New Models for Fault Tolerance:

Erasure Coded Computations

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Background and Motivation

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Distributed Fault Tolerant Linear System Solver

Adaptive Fault Tolerant Linear System Solver

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Erasure Coded Eigensolver

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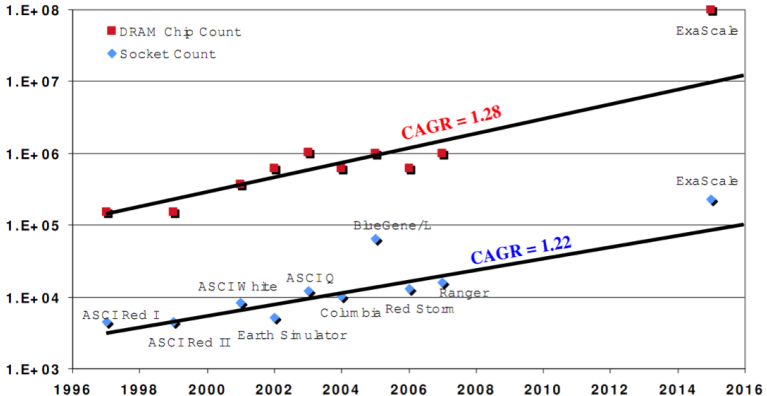
Faults in Parallel and Distributed System

- As parallel systems scale to millions of cores, faults become one of the most critical challenges.
- As data centers scale to hundreds of thousands of nodes, faults are a prime consideration for distributed computations.
- As networks scale from data center to wide area, network faults and partitions constitute a major consideration for wide area distributed computations.

Estimated Chip Counts in Exascale Systems



Source: *DARPA Exascale Technology Study [Kogge et al.]*

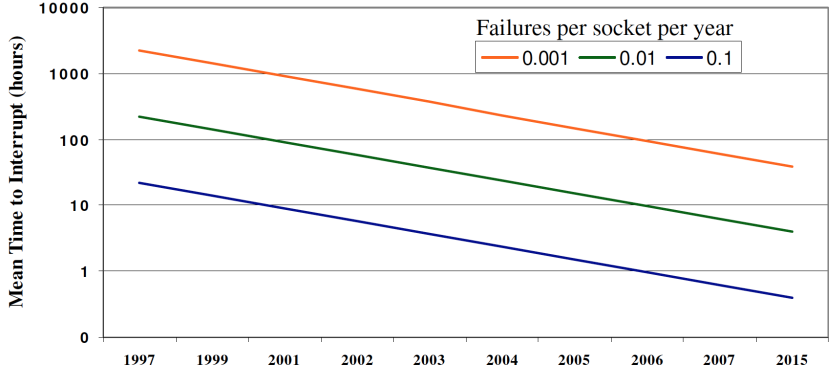
BlueGene Failure In Time (FIT) budget

Component	*FIT per component*[†]	*Components per 64Ki compute node partition*	*FITs per system* (K)	*Failure rate per week*
Control–FPGA complex	160	3,024	484	0.08
DRAM	5	608,256	3,041	0.51
Compute + I/O ASIC	20	66,560	1,331	0.22
Link ASIC	25	3,072	77	0.012
Clock chip	6.5	~1,200	8	0.0013
Nonredundant power supply	500	384	384	0.064
Total (65,536 compute nodes)			5,315	0.89

[†] T = 60°C, V = Nominal, 40K POH. *FIT* = Failures in ppm/KPOH. One FIT = 0.168×16^{-6} fails per week if the machine runs 24 hours a day.

Source: *P. COTEUS ET AL., IBM J. RES. & DEV. VOL. 49 NO. 2/3*

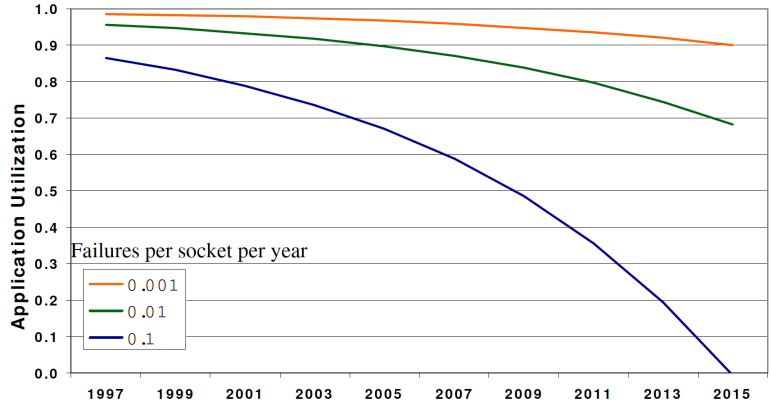
Scaling trends for environmental factors that affect resiliency



Source: *DARPA Exascale Technology Study [Kogge et al.]*

Application Utilization for checkpoint overheads

If one socket fails on average every 10 years, application utilization drops to 0 at 220K sockets!



Source: *DARPA Exascale Technology Study [Kogge et al.]*

Faults in Distributed Systems

- Profile and rate of faults in distributed systems is different.
- Disk, network, and system stack contribute significantly.
- The nature of faults is different as well – network partitions may render large parts of the system inaccessible.

Faults and Failure Models

Types of Faults

(i) Permanent; (ii) Transient; (iii) Intermittent.

Failure Model

Failure model is an abstraction of system behavior in the presence of a fault.

- Byzantine: a component can exhibit arbitrary and malicious behavior, perhaps involving collusion with other faulty components.
- Fail-stop: a component changes to a state that permits other components to detect the failure and then stops.

