Dynamic Programming

Ananth Grama, Anshul Gupta, George Karypis, and Vipin Kumar

To accompany the text "Introduction to Parallel Computing",
Addison Wesley, 2003.

Topic Overview

- Overview of Serial Dynamic Programming
- Serial Monadic DP Formulations
- Nonserial Monadic DP Formulations
- Serial Polyadic DP Formulations
- Nonserial Polyadic DP Formulations

Overview of Serial Dynamic Programming

- *Dynamic programming* (DP) is used to solve a wide variety of discrete optimization problems such as scheduling, string-editing, packaging, and inventory management.

- Break problems into subproblems and combine their solutions into solutions to larger problems.

- In contrast to divide-and-conquer, there may be relationships across subproblems.

Dynamic Programming: Example

- Consider the problem of finding a shortest path between a pair of vertices in an acyclic graph.

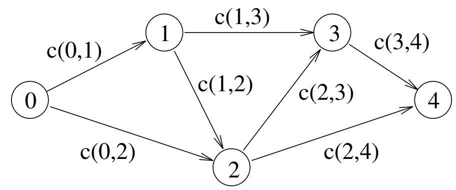
- An edge connecting node i to node j has cost $c(i,j)$.

- The graph contains n nodes numbered $0, 1, \ldots, n-1$, and has an edge from node i to node j only if $i < j$. Node 0 is source and node $n-1$ is the destination.

- Let $f(x)$ be the cost of the shortest path from node 0 to node x.

$$f(x) = \begin{cases} 0 & x = 0 \\ \min\limits_{0 \leq j < x} \{f(j) + c(j,x)\} & 1 \leq x \leq n-1 \end{cases}$$

Dynamic Programming: Example



A graph for which the shortest path between nodes 0 and 4 is to be computed.

$$f(4) = \min\{f(3) + c(3,4), f(2) + c(2,4)\}.$$

Dynamic Programming

- The solution to a DP problem is typically expressed as a minimum (or maximum) of possible alternate solutions.

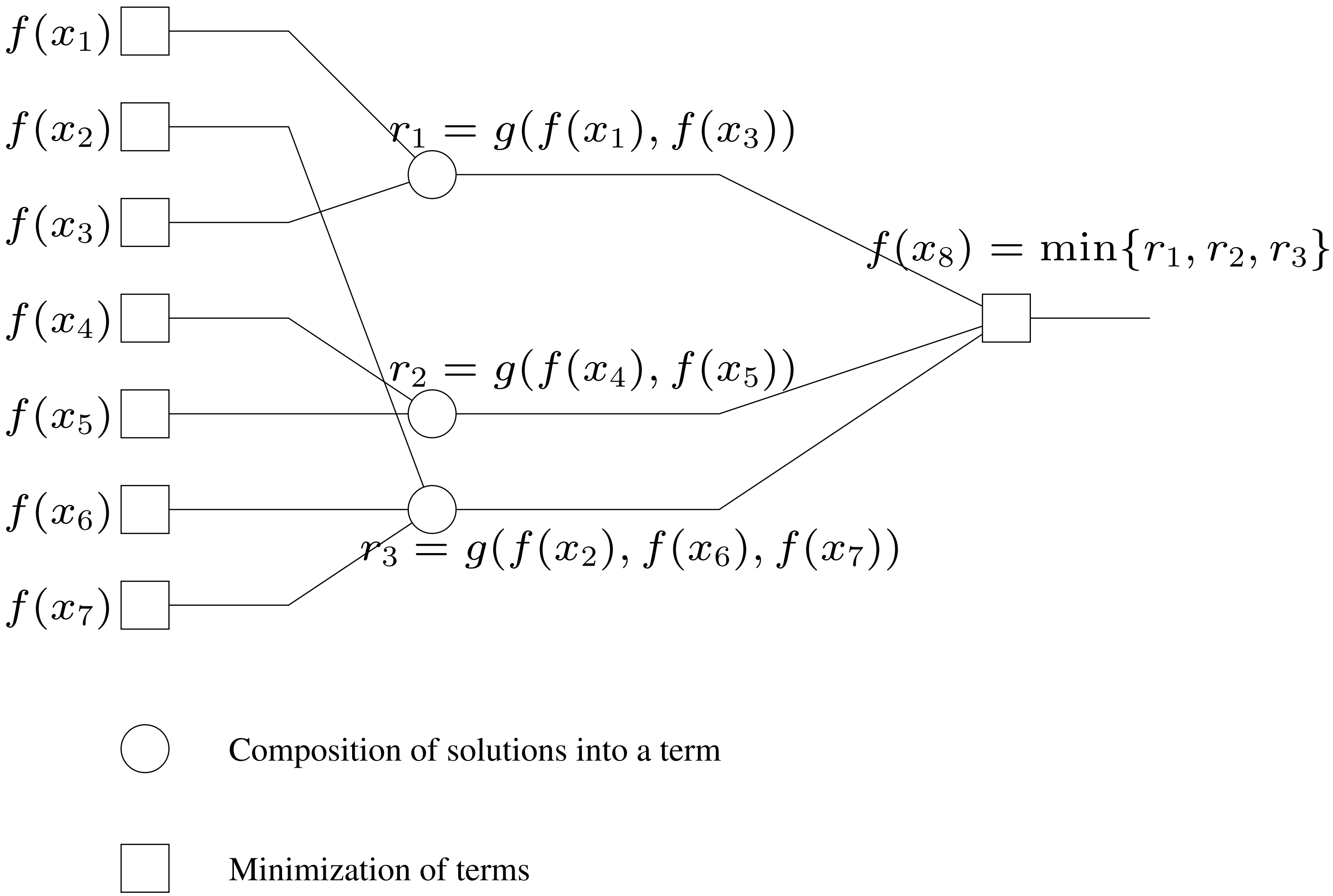
- If r represents the cost of a solution composed of subproblems x_1, x_2, ..., x_l, then r can be written as

