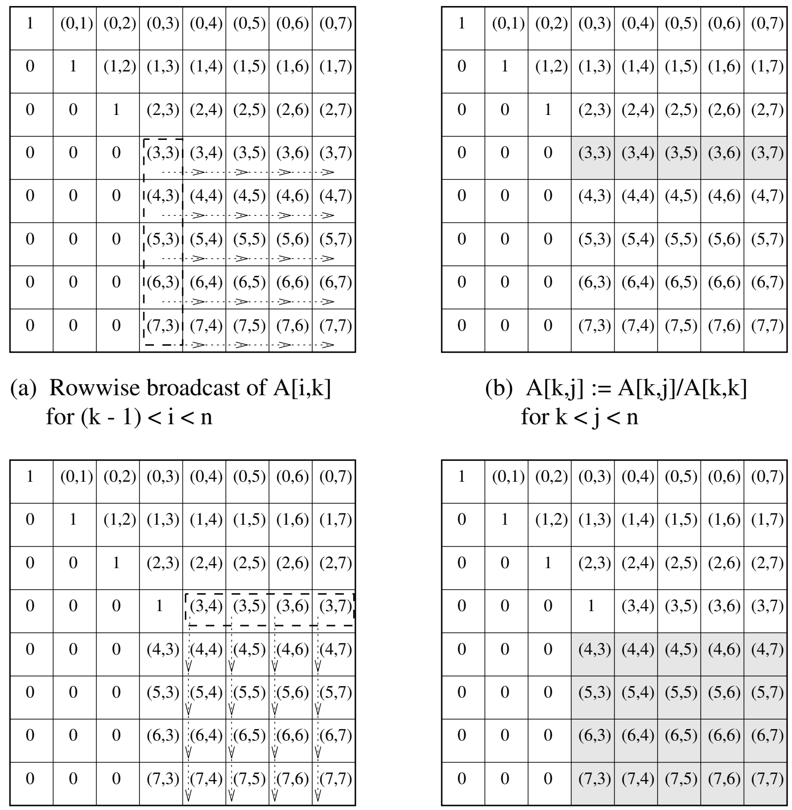
Parallel Gaussian Elimination: Cyclic 1D Mapping

- The load imbalance problem can be alleviated by using a cyclic mapping.
- In this case, other than processing of the last p rows, there is no load imbalance.
- This corresponds to a cumulative load imbalance overhead of $O(n^2p)$ (instead of $O(n^3)$ in the previous case).

Parallel Gaussian Elimination: 2-D Mapping

- Assume an $n \times n$ matrix A mapped onto an $n \times n$ mesh of processors.
- Each update of the partial matrix can be thought of as a scaled rank-one update (scaling by the pivot element).
- In the first step, the pivot is broadcast to the row of processors.
- In the second step, each processor locally updates its value. For this it needs the corresponding value from the pivot row, and the scaling value from its own row.
- This requires two broadcasts, each of which takes $\log n$ time.
- This results in a non-cost-optimal algorithm.

Parallel Gaussian Elimination: 2-D Mapping



(a) Rowwise broadcast of A[i,k] for (k - 1) < i < n

(b) A[k,j] := A[k,j]/A[k,k] for k < j < n

(c) Columnwise broadcast of A[k,j] for k < j < n

(d) A[i,j] := A[i,j]-A[i,k] $\times$ A[k,j] for k < i < n and k < j < n

Various steps in the Gaussian elimination iteration corresponding to k = 3 for an 8 $\times$ 8 matrix on 64 processes arranged in a logical two-dimensional mesh.

Parallel Gaussian Elimination: 2-D Mapping with Pipelining

- We pipeline along two dimensions. First, the pivot value is pipelined along the row. Then the scaled pivot row is pipelined down.
- Processor $P_{i,j}$ (not on the pivot row) performs the elimination step $A[i,j] := A[i,j] - A[i,k] \times A[k,j]$ as soon as $A[i,k]$ and $A[k,j]$ are available.
- The computation and communication for each iteration moves through the mesh from top-left to bottom-right as a "front."
- After the front corresponding to a certain iteration passes through a process, the process is free to perform subsequent iterations.
- Multiple fronts that correspond to different iterations are active simultaneously.

