Graph Algorithms

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Topic Overview

- Definitions and Representation
- Minimum Spanning Tree: Prim's Algorithm
- Single-Source Shortest Paths: Dijkstra's Algorithm
- All-Pairs Shortest Paths
- Transitive Closure
- Connected Components
- Algorithms for Sparse Graphs

Definitions and Representation

- An *undirected graph* G is a pair (V, E), where V is a finite set of points called *vertices* and E is a finite set of *edges*.

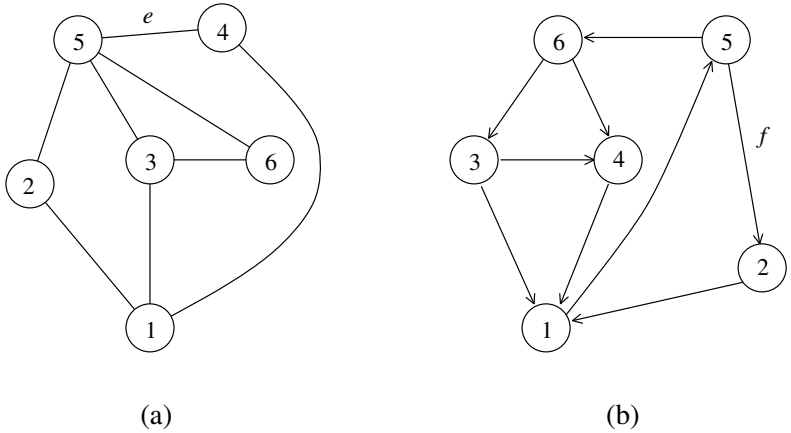
- An edge $e \in E$ is an unordered pair (u, v), where $u, v \in V$.

- In a directed graph, the edge e is an ordered pair (u, v). An edge (u, v) is *incident from* vertex u and is *incident to* vertex v.

- A *path* from a vertex v to a vertex u is a sequence $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ of vertices where $v_0 = v$, $v_k = u$, and $(v_i, v_{i+1}) \in E$ for $i = 0, 1, \ldots, k \quad 1$.

- The length of a path is defined as the number of edges in the path.

Definitions and Representation



(a) An undirected graph and (b) a directed graph.

Definitions and Representation

- An undirected graph is *connected* if every pair of vertices is connected by a path.

- A *forest* is an acyclic graph, and a *tree* is a connected acyclic graph.

- A graph that has weights associated with each edge is called a *weighted graph*.

Definitions and Representation

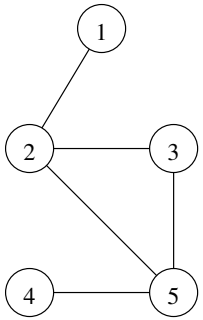
- Graphs can be represented by their adjacency matrix or an edge (or vertex) list.

- Adjacency matrices have a value $a_{i,j} = 1$ if nodes i and j share an edge; 0 otherwise. In case of a weighted graph, $a_{i,j} = w_{i,j}$, the weight of the edge.

- The *adjacency list* representation of a graph $G = (V, E)$ consists of an array $Adj[1..|V|]$ of lists. Each list $Adj[v]$ is a list of all vertices adjacent to v.

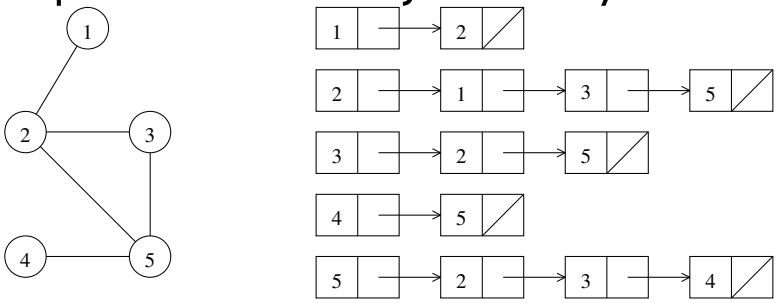
- For a grapn with n nodes, adjacency matrices take $Theta(n^2)$ space and adjacency list takes $\Theta(|E|)$ space.

Definitions and Representation



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

An undirected graph and its adjacency matrix representation.



An undirected graph and its adjacency list representation.

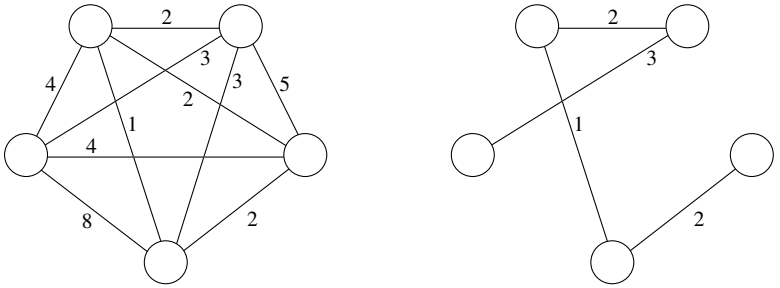
Minimum Spanning Tree

- A *spanning tree* of an undirected graph G is a subgraph of G that is a tree containing all the vertices of G.

- In a weighted graph, the weight of a subgraph is the sum of the weights of the edges in the subgraph.

- A *minimum spanning tree* (MST) for a weighted undirected graph is a spanning tree with minimum weight.

Minimum Spanning Tree



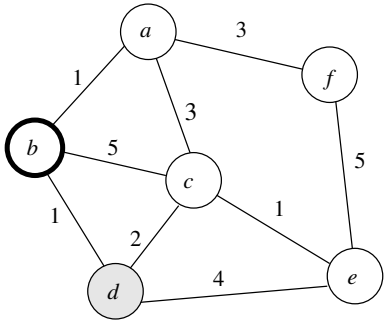
An undirected graph and its minimum spanning tree.

Minimum Spanning Tree: Prim's Algorithm

- Prim's algorithm for finding an MST is a greedy algorithm.
- Start by selecting an arbitrary vertex, include it into the current MST.
- Grow the current MST by inserting into it the vertex closest to one of the vertices already in current MST.