$$r = g(f(x_1), f(x_2), \ldots, f(x_l)).$$

  Here, g is the *composition function*.

- If the optimal solution to each problem is determined by composing optimal solutions to the subproblems and selecting the minimum (or maximum), the formulation is said to be a DP formulation.

Dynamic Programming: Example



The computation and composition of subproblem solutions to solve problem $f(x_8)$.

Dynamic Programming

- The recursive DP equation is also called the *functional equation* or *optimization equation*.

- In the equation for the shortest path problem the composition function is $f(j) + c(j, x)$. This contains a single recursive term ($f(j)$). Such a formulation is called monadic.

- If the RHS has multiple recursive terms, the DP formulation is called polyadic.

Dynamic Programming

- The dependencies between subproblems can be expressed as a graph.
- If the graph can be levelized (i.e., solutions to problems at a level depend only on solutions to problems at the previous level), the formulation is called serial, else it is called non-serial.
- Based on these two criteria, we can classify DP formulations into four categories – serial-monadic, serial-polyadic, non-serial-monadic, non-serial-polyadic.
- This classification is useful since it identifies concurrency and dependencies that guide parallel formulations.

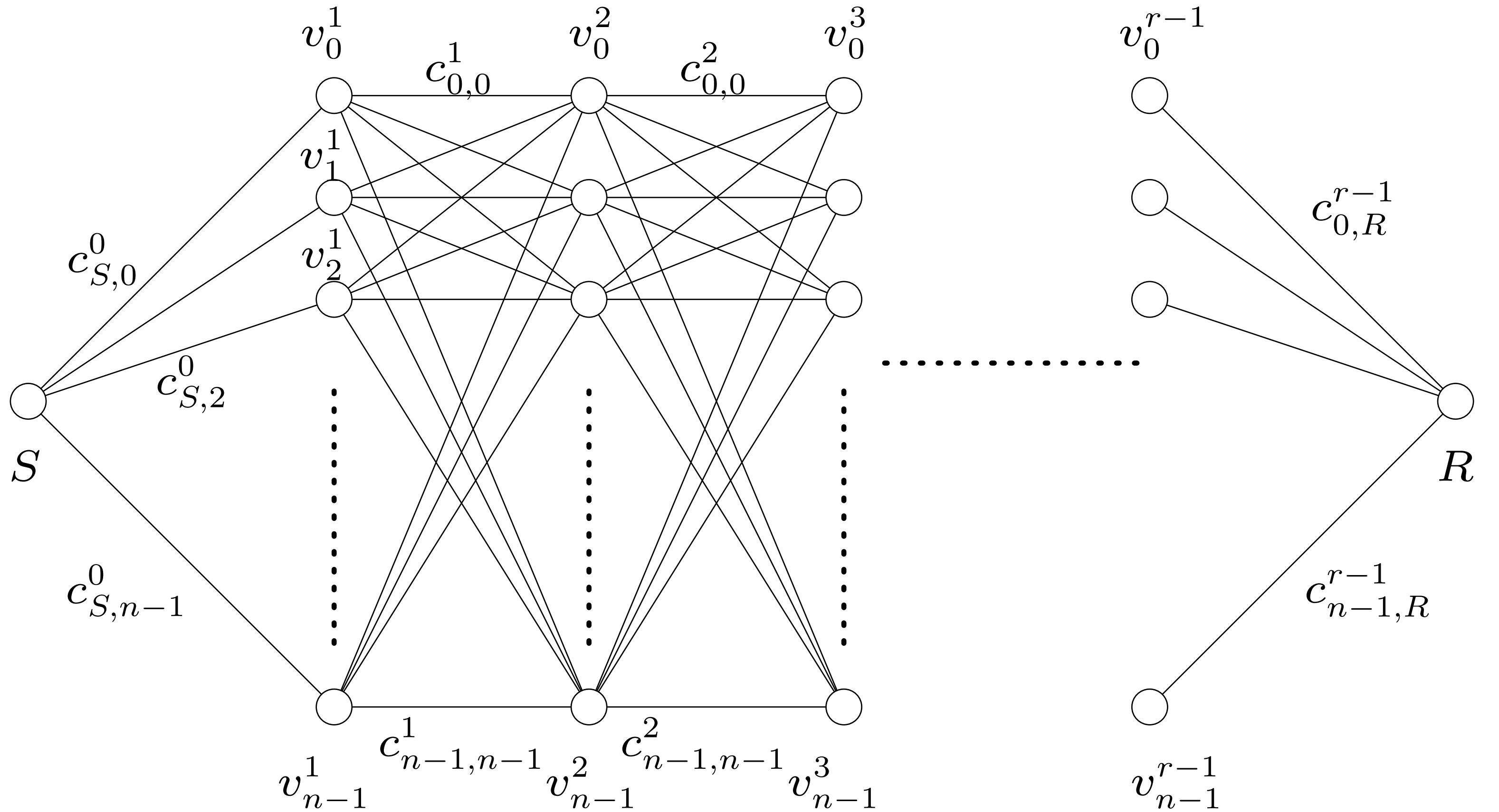
Serial Monadic DP Formulations

- It is difficult to derive canonical parallel formulations for the entire class of formulations.
- For this reason, we select two representative examples, the shortest-path problem for a multistage graph and the 0/1 knapsack problem.
- We derive parallel formulations for these problems and identify common principles guiding design within the class.

Shortest-Path Problem

- Special class of shortest path problem where the graph is a weighted multistage graph of $r + 1$ levels.
- Each level is assumed to have n levels and every node at level i is connected to every node at level $i + 1$.
- Levels zero and r contain only one node, the source and destination nodes, respectively.
- The objective of this problem is to find the shortest path from S to R.