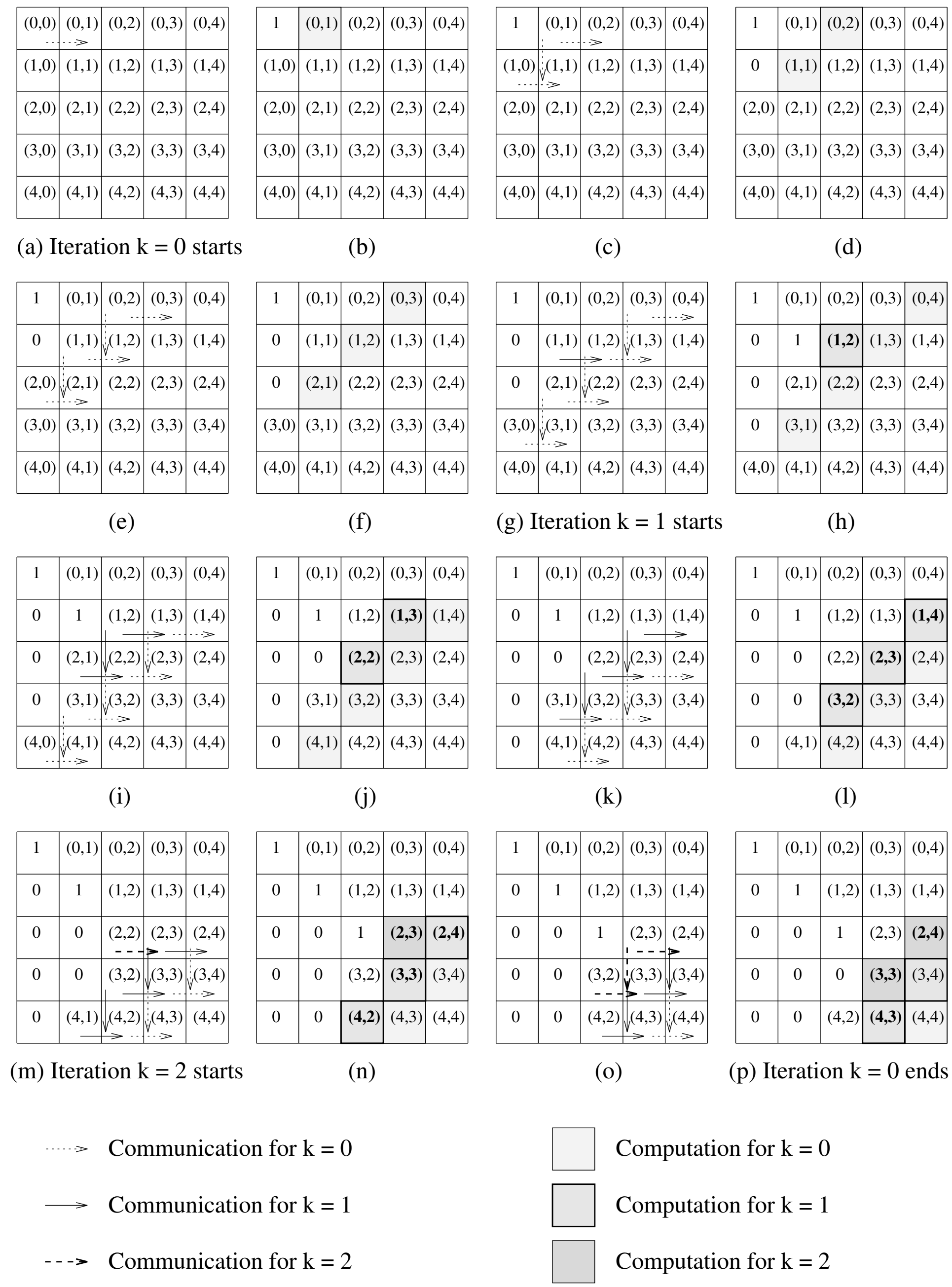
Parallel Gaussian Elimination: 2-D Mapping with Pipelining

- If each step (division, elimination, or communication) is assumed to take constant time, the front moves a single step in this time. The front takes $\Theta(n)$ time to reach P$_{n-1,n-1}$.

- Once the front has progressed past a diagonal processor, the next front can be initiated. In this way, the last front passes the bottom-right corner of the matrix $\Theta(n)$ steps after the first one.

- The parallel time is therefore $O(n)$, which is cost-optimal.

2-D Mapping with Pipelining



Pipelined Gaussian elimination for a 5×5 matrix with 25 processors.

Parallel Gaussian Elimination: 2-D Mapping with Pipelining and $p < n$

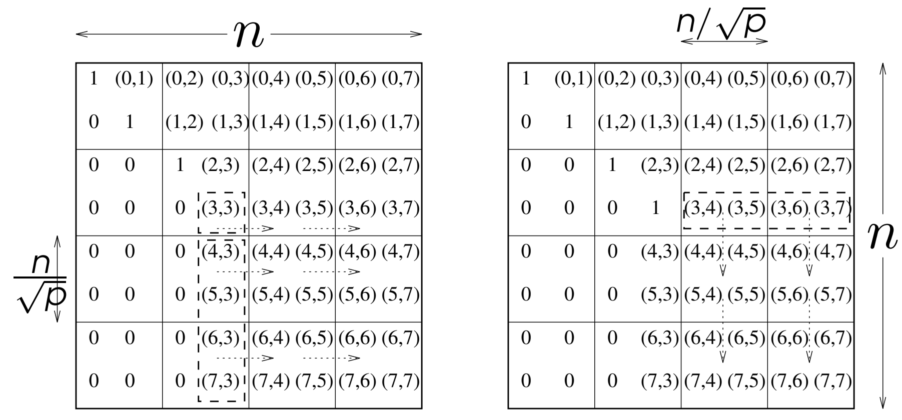
- In this case, a processor containing a completely active part of the matrix performs n^2/p multiplications and subtractions, and communicates $n/\sqrt{p}$ words along its row and its column.