Minimum Spanning Tree: Prim's Algorithm

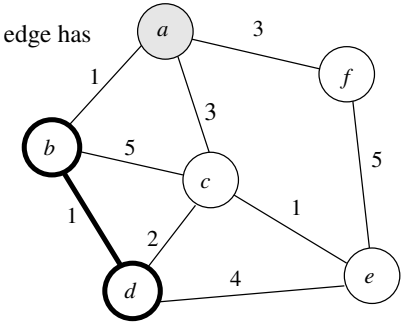
(a) Original graph



	a	b	c	d	e	f
d[]	1	0	5	1	∞	∞

a	0	1	3	∞	∞	3
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

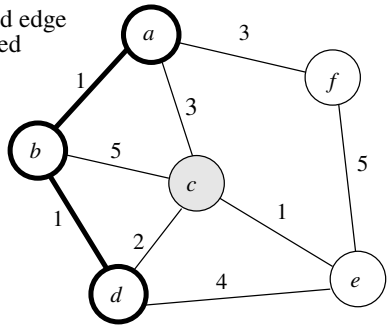
(b) After the first edge has been selected



	a	b	c	d	e	f
d[]	1	0	2	1	4	∞

a	0	1	3	∞	∞	3
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

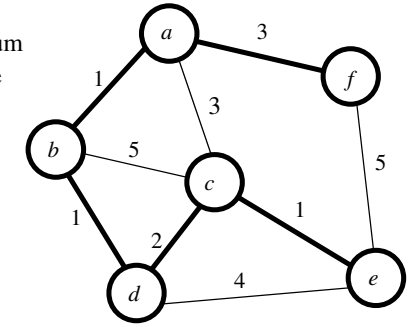
(c) After the second edge has been selected



	a	b	c	d	e	f
d[]	1	0	2	1	4	3

a	0	1	3	∞	∞	3
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

(d) Final minimum spanning tree



	a	b	c	d	e	f
d[]	1	0	2	1	1	3

a	0	1	3	∞	∞	3
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

Prim’s minimum spanning tree algorithm.

Minimum Spanning Tree: Prim's Algorithm

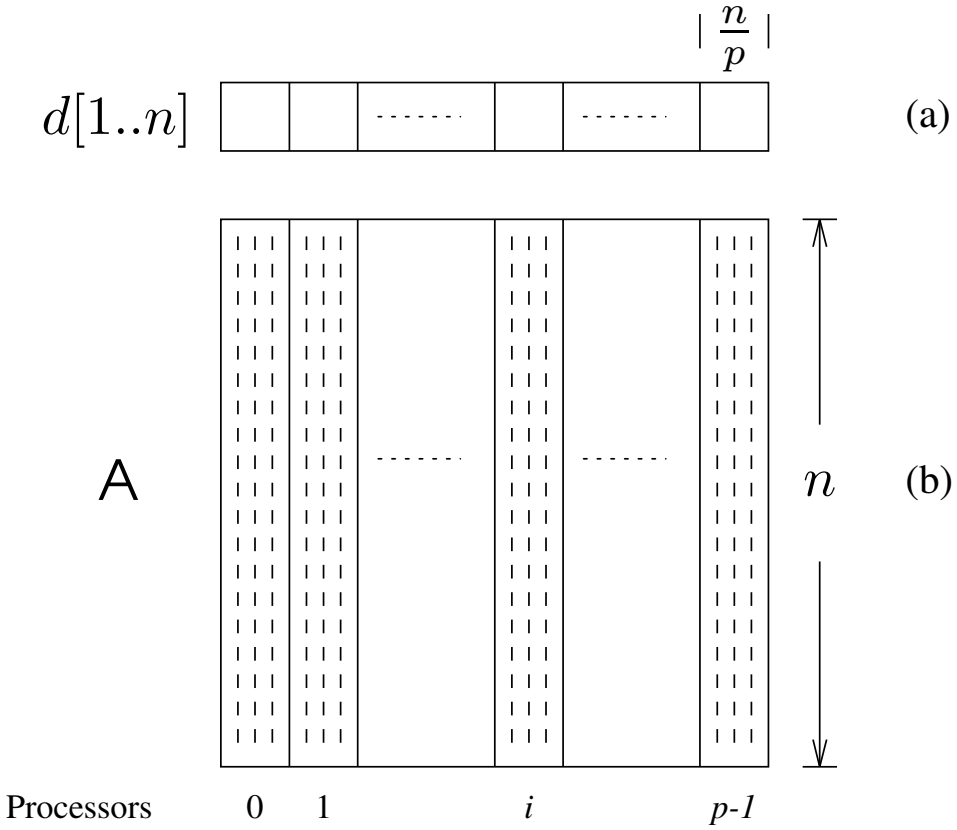
```
1.    procedure PRIM_MST(V, E, w, r)
2.    begin
3.        V_T := {r};
4.        d[r] := 0;
5.        for all v ∈ (V     V_T) do
6.            if edge (r, v) exists set d[v] := w(r, v);
7.            else set d[v] := ∞;
8.        while V_T ≠ V do
9.        begin
10.           find a vertex u such that d[u] := min{d[v]|v ∈ (V     V_T)};
11.           V_T := V_T ∪ {u};
12.           for all v ∈ (V     V_T) do
13.               d[v] := min{d[v], w(u, v)};
14.       endwhile
15.   end PRIM_MST
```

Prim's sequential minimum spanning tree algorithm.

Prim's Algorithm: Parallel Formulation

- The algorithm works in n outer iterations – it is hard to execute these iterations concurrently.
- The inner loop is relatively easy to parallelize. Let p be the number of processes, and let n be the number of vertices.
- The adjacency matrix is partitioned in a 1-D block fashion, with distance vector d partitioned accordingly.
- In each step, a processor selects the locally closest node. followed by a global reduction to select globally closest node.
- This node is inserted into MST, and the choice broadcast to all processors.
- Each processor updates its part of the d vector locally.

Prim's Algorithm: Parallel Formulation



The partitioning of the distance array d and the adjacency matrix A among p processes.

Prim's Algorithm: Parallel Formulation

- The cost to select the minimum entry is $O(n/p + \log p)$.
- The cost of a broadcast is $O(\log p)$.
- The cost of local updation of the d vector is $O(n/p)$.
- The parallel time per iteration is $O(n/p + \log p)$.
- The total parallel time is given by $O(n^2/p + n \log p)$.
- The corresponding isoefficiency is $O(p^2 \log^2 p)$.

Single-Source Shortest Paths

- For a weighted graph $G = (V, E, w)$, the *single-source shortest paths* problem is to find the shortest paths from a vertex $v \in V$ to all other vertices in V.

- Dijkstra's algorithm is similar to Prim's algorithm. It maintains a set of nodes for which the shortest paths are known.

- It grows this set based on the node closest to source using one of the nodes in the current shortest path set.

Single-Source Shortest Paths: Dijkstra's Algorithm

```
1.    procedure DIJKSTRA_SINGLE_SOURCE_SP(V, E, w, s)
2.    begin
3.        V_T := {s};
4.        for all v ∈ (V  V_T) do
5.            if (s, v) exists set l[v] := w(s, v);
6.            else set l[v] := ∞;
7.        while V_T ≠ V do
8.        begin
9.            find a vertex u such that l[u] := min{l[v]|v ∈ (V  V_T)};
10.           V_T := V_T ∪ {u};
11.           for all v ∈ (V  V_T) do
12.               l[v] := min{l[v], l[u] + w(u, v)};
13.       endwhile
14.   end  DIJKSTRA_SINGLE_SOURCE_SP
```

Dijkstra's sequential single-source shortest paths algorithm.