Shortest-Path Problem



An example of a serial monadic DP formulation for finding the shortest path in a graph whose nodes can be organized into levels.

Shortest Path Problem

- The i^{th} node at level l in the graph is labeled v_i^l and the cost of an edge connecting v_i^l to node v_j^{l+1} is labeled $c_{i,j}^l$.
- The cost of reaching the goal node R from any node v_i^l is represented by C_i^l.
- If there are n nodes at level l, the vector $[C_0^l, C_1^l, \ldots, C_{n-1}^l]^T$ is referred to as $\mathcal{C}^l$. Note that $\mathcal{C}^0 = [C_0^0]$.
- We have

$$C_i^l = \text{min } \left\{(c_{i,j}^l + C_j^{l+1}) | j \text{ is a node at level } l+1\right\}. \tag{1}$$

Shortest Path Problem

- Since all nodes v_j^{r-1} have only one edge connecting them to the goal node R at level r, the cost C_j^{r-1} is equal to $c_{j,R}^{r-1}$.

- We have:

$$\mathcal{C}^{r-1} = [c_{0,R}^{r-1}, c_{1,R}^{r-1}, \ldots, c_{n-1,R}^{r-1}]. \tag{2}$$

  Notice that this problem is serial and monadic.

Shortest Path Problem

The cost of reaching the goal node R from any node at level l $(0 < l < r-1)$ is

$$
\begin{aligned}
C_0^l &= \min\{(c_{0,0}^l + C_0^{l+1}), (c_{0,1}^l + C_1^{l+1}), \ldots, (c_{0,n-1}^l + C_{n-1}^{l+1})\}, \\
C_1^l &= \min\{(c_{1,0}^l + C_0^{l+1}), (c_{1,1}^l + C_1^{l+1}), \ldots, (c_{1,n-1}^l + C_{n-1}^{l+1})\}, \\
&\vdots \\
C_{n-1}^l &= \min\{(c_{n-1,0}^l + C_0^{l+1}), (c_{n-1,1}^l + C_1^{l+1}), \ldots, (c_{n-1,n-1}^l + C_{n-1}^{l+1})\}.
\end{aligned}
$$

Shortest Path Problem

- We can express the solution to the problem as a modified sequence of matrix-vector products.
- Replacing the addition operation by minimization and the multiplication operation by addition, the preceding set of equations becomes:

$$\mathcal{C}^l = M_{l,l+1} \times \mathcal{C}^{l+1}, \tag{3}$$

where $\mathcal{C}^l$ and $\mathcal{C}^{l+1}$ are $n \times 1$ vectors representing the cost of reaching the goal node from each node at levels l and $l+1$.

Shortest Path Problem

- Matrix $M_{l,l+1}$ is an $n \times n$ matrix in which entry (i, j) stores the cost of the edge connecting node i at level l to node j at level $l+1$.