- The computation dominantes communication for $n >> p$.

- The total parallel run time of this algorithm is $(2n^2/p) \times n$, since there are n iterations. This is equal to $2n^3/p$.

- This is three times the serial operation count!

Parallel Gaussian Elimination: 2-D Mapping with Pipelining and $p < n$



(a) Rowwise broadcast of A[i,k] for i = k to (n - 1)

(b) Columnwise broadcast of A[k,j] for j = (k + 1) to (n - 1)

The communication steps in the Gaussian elimination iteration corresponding to k = 3 for an 8 $\times$ 8 matrix on 16 processes of a two-dimensional mesh.

Parallel Gaussian Elimination: 2-D Mapping with Pipelining and $p < n$

1	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
0	1	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)
0	0	1	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)
0	0	0	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)
0	0	0	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)
0	0	0	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)
0	0	0	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)
0	0	0	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)

(a) Block-checkerboard mapping

1	(0,4)	(0,1)	(0,5)	(0,2)	(0,6)	(0,3)	(0,7)
0	(4,4)	0	(4,5)	0	(4,6)	(4,3)	(4,7)
0	(1,4)	1	(1,5)	(1,2)	(1,6)	(1,3)	(1,7)
0	(5,4)	0	(5,5)	0	(5,6)	(5,3)	(5,7)
0	(2,4)	0	(2,5)	1	(2,6)	(2,3)	(2,7)
0	(6,4)	0	(6,5)	0	(6,6)	(6,3)	(6,7)
0	(3,4)	0	(3,5)	0	(3,6)	(3,3)	(3,7)
0	(7,4)	0	(7,5)	0	(7,6)	(7,3)	(7,7)

(b) Cyclic-checkerboard mapping

Computational load on different processes in block and cyclic 2-D mappings of an 8×8 matrix onto 16 processes during the Gaussian elimination iteration corresponding to $k = 3$.

Parallel Gaussian Elimination: 2-D Cyclic Mapping

- The idling in the block mapping can be alleviated using a cyclic mapping.
- The maximum difference in computational load between any two processes in any iteration is that of one row and one column update.
- This contributes $\Theta(n\sqrt{p})$ to the overhead function. Since there are n iterations, the total overhead is $\Theta(n^2\sqrt{p})$.

Gaussian Elimination with Partial Pivoting

- For numerical stability, one generally uses partial pivoting.
- In the kth iteration, we select a column i (called the *pivot* column) such that $A[k, i]$ is the largest in magnitude among all $A[k, j]$ such that $k \leq j < n$.
- The kth and the ith columns are interchanged.
- Simple to implement with row-partitioning and does not add overhead since the division step takes the same time as computing the max.
- Column-partitioning, however, requires a global reduction, adding a $\log p$ term to the overhead.
- Pivoting precludes the use of pipelining.

Gaussian Elimination with Partial Pivoting: 2-D Partitioning

- Partial pivoting restricts use of pipelining, resulting in performance loss.
- This loss can be alleviated by restricting pivoting to specific columns.
- Alternately, we can use faster algorithms for broadcast.

Solving a Triangular System: Back-Substitution

- The upper triangular matrix U undergoes back-substitution to determine the vector x.

```
procedure BACK_SUBSTITUTION (U, x, y)
begin
    for k := n - 1 downto 0 do    /* Main loop */
        begin
            x[k] := y[k];
            for i := k - 1 downto 0 do
                y[i] := y[i] - x[k] × U[i, k];
        endfor;
end BACK_SUBSTITUTION
```

A serial algorithm for back-substitution.

Solving a Triangular System: Back-Substitution

- The algorithm performs approximately $n^2/2$ multiplications and subtractions.
- Since complexity of this part is asymptotically lower, we should optimize the data distribution for the factorization part.
- Consider a rowwise block 1-D mapping of the $n \times n$ matrix U with vector y distributed uniformly.
- The value of the variable solved at a step can be pipelined back.
- Each step of a pipelined implementation requires a constant amount of time for communication and $\Theta(n/p)$ time for computation.
- The parallel run time of the entire algorithm is $\Theta(n^2/p)$.

Solving a Triangular System: Back-Substitution

- If the matrix is partitioned by using 2-D partitioning on a $\sqrt{p} \times \sqrt{p}$ logical mesh of processes, and the elements of the vector are distributed along one of the columns of the process mesh, then only the $\sqrt{p}$ processes containing the vector perform any computation.

- Using pipelining to communicate the appropriate elements of U to the process containing the corresponding elements of y for the substitution step (line 7), the algorithm can be executed in $\Theta(n^2/\sqrt{p})$ time.

- While this is not cost optimal, since this does not dominante the overall computation, the cost optimality is determined by the factorization.