Dijkstra's Algorithm: Parallel Formulation

- Very similar to the parallel formulation of Prim's algorithm for minimum spanning trees.
- The weighted adjacency matrix is partitioned using the 1-D block mapping.
- Each process selects, locally, the node closest to the source, followed by a global reduction to select next node.
- The node is broadcast to all processors and the l-vector updated.
- The parallel performance of Dijkstra's algorithm is identical to that of Prim's algorithm.

All-Pairs Shortest Paths

- Given a weighted graph $G(V, E, w)$, the *all-pairs shortest paths* problem is to find the shortest paths between all pairs of vertices $v_i, v_j \in V$.

- A number of algorithms are known for solving this problem.

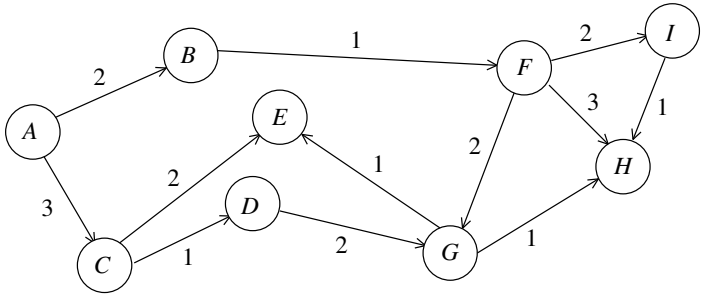
All-Pairs Shortest Paths: Matrix-Multiplication Based Algorithm

- Consider the multiplication of the weighted adjacency matrix with itself – except, in this case, we replace the multiplication operation in matrix multiplication by addition, and the addition operation by minimization.

- Notice that the product of weighted adjacency matrix with itself returns a matrix that contains shortest paths of length 2 between any pair of nodes.

- It follows from this argument that A^n contains all shortest paths.

Matrix-Multiplication Based Algorithm



$$A^1 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & 0 & 1 & 2 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & 2 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & \infty & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^8 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

Matrix-Multiplication Based Algorithm

- A^n is computed by doubling powers – i.e., as A, A^2, A^4, A^8, and so on.
- We need $\log n$ matrix multiplications, each taking time $O(n^3)$.
- The serial complexity of this procedure is $O(n^3 \log n)$.
- This algorithm is not optimal, since the best known algorithms have complexity $O(n^3)$.

Matrix-Multiplication Based Algorithm: Parallel Formulation

- Each of the $\log n$ matrix multiplications can be performed in parallel.
- We can use $n^3/\log n$ processors to compute each matrix-matrix product in time $\log n$.
- The entire process takes $O(\log^2 n)$ time.

Dijkstra's Algorithm

- Execute n instances of the single-source shortest path problem, one for each of the n source vertices.

- Complexity is $O(n^3)$.

Dijkstra's Algorithm: Parallel Formulation

- Two parallelization strategies – execute each of the n shortest path problems on a different processor (source partitioned), or use a parallel formulation of the shortest path problem to increase concurrency (source parallel).

Dijkstra's Algorithm: Source Partitioned Formulation

- Use n processors, each processor P_i finds the shortest paths from vertex v_i to all other vertices by executing Dijkstra's sequential single-source shortest paths algorithm.

- It requires no interprocess communication (provided that the adjacency matrix is replicated at all processes).

- The parallel run time of this formulation is: $\Theta(n^2)$.

- While the algorithm is cost optimal, it can only use n processors. Therefore, the isoefficiency due to concurrency is p^3.

Dijkstra's Algorithm: Source Parallel Formulation

- In this case, each of the shortest path problems is further executed in parallel. We can therefore use up to n^2 processors.

- Given p processors ($p > n$), each single source shortest path problem is executed by p/n processors.

- Using previous results, this takes time:

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n \log p).}^{\text{communication}} \tag{1}$$

- For cost optimality, we have $p = O(n^2/\log n)$ and the isoefficiency is $\Theta((p \log p)^{1.5})$.

Floyd's Algorithm

- For any pair of vertices $v_i, v_j \in V$, consider all paths from v_i to v_j whose intermediate vertices belong to the set $\{v_1, v_2, \ldots, v_k\}$. Let $p_{i,j}^{(k)}$ (of weight $d_{i,j}^{(k)}$ be the minimum-weight path among them.

- If vertex v_k is not in the shortest path from v_i to v_j, then $p_{i,j}^{(k)}$ is the same as $p_{i,j}^{(k \ \ 1)}$.

- If f v_k is in $p_{i,j}^{(k)}$, then we can break $p_{i,j}^{(k)}$ into two paths – one from v_i to v_k and one from v_k to v_j. Each of these paths uses vertices from $\{v_1, v_2, \ldots, v_{k \ \ 1}\}$.

Floyd's Algorithm

From our observations, the following recurrence relation follows:

$$d_{i,j}^{(k)} = \begin{cases} w(v_i, v_j) & \text{if } k = 0 \\ \min\left\{d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}\right\} & \text{if } k \geq 1 \end{cases} \tag{2}$$

This equation must be computed for each pair of nodes and for $k = 1, n$. The serial complexity is $O(n^3)$.