- $$M_{l,l+1} = \begin{bmatrix} c^l_{0,0} & c^l_{0,1} & \cdots & c^l_{0,n-1} \\ c^l_{1,0} & c^l_{1,1} & \cdots & c^l_{1,n-1} \\ \vdots & \vdots & & \vdots \\ c^l_{n-1,0} & c^l_{n-1,1} & \cdots & c^l_{n-1,n-1} \end{bmatrix}.$$

- The shortest path problem has been formulated as a sequence of r matrix-vector products.

Parallel Shortest Path

- We can parallelize this algorithm using the parallel algorithms for the matrix-vector product.
- $\Theta(n)$ processing elements can compute each vector $\mathcal{C}^l$ in time $\Theta(n)$ and solve the entire problem in time $\Theta(rn)$.
- In many instances of this problem, the matrix M may be sparse. For such problems, it is highly desirable to use sparse matrix techniques.

0/1 Knapsack Problem

- We are given a knapsack of capacity c and a set of n objects numbered $1, 2, \ldots, n$. Each object i has weight w_i and profit p_i.
- Let $v = [v_1, v_2, \ldots, v_n]$ be a solution vector in which $v_i = 0$ if object i is not in the knapsack, and $v_i = 1$ if it is in the knapsack.
- The goal is to find a subset of objects to put into the knapsack so that
$$\sum_{i=1}^{n} w_i v_i \leq c$$
(that is, the objects fit into the knapsack) and
$$\sum_{i=1}^{n} p_i v_i$$
is maximized (that is, the profit is maximized).

0/1 Knapsack Problem

- The naive method is to consider all 2^n possible subsets of the n objects and choose the one that fits into the knapsack and maximizes the profit.

- Let $F[i, x]$ be the maximum profit for a knapsack of capacity x using only objects $\{1, 2, \ldots, i\}$. The DP formulation is:

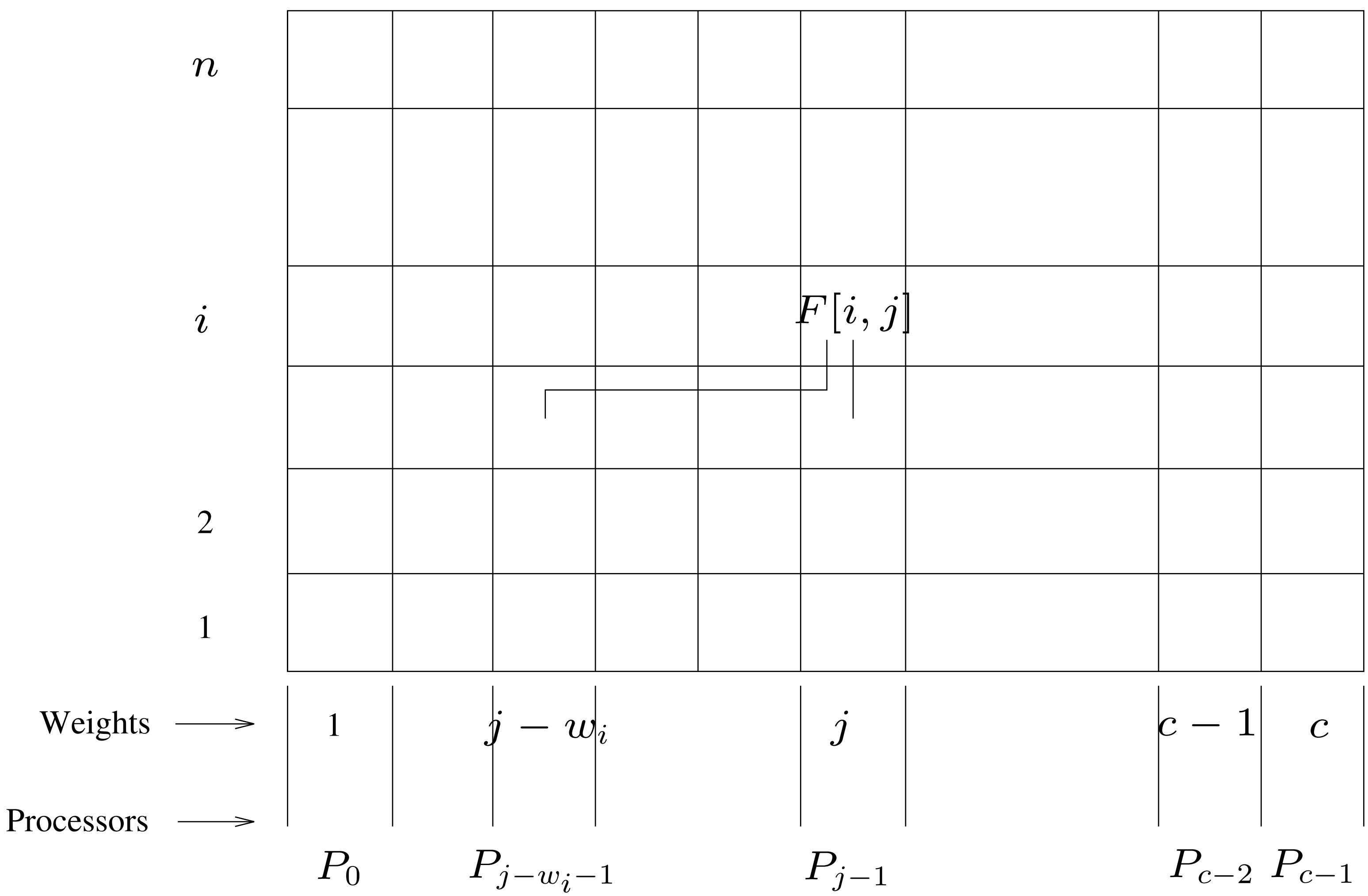
$$F[i,x] = \begin{cases} 0 & x \geq 0, i = 0 \\ -\infty & x < 0, i = 0 \\ \max\{F[i-1,x], (F[i-1,x-w_i]+p_i)\} & 1 \leq i \leq n \end{cases}$$