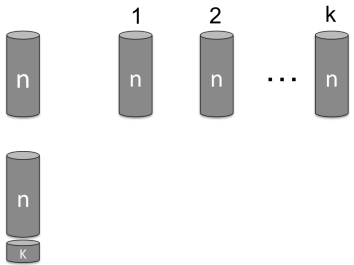
Fault Tolerance

Algorithm-based methods and System-supported methods.

- Algorithmic methods alter the algorithm to make it robust to faults.
- System-supported methods include checkpoint-restart, active replicas, and deterministic replay.
  - Checkpoint-restart schemes involve the overhead of consistent checkpointing and I/O.
  - Active replicas execute multiple replicas of each task.
  - Tasks in deterministic replay are scheduled at different execution units and monitored for successful completion. They are rescheduled at other execution units if failures are detected.

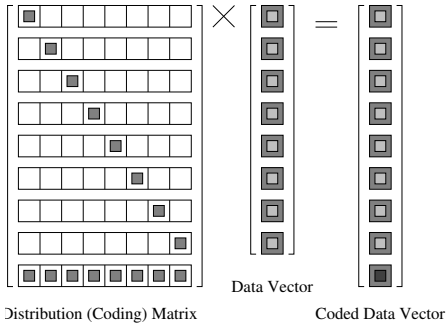
Fault Tolerant Storage--Replication v.s. Erasure Coding

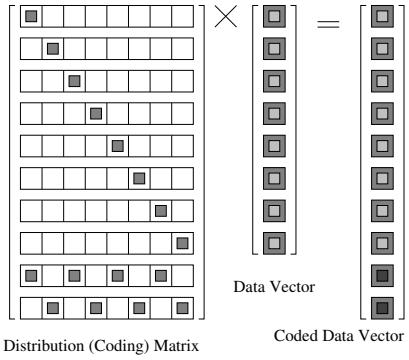
- Replication based schemes maintain as many copies of data as are needed to guarantee required tolerance. To tolerate $k-1$ disk failures on n data items (disks), total storage is nk.
- Erasure coding schemes transform the data so that the original data can be reconstructed from (a subset of) the available coded data. To tolerate $k-1$ disk failures (erasures) on n data items, total storage is $n+k$.



Erasure Coded Storage

Algebraic view of erasure coding:



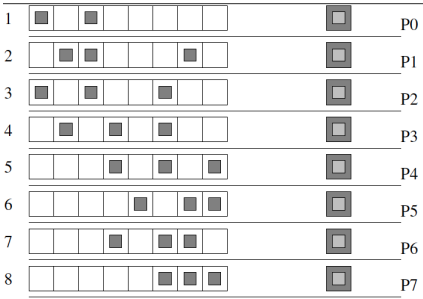


Some Notes on Erasure Coded Storage

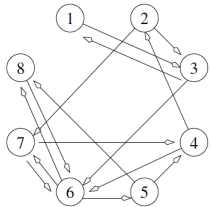
- All arithmetic must be performed over a finite field (solvers can become expensive)
- Coding and decoding require communication. Codes must consider sparsity and reconstruction cost.
- Several current systems use erasure coding: RAID 4/5 uses parity, RAID 6 uses a vanDerMonde coding block, CFS uses Reed-Solomon.

Erasure Coded Computation: Basic Kernels

Coding the result of a sparse matrix-vector product.



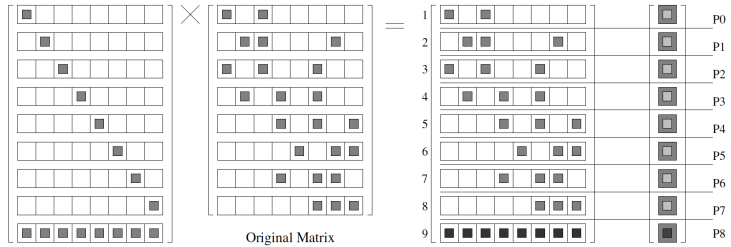
(a) Original Matrix



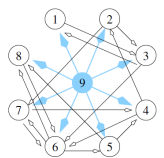
(b) Graph of Original Matrix

Erasure Coded Computation: Basic Kernels

Multiply the distribution/ coding matrix with the given sparse matrix. This results in an augmented matrix.



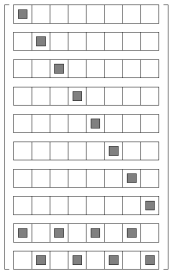
(c) Augmentation Process: Multiplication by a distribution/ coding matrix. Augmented matrix is tolerant to one node (row) failure.



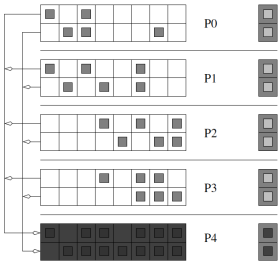
(d) Graph of augmented mat

Erasure Coded Computation Basic Kernels

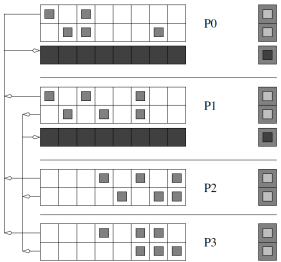
Example of an alternate distribution matrix, which allows us to control the fill in the augmentation rows. We also show the augmentation block distribution across processors.



Distribution (Coding) Matrix



Controlling density of augmented blocks by combining blocks (first row at each processor is combined into first row of the augmented block). Note the reduced fill. Matvec is still tolerant to one process failure.



Augmentation block is now distributed across processors. This addresses problem of load imbalance. Note higher communication cost of this scheme. This can be amortized through coarse–grained processor partitions.

Goals and Challenges

Goal

- Code the result of a computation in a fault tolerant manner (in storage or communication, the computation is an identity operator).
- Data is not a linear bit string or a block – but rather