Floyd's Algorithm

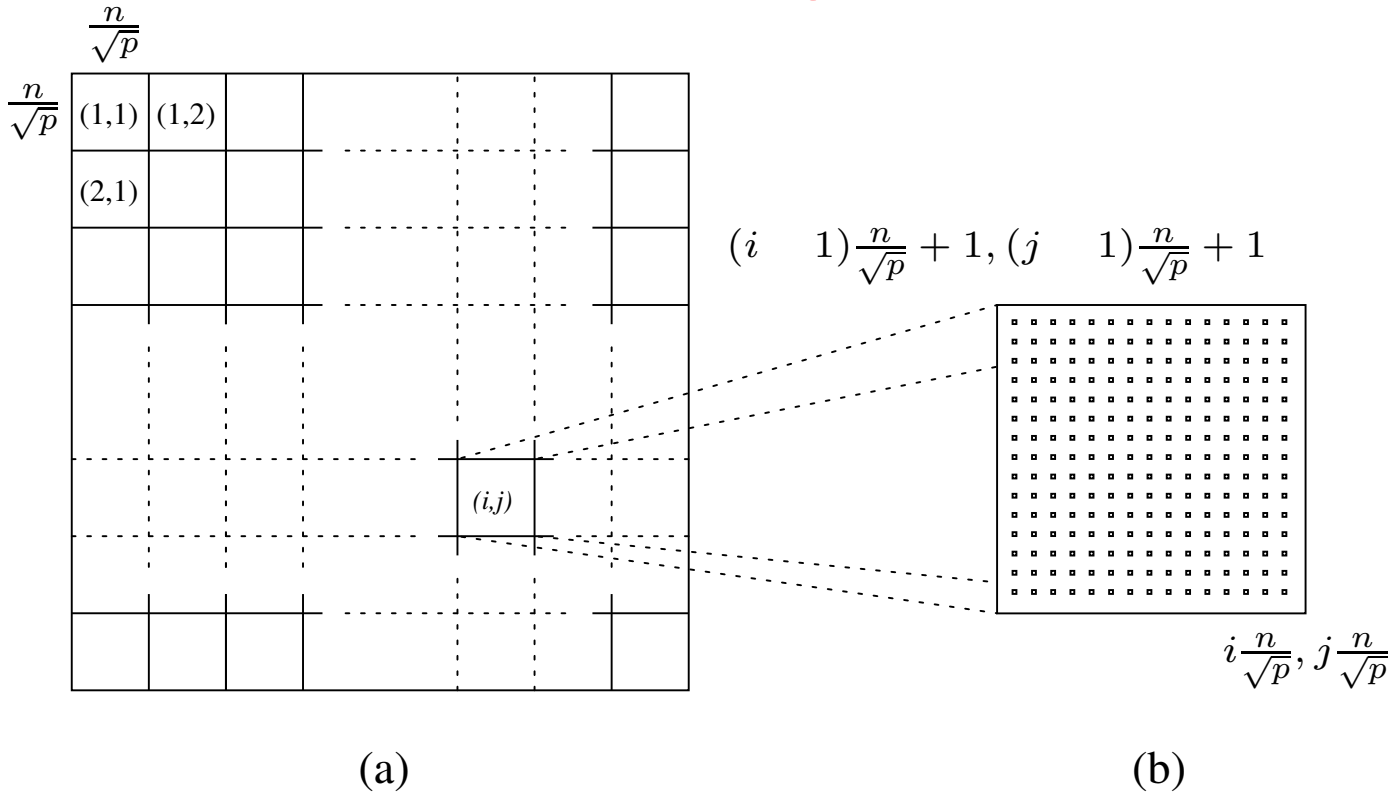
```
1.    procedure FLOYD_ALL_PAIRS_SP(A)
2.    begin
3.        D^(0) = A;
4.        for k := 1 to n do
5.            for i := 1 to n do
6.                for j := 1 to n do
7.                    d_{i,j}^(k) := min( d_{i,j}^(k-1), d_{i,k}^(k-1) + d_{k,j}^(k-1) );
8.    end FLOYD_ALL_PAIRS_SP
```

Floyd's all-pairs shortest paths algorithm. This program computes the all-pairs shortest paths of the graph $G = (V, E)$ with adjacency matrix A.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

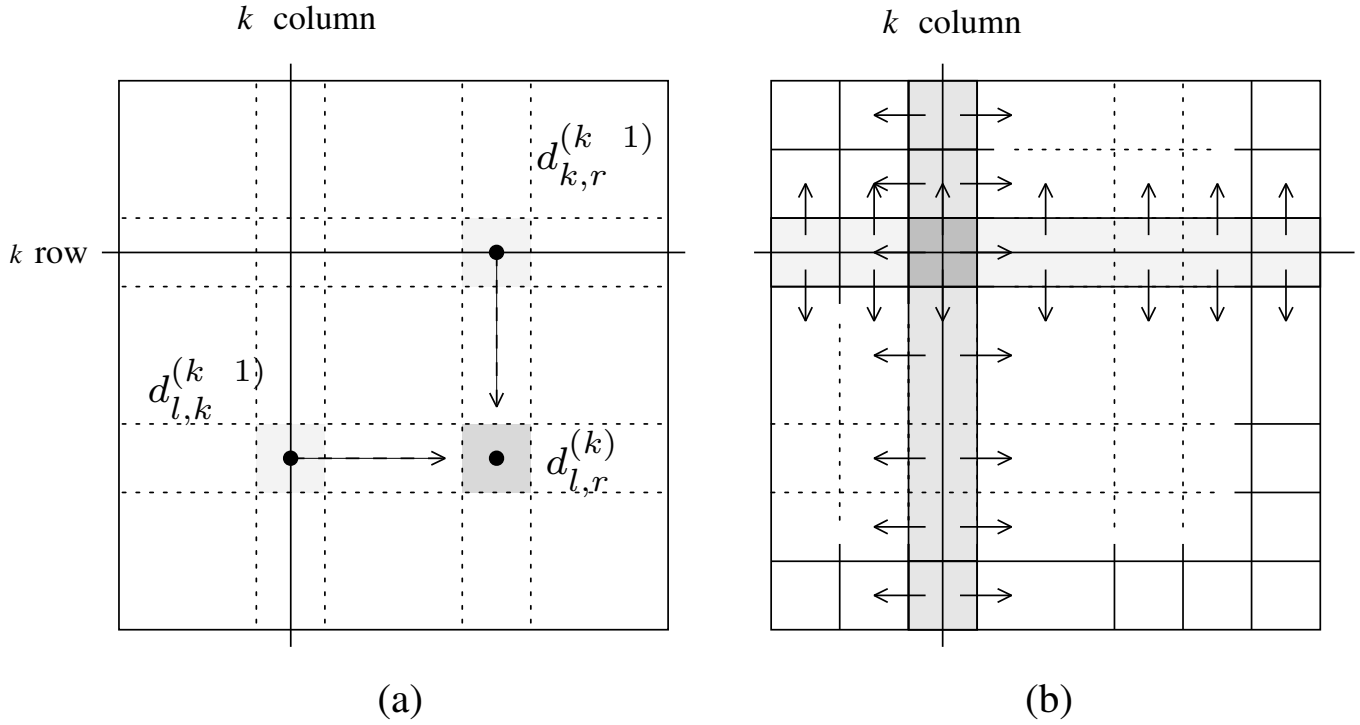
- Matrix $D^{(k)}$ is divided into p blocks of size $(n/\sqrt{p}) \times (n/\sqrt{p})$.
- Each processor updates its part of the matrix during each iteration.
- To compute $d_{l,r}^{(k)}$ processor $P_{i,j}$ must get $d_{l,k}^{(k-1)}$ and $d_{k,r}^{(k-1)}$.
- In general, during the k^{th} iteration, each of the $\sqrt{p}$ processes containing part of the k^{th} row send it to the $\sqrt{p} - 1$ processes in the same column.
- Similarly, each of the $\sqrt{p}$ processes containing part of the k^{th} column sends it to the $\sqrt{p} - 1$ processes in the same row.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping



(a) Matrix $D^{(k)}$ distributed by 2-D block mapping into $\sqrt{p} \times \sqrt{p}$ subblocks, and (b) the subblock of $D^{(k)}$ assigned to process $P_{i,j}$.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping



(a) Communication patterns used in the 2-D block mapping. When computing $d_{i,j}^{(k)}$, information must be sent to the highlighted process from two other processes along the same row and column. (b) The row and column of $\sqrt{p}$ processes that contain the k^{th} row and column send them along process columns and rows.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

```
1.    procedure FLOYD_2DBLOCK(D^(0))
2.    begin
3.        for k := 1 to n do
4.        begin
5.            each process P_{i,j} that has a segment of the k^th row of D^(k-1);
                  broadcasts it to the P_{*,j} processes;
6.            each process P_{i,j} that has a segment of the k^th column of D^(k-1);
                  broadcasts it to the P_{i,*} processes;
7.            each process waits to receive the needed segments;
8.            each process P_{i,j} computes its part of the D^(k) matrix;
9.        end
10.   end  FLOYD_2DBLOCK
```

Floyd's parallel formulation using the 2-D block mapping. $P_{*,j}$ denotes all the processes in the j^{th} column, and $P_{i,*}$ denotes all the processes in the i^{th} row. The matrix $D^{(0)}$ is the adjacency matrix.