0/1 Knapsack Problem

- Construct a table F of size $n \times c$ in row-major order.
- Filling an entry in a row requires two entries from the previous row: one from the same column and one from the column offset by the weight of the object corresponding to the row.
- Computing each entry takes constant time; the sequential run time of this algorithm is $\Theta(nc)$.
- The formulation is serial-monadic.

0/1 Knapsack Problem

Table F



Computing entries of table F for the 0/1 knapsack problem. The computation of entry $F[i, j]$ requires communication with processing elements containing entries $F[i-1, j]$ and $F[i-1, j-w_i]$.

0/1 Knapsack Problem

- Using c processors in a PRAM, we can derive a simple parallel algorithm that runs in $O(n)$ time by partitioning the columns across processors.

- In a distributed memory machine, in the j^{th} iteration, for computing $F[j, r]$ at processing element P_{r-1}, $F[j-1, r]$ is available locally but $F[j-1, r-w_j]$ must fetched.

- The communication operation is a circular shift and the time is given by $(t_s + t_w)\log c$. The total time is therefore $t_c + (t_s + t_w)\log c$.

- Across all n iterations (rows), the parallel time is $O(n \log c)$. Note that this is not cost optimal.

0/1 Knapsack Problem

- Using p-processing elements, each processing element computes c/p elements of the table in each iteration.
- The corresponding shift operation takes time $(2t_s + t_w c/p)$, since the data block may be partitioned across two processors, but the total volume of data is c/p.
- The corresponding parallel time is $n(t_c c/p + 2t_s + t_w c/p)$, or $O(nc/p)$ (which is cost-optimal).
- Note that there is an upper bound on the efficiency of this formulation.

Nonserial Monadic DP Formulations: Longest-Common-Subsequence

- Given a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$, a subsequence of A can be formed by deleting some entries from A.

- Given two sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, find the longest sequence that is a subsequence of both A and B.

- If $A = \langle c, a, d, b, r, z \rangle$ and $B = \langle a, s, b, z \rangle$, the longest common subsequence of A and B is $\langle a, b, z \rangle$.

Longest-Common-Subsequence Problem

- Let $F[i,j]$ denote the length of the longest common subsequence of the first i elements of A and the first j elements of B. The objective of the LCS problem is to find $F[n,m]$.

- We can write:

$$F[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ F[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \text{max}\, \{F[i, j-1], F[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

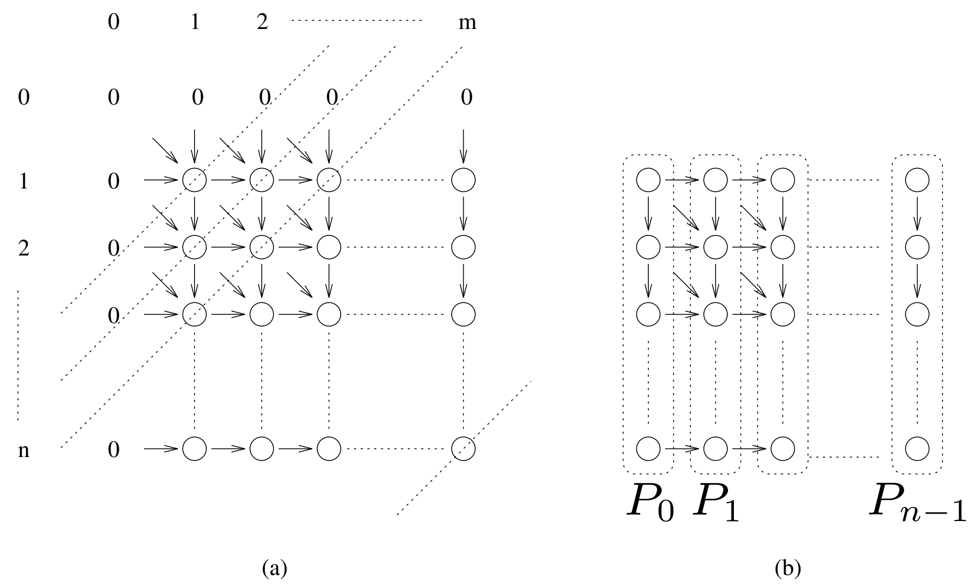
Longest-Common-Subsequence Problem

- The algorithm computes the two-dimensional F table in a row- or column-major fashion. The complexity is $\Theta(nm)$.

- Treating nodes along a diagonal as belonging to one level, each node depends on two subproblems at the preceding level and one subproblem two levels prior.

- This DP formulation is nonserial monadic.

Longest-Common-Subsequence Problem



(a) Computing entries of table F for the longest-common-subsequence problem. Computation proceeds along the dotted diagonal lines. (b) Mapping elements of the table to processing elements.

Longest-Common-Subsequence: Example

Consider the LCS of two amino-acid sequences H E A G A W G H E E and P A W H E A E. For the interested reader, the names of the corresponding amino-acids are A: Alanine, E: Glutamic acid, G: Glycine, H: Histidine, P: Proline, and W: Tryptophan.

		H	E	A	G	A	W	G	H	E	E
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	1	1	1	1	1	1	1	1
W	0	0	0	1	1	1	2	2	2	2	2
H	0	1	1	1	1	1	2	2	3	3	3
E	0	1	2	2	2	2	2	2	3	4	4
A	0	1	2	3	3	3	3	3	3	4	4
E	0	1	2	3	3	3	3	3	3	4	5

The F table for computing the LCS of the sequences. The LCS is A W H E E.

Parallel Longest-Common-Subsequence

- Table entries are computed in a diagonal sweep from the top-left to the bottom-right corner.

- Using n processors in a PRAM, each entry in a diagonal can be computed in constant time.

- For two sequences of length n, there are $2n-1$ diagonals.

- The parallel run time is $\Theta(n)$ and the algorithm is cost-optimal.

Parallel Longest-Common-Subsequence

- Consider a (logical) linear array of processors. Processing element P_i is responsible for the $(i+1)$th column of the table.

- To compute $F[i,j]$, processing element P_{j-1} may need either $F[i-1,j-1]$ or $F[i,j-1]$ from the processing element to its left. This communication takes time $t_s + t_w$.

- The computation takes constant time (t_c).

- We have:

$$T_P = (2n-1)(t_s + t_w + t_c).$$

- Note that this formulation is cost-optimal, however, its efficiency is upper-bounded by 0.5!

- Can you think of how to fix this?

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- Given weighted graph $G(V, E)$, Floyd's algorithm determines the cost $d_{i,j}$ of the shortest path between each pair of nodes in V.

- Let $d^k_{i,j}$ be the minimum cost of a path from node i to node j, using only nodes $v_0, v_1, \ldots, v_{k-1}$.

- We have:

$$d^k_{i,j} = \begin{cases} c_{i,j} & k = 0 \\ \text{min } \{d^{k-1}_{i,j}, (d^{k-1}_{i,k} + d^{k-1}_{k,j})\} & 0 \leq k \leq n-1 \end{cases}. \qquad (4)$$

- Each iteration requires time $\Theta(n^2)$ and the overall run time of the sequential algorithm is $\Theta(n^3)$.

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- A PRAM formulation of this algorithm uses n^2 processors in a logical 2D mesh. Processor $P_{i,j}$ computes the value of $d_{i,j}^k$ for $k = 1, 2, \ldots, n$ in constant time.
- The parallel runtime is $\Theta(n)$ and it is cost-optimal.
- The algorithm can easily be adapted to practical architectures, as discussed in our treatment of Graph Algorithms.