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

- During each iteration of the algorithm, the k^{th} row and k^{th} column of processors perform a one-to-all broadcast along their rows/columns.

- The size of this broadcast is $n/\sqrt{p}$ elements, taking time $\Theta((n \log p)/\sqrt{p})$.

- The synchronization step takes time $\Theta(\log p)$.

- The computation time is $\Theta(n^2/p)$.

- The parallel run time of the 2-D block mapping formulation of Floyd's algorithm is

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta\left(\frac{n^2}{\sqrt{p}} \log p\right)}^{\text{communication}}.$$

Floyd's Algorithm: Parallel Formulation Using 2-D Block Mapping

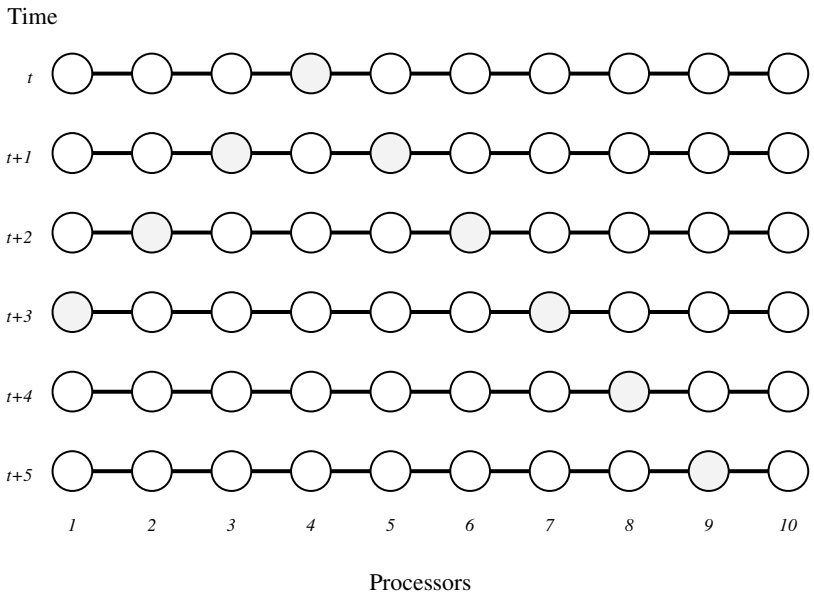
- The above formulation can use $O(n^2/\log^2 n)$ processors cost-optimally.
- The isoefficiency of this formulation is $\Theta(p^{1.5}\log^3 p)$.
- This algorithm can be further improved by relaxing the strict synchronization after each iteration.

Floyd's Algorithm: Speeding Things Up by Pipelining

- The synchronization step in parallel Floyd's algorithm can be removed without affecting the correctness of the algorithm.

- A process starts working on the k^{th} iteration as soon as it has computed the $(k \quad 1)^{th}$ iteration and has the relevant parts of the $D^{(k \quad 1)}$ matrix.

Floyd's Algorithm: Speeding Things Up by Pipelining



Communication protocol followed in the pipelined 2-D block mapping formulation of Floyd's algorithm. Assume that process 4 at time t has just computed a segment of the k^{th} column of the $D^{(k \quad 1)}$ matrix. It sends the segment to processes 3 and 5. These processes receive the segment at time $t + 1$ (where the time unit is the time it takes for a matrix segment to travel over the communication link between adjacent processes). Similarly, processes farther away from process 4 receive the segment later. Process 1 (at the boundary) does not forward the segment after receiving it.

Floyd's Algorithm: Speeding Things Up by Pipelining

- In each step, $n/\sqrt{p}$ elements of the first row are sent from process $P_{i,j}$ to $P_{i+1,j}$.
- Similarly, elements of the first column are sent from process $P_{i,j}$ to process $P_{i,j+1}$.
- Each such step takes time $\Theta(n/\sqrt{p})$.
- After $\Theta(\sqrt{p})$ steps, process $P_{\sqrt{p},\sqrt{p}}$ gets the relevant elements of the first row and first column in time $\Theta(n)$.
- The values of successive rows and columns follow after time $\Theta(n^2/p)$ in a pipelined mode.
- Process $P_{\sqrt{p},\sqrt{p}}$ finishes its share of the shortest path computation in time $\Theta(n^3/p) + \Theta(n)$.
- When process $P_{\sqrt{p},\sqrt{p}}$ has finished the $(n \quad 1)^{th}$ iteration, it sends the relevant values of the n^{th} row and column to the other processes.

Floyd's Algorithm: Speeding Things Up by Pipelining

- The overall parallel run time of this formulation is

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n).}^{\text{communication}}$$

- The pipelined formulation of Floyd's algorithm uses up to $O(n^2)$ processes efficiently.

- The corresponding isoefficiency is $\Theta(p^{1.5})$.

All-pairs Shortest Path: Comparison

The performance and scalability of the all-pairs shortest paths algorithms on various architectures with $O(p)$ bisection bandwidth. Similar run times apply to all k d cube architectures, provided that processes are properly mapped to the underlying processors.

	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Dijkstra source-partitioned	$\Theta(n)$	$\Theta(n^2)$	$\Theta(p^3)$
Dijkstra source-parallel	$\Theta(n^2/\log n)$	$\Theta(n \log n)$	$\Theta((p \log p)^{1.5})$
Floyd 1-D block	$\Theta(n/\log n)$	$\Theta(n^2 \log n)$	$\Theta((p \log p)^3)$
Floyd 2-D block	$\Theta(n^2/\log^2 n)$	$\Theta(n \log^2 n)$	$\Theta(p^{1.5} \log^3 p)$
Floyd pipelined 2-D block	$\Theta(n^2)$	$\Theta(n)$	$\Theta(p^{1.5})$

Transitive Closure

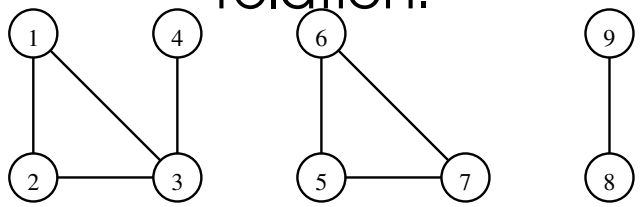
- If $G = (V, E)$ is a graph, then the /em transitive closure of G is defined as the graph $G^* = (V, E^*)$, where $E^* = \{(v_i, v_j) |$ there is a path from v_i to v_j in $G\}$.