Nonserial Polyadic DP Formulation: Optimal Matrix-Parenthesization Problem

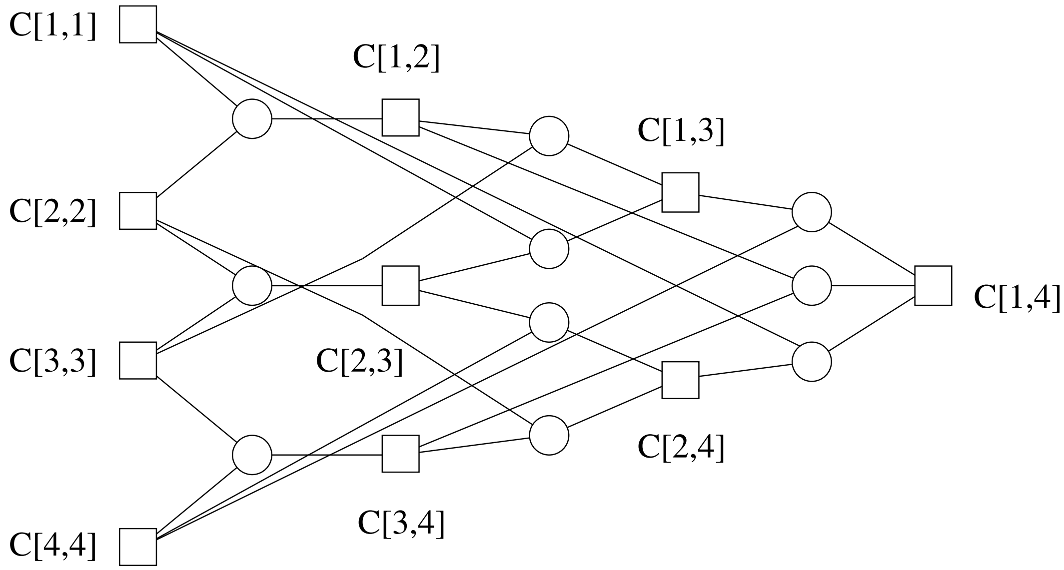
- When multiplying a sequence of matrices, the order of multiplication significantly impacts operation count.
- Let $C[i,j]$ be the optimal cost of multiplying the matrices $A_i, \ldots, A_j$.
- The chain of matrices can be expressed as a product of two smaller chains, $A_i, A_{i+1}, \ldots, A_k$ and $A_{k+1}, \ldots, A_j$.
- The chain $A_i, A_{i+1}, \ldots, A_k$ results in a matrix of dimensions $r_{i-1} \times r_k$, and the chain $A_{k+1}, \ldots, A_j$ results in a matrix of dimensions $r_k \times r_j$.
- The cost of multiplying these two matrices is $r_{i-1} r_k r_j$.

Optimal Matrix-Parenthesization Problem

- We have:

$$C[i,j] = \begin{cases} \min\limits_{i \leq k < j} \{C[i,k] + C[k+1,j] + r_{i-1}r_k r_j\} & 1 \leq i < j \leq n \\ 0 & j = i, 0 < i \leq n \end{cases} \tag{5}$$

Optimal Matrix-Parenthesization Problem



A nonserial polyadic DP formulation for finding an optimal matrix parenthesization for a chain of four matrices. A square node represents the optimal cost of multiplying a matrix chain. A circle node represents a possible parenthesization.

Optimal Matrix-Parenthesization Problem

- The goal of finding $C[1, n]$ is accomplished in a bottom-up fashion.

- Visualize this by thinking of filling in the C table diagonally. Entries in diagonal l corresponds to the cost of multiplying matrix chains of length $l + 1$.

- The value of $C[i, j]$ is computed as $\min\{C[i, k] + C[k + 1, j] + r_{i-1}r_kr_j\}$, where k can take values from i to $j - 1$.

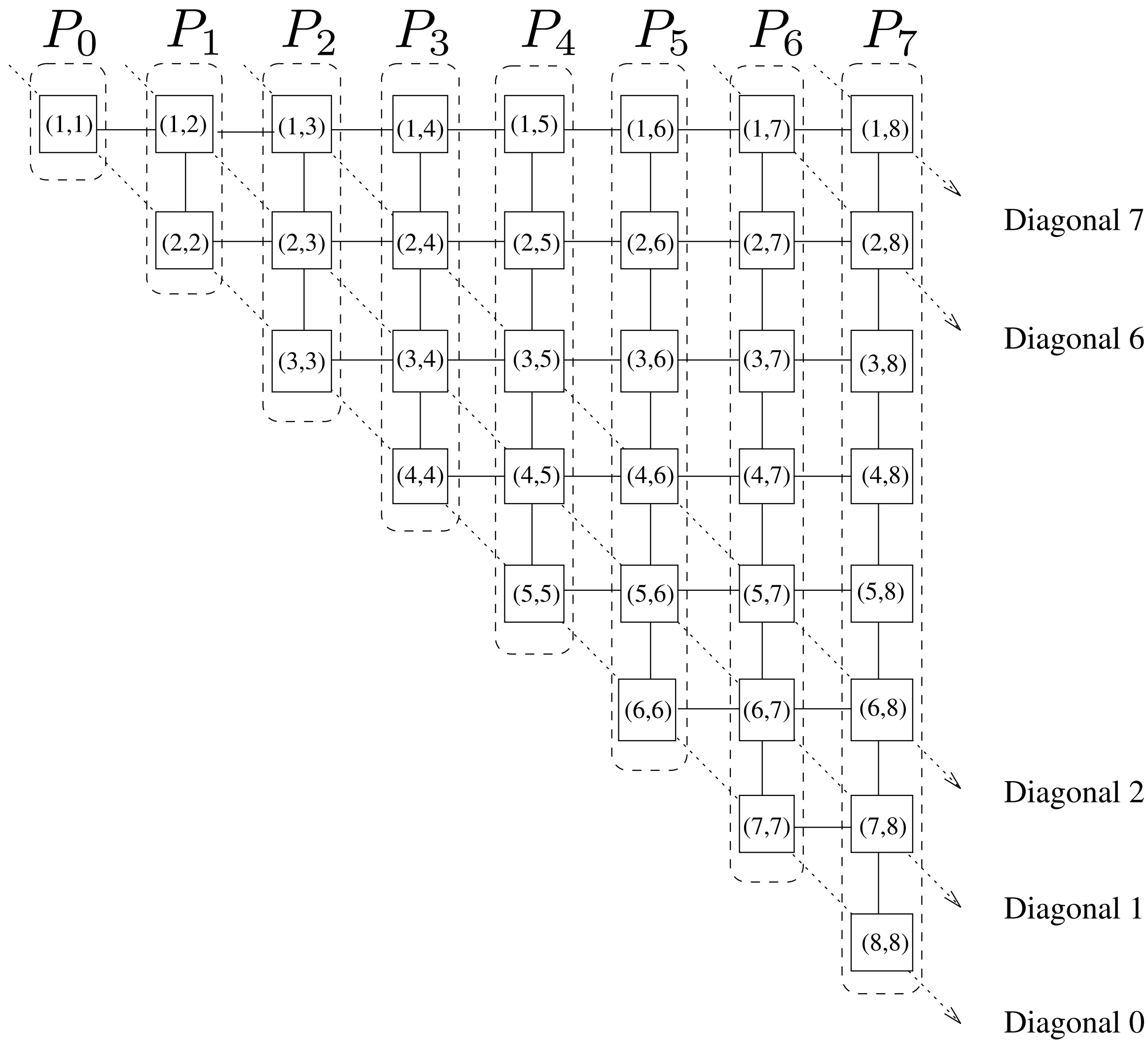
- Computing $C[i, j]$ requires that we evaluate $(j - i)$ terms and select their minimum.

- The computation of each term takes time t_c, and the computation of $C[i, j]$ takes time $(j-i)t_c$. Each entry in diagonal l can be computed in time lt_c.

Optimal Matrix-Parenthesization Problem

- The algorithm computes $(n-1)$ chains of length two. This takes time $(n-1)t_c$; computing $(n-2)$ chains of length three takes time $(n-2)2t_c$. In the final step, the algorithm computes one chain of length n in time $(n-1)t_c$.
- It follows that the serial time is $\Theta(n^3)$.

Optimal Matrix-Parenthesization Problem



The diagonal order of computation for the optimal matrix-parenthesization problem.

Parallel Optimal Matrix-Parenthesization Problem

- Consider a logical ring of processors. In step l, each processor computes a single element belonging to the l^{th} diagonal.
- On computing the assigned value of the element in table C, each processor sends its value to all other processors using an all-to-all broadcast.
- The next value can then be computed locally.
- The total time required to compute the entries along diagonal l is $lt_c + t_s \log n + t_w(n-1)$.
- The corresponding parallel time is given by:

$$
\begin{aligned}
T_P &= \sum_{l=1}^{n-1} (lt_c + t_s \log n + t_w(n-1)), \\
&= \frac{(n-1)(n)}{2} t_c + t_s(n-1)\log n + t_w(n-1)^2.
\end{aligned}
$$

Parallel Optimal Matrix-Parenthesization Problem

- When using p ($< n$) processors, each processor stores n/p nodes.

- The time taken for all-to-all broadcast of n/p words is $t_s \log p + t_w n(p-1)/p \approx t_s \log p + t_w n$ and the time to compute n/p entries of the table in the l^{th} diagonal is $l t_c n/p$.

- The parallel run time is

$$
\begin{aligned}
T_P &= \sum_{l=1}^{n-1} (l t_c n/p + t_s \log p + t_w n), \\
&= \frac{n^2(n-1)}{2p} t_c + t_s(n-1)\log p + t_w n(n-1).
\end{aligned}
$$

- $T_P = \Theta(n^3/p) + \Theta(n^2)$.

- This formulation can be improved to use up to $n(n+1)/2$ processors using pipelining.

Discussion of Parallel Dynamic Programming Algorithms

- By representing computation as a graph, we identify three sources of parallelism: parallelism within nodes, parallelism across nodes at a level, and pipelining nodes across multiple levels. The first two are available in serial formulations and the third one in non-serial formulations.

- Data locality is critical for performance. Different DP formulations, by the very nature of the problem instance, have different degrees of locality.