- The /em connectivity matrix of G is a matrix $A^* = (a^*_{i,j})$ such that $a^*_{i,j} = 1$ if there is a path from v_i to v_j or $i = j$, and $a^*_{i,j} = \infty$ otherwise.

- To compute A^* we assign a weight of 1 to each edge of E and use any of the all-pairs shortest paths algorithms on this weighted graph.

Connected Components

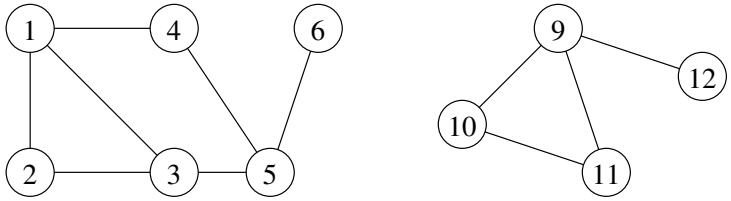
The connected components of an undirected graph are the equivalence classes of vertices under the "is reachable from" relation.



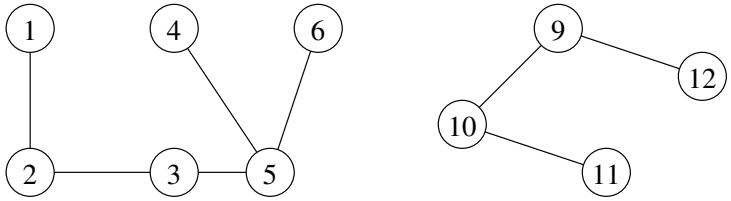
A graph with three connected components: $\{1, 2, 3, 4\}$, $\{5, 6, 7\}$, and $\{8, 9\}$.

Connected Components: Depth-First Search Based Algorithm

Perform DFS on the graph to get a forest – eac tree in the forest corresponds to a separate connected component.



(a)



(b)

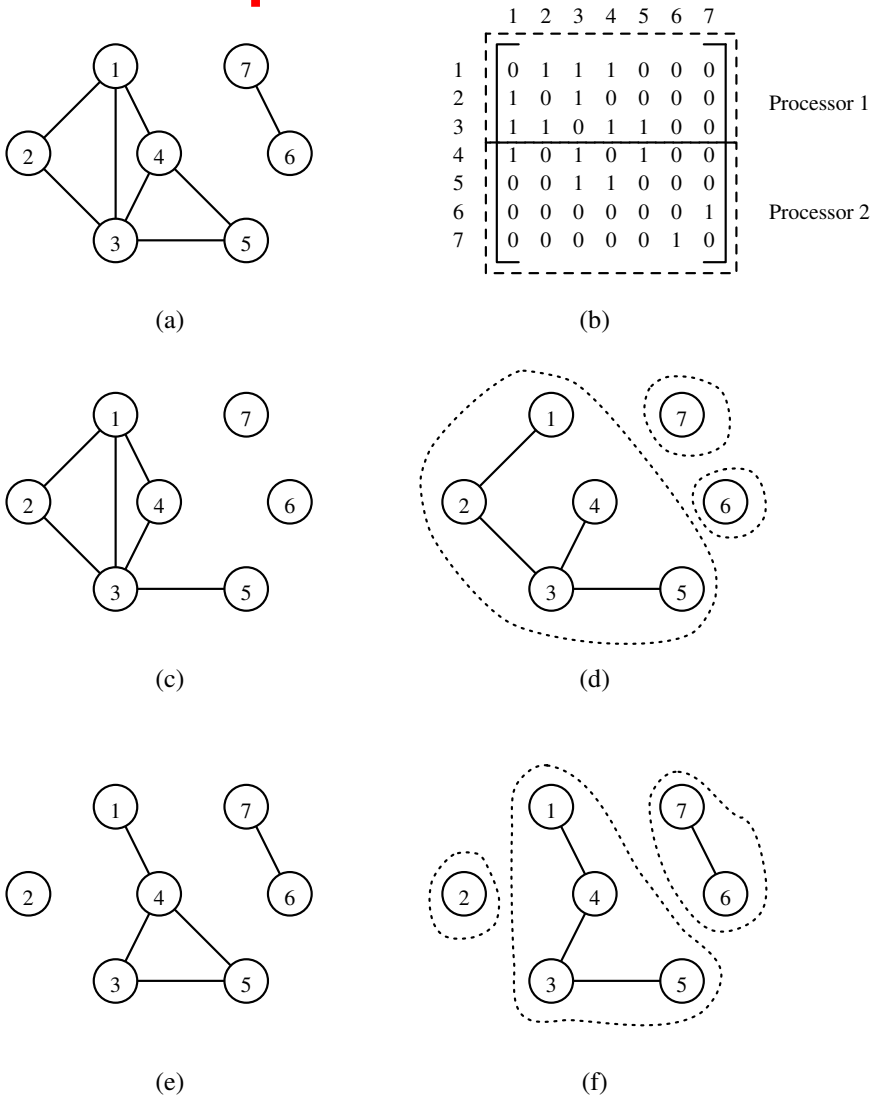
Part (b) is a depth-first forest obtained from depth-first traversal of the graph in part (a). Each of these trees is a connected component of the graph in part (a).

Connected Components: Parallel Formulation

- Partition the graph across processors and run independent connected component algorithms on each processor. At this point, we have p spanning forests.

- In the second step, spanning forests are merged pairwise until only one spanning forest remains.

Connected Components: Parallel Formulation



Computing connected components in parallel. The adjacency matrix of the graph G in (a) is partitioned into two parts (b). Each process gets a subgraph of G ((c) and (e)). Each process then computes the spanning forest of the subgraph ((d) and (f)). Finally, the two spanning trees are merged to form the solution.

Connected Components: Parallel Formulation

- To merge pairs of spanning forests efficiently, the algorithm uses disjoint sets of edges.

- We define the following operations on the disjoint sets:

  find(x) returns a pointer to the representative element of the set containing x. Each set has its own unique representative.
  union(x, y) unites the sets containing the elements x and y. The two sets are assumed to be disjoint prior to the operation.

Connected Components: Parallel Formulation

- For merging forest A into forest B, for each edge (u, v) of A, a find operation is performed to determine if the vertices are in the same tree of B.
- If not, then the two trees (sets) of B containing u and v are united by a union operation.
- Otherwise, no union operation is necessary.
- Hence, merging A and B requires at most $2(n \quad 1)$ find operations and $(n \quad 1)$ union operations.

Connected Components: Parallel 1-D Block Mapping

- The $n \times n$ adjacency matrix is partitioned into p blocks.
- Each processor can compute its local spanning forest in time $\Theta(n^2/p)$.
- Merging is done by embedding a logical tree into the topology. There are $\log p$ merging stages, and each takes time $\Theta(n)$. Thus, the cost due to merging is $\Theta(n \log p)$.
- During each merging stage, spanning forests are sent between nearest neighbors. Recall that $\Theta(n)$ edges of the spanning forest are transmitted.

Connected Components: Parallel 1-D Block Mapping

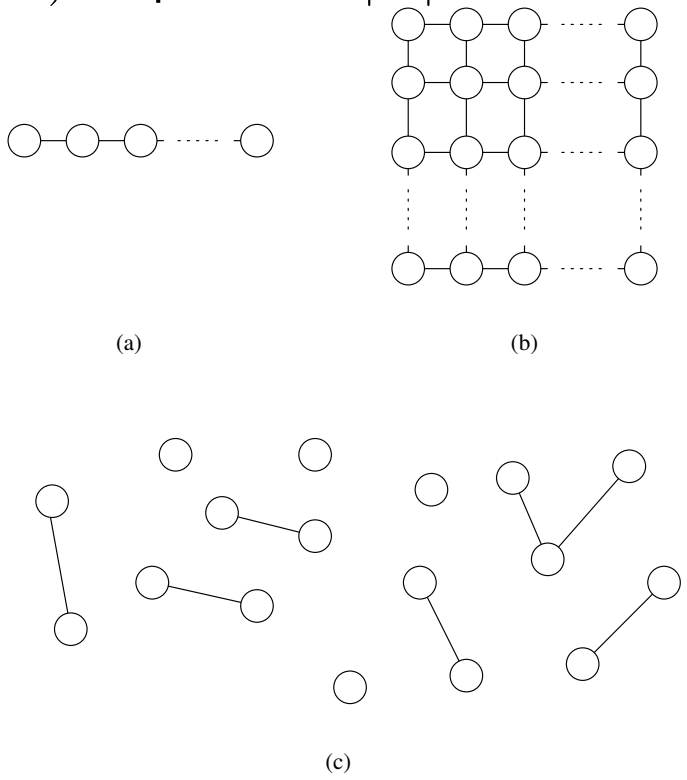
- The parallel run time of the connected-component algorithm is

$$T_P = \overbrace{\Theta\left(\frac{n^2}{p}\right)}^{\text{local computation}} + \overbrace{\Theta(n \log p).}^{\text{forest merging}}$$

- For a cost-optimal formulation $p = O(n/\log n)$. The corresponding isoefficiency is $\Theta(p^2 \log^2 p)$.

Algorithms for Sparse Graphs

A graph $G = (V, E)$ is sparse if $|E|$ is much smaller than $|V|^2$.



Examples of sparse graphs: (a) a linear graph, in which each vertex has two incident edges; (b) a grid graph, in which each vertex has four incident vertices; and (c) a random sparse graph.

Algorithms for Sparse Graphs

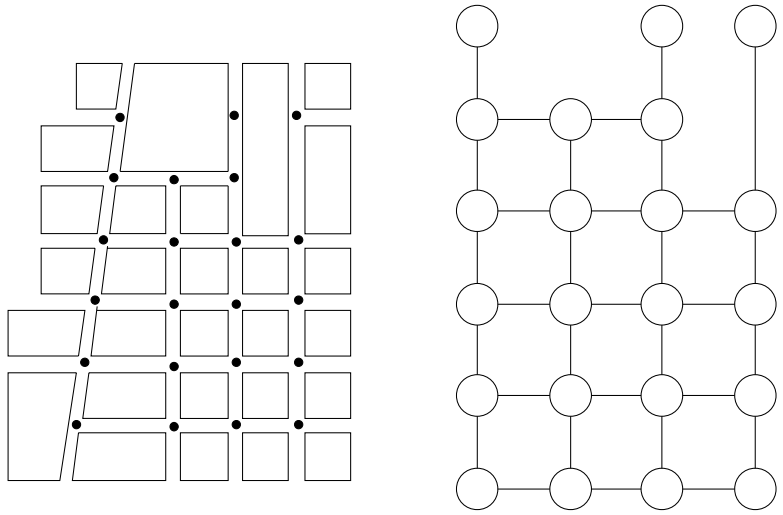
- Dense algorithms can be improved significantly if we make use of the sparseness. For example, the run time of Prim's minimum spanning tree algorithm can be reduced from $\Theta(n^2)$ to $\Theta(|E| \log n)$.

- Sparse algorithms use adjacency list instead of an adjacency matrix.

- Partitioning adjacency lists is more difficult for sparse graphs – do we balance number of vertices or edges?

- Parallel algorithms typically make use of graph structure or degree information for performance.

Algorithms for Sparse Graphs

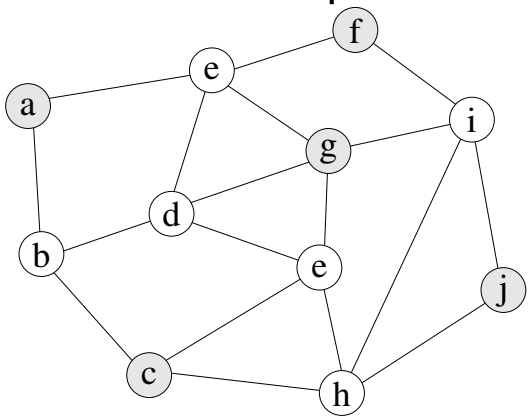


(a) (b)

A street map (a) can be represented by a graph (b). In the graph shown in (b), each street intersection is a vertex and each edge is a street segment. The vertices of (b) are the intersections of (a) marked by dots.

Finding a Maximal Independent Set

A set of vertices $I \subset V$ is called /em independent if no pair of vertices in I is connected via an edge in G. An independent set is called /em maximal if by including any other vertex not in I, the independence property is violated.



{a, d, i, h} is an independent set

{a, c, j, f, g} is a maximal independent set

{a, d, h, f} is a maximal independent set

Examples of independent and maximal independent sets.

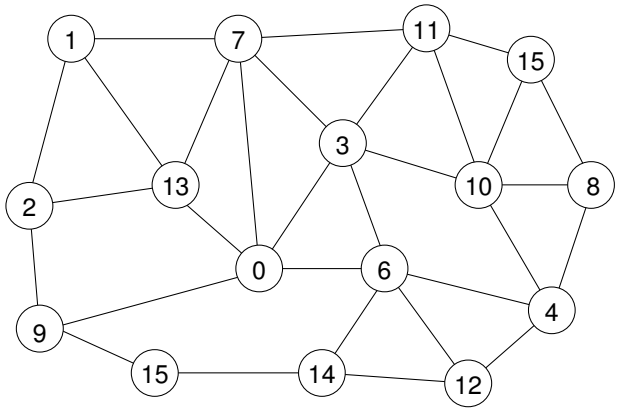
Finding a Maximal Independent Set (MIS)

- Simple algorithms start by MIS I to be empty, and assigning all vertices to a candidate set C.
- Vertex v from C is moved into I and all vertices adjacent to v are removed from C.
- This process is repeated until C is empty.
- This process is inherently serial!

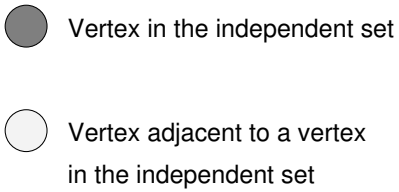
Finding a Maximal Independent Set (MIS)

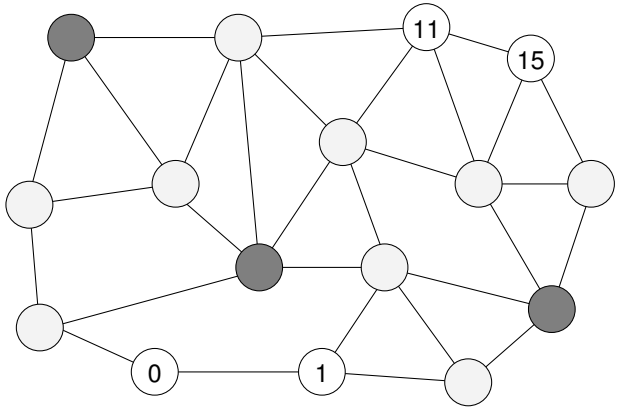
- Parallel MIS algorithms use randimization to gain concurrency (Luby's algorithm for graph coloring).
- Initially, each node is in the candidate set C. Each node generates a (unique) random number and communicates it to its neighbors.
- If a nodes number exceeds that of all its neighbors, it joins set I. All of its neighbors are removed from C.
- This process continues until C is empty.
- On average, this algorithm converges after $O(\log |V|)$ such steps.

Finding a Maximal Independent Set (MIS)

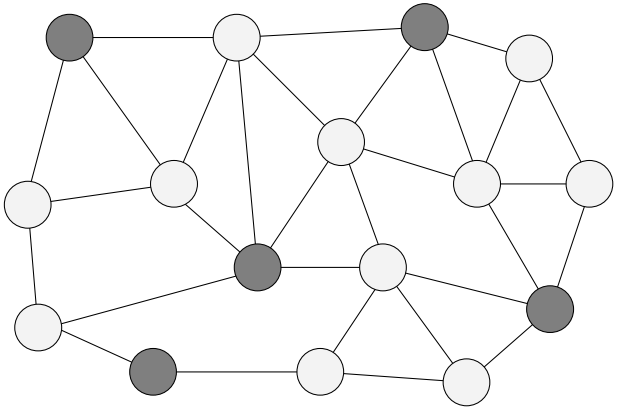


(a) After the 1st random number assignment





(b) After the 2nd random number assignment



(c) Final maximal independent set

The different augmentation steps of Luby's randomized maximal independent set algorithm. The numbers inside each vertex correspond to the random number assigned to the vertex.

Finding a Maximal Independent Set (MIS): Parallel Formulation

- We use three arrays, each of length n – I, which stores nodes in MIS, C, which stores the candidate set, and R, the random numbers.

- Partition C across p processors. Each processor generates the corresponding values in the R array, and from this, computes which candidate vertices can enter MIS.

- The C array is updated by deleting all the neighbors of vertices that entered MIS.

- The performance of this algorithm is dependent on the structure of the graph.

Single-Source Shortest Paths

- Dijkstra's algorithm, modified to handle sparse graphs is called Johnson's algorithm.

- The modification accounts for the fact that the minimization step in Dijkstra's algorithm needs to be performed only for those nodes adjacent to the previously selected nodes.

- Johnson's algorithm uses a priority queue Q to store the value $l[v]$ for each vertex $v \in (V \quad V_T)$.

Single-Source Shortest Paths: Johnson's Algorithm

```
1.    procedure JOHNSON_SINGLE_SOURCE_SP(V, E, s)
2.    begin
3.        Q := V;
4.        for all v ∈ Q do
5.            l[v] := ∞;
6.        l[s] := 0;
7.        while Q ≠ ∅ do
8.        begin
9.            u := extract_min(Q);
10.           for each v ∈ Adj[u] do
11.               if v ∈ Q and l[u] + w(u, v) < l[v] then
12.                   l[v] := l[u] + w(u, v);
13.       endwhile
14.   end JOHNSON_SINGLE_SOURCE_SP
```

Johnson's sequential single-source shortest paths algorithm.

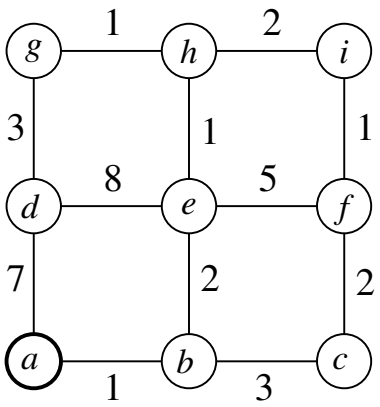
Single-Source Shortest Paths: Parallel Johnson's Algorithm

- Maintaining strict order of Johnson's algorithm generally leads to a very restrictive class of parallel algorithms.

- We need to allow exploration of multiple nodes concurrently. This is done by simultaneously extracting p nodes from the priority queue, updating the neighbors' cost, and augmenting the shortest path.

- If an error is made, it can be discovered (as a shorter path) and the node can be reinserted with this shorter path.

Single-Source Shortest Paths: Parallel Johnson's Algorithm



	Priority Queue	Array l[]
(1)	*b:1, d:7, c:inf, e:inf, f:inf, g:inf, h:inf, i:inf*	
(2)	*e:3, c:4, g:10, f:inf, h:inf, i:inf*	
(3)	*h:4, f:6, i:inf*	
(4)	*g:5, i:6*	

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
0	1	∞	7	∞	∞	∞	∞	∞
0	1	4	7	3	∞	10	∞	∞
0	1	4	7	3	6	10	4	∞
0	1	4	7	3	6	5	4	6

An example of the modified Johnson's algorithm for processing unsafe vertices concurrently.

Single-Source Shortest Paths: Parallel Johnson's Algorithm

- Even if we can extract and process multiple nodes from the queue, the queue itself is a major bottleneck.

- For this reason, we use multiple queues, one for each processor. Each processor builds its priority queue only using its own vertices.

- When process P_i extracts the vertex $u \in V_i$, it sends a message to processes that store vertices adjacent to u.

- Process P_j, upon receiving this message, sets the value of $l[v]$ stored in its priority queue to $\min\{l[v], l[u] + w(u, v)\}$.

Single-Source Shortest Paths: Parallel Johnson's Algorithm

- If a shorter path has been discovered to node v, it is reinserted back into the local priority queue.
- The algorithm terminates only when all the queues become empty.
- A number of node paritioning schemes can be used to exploit graph structure for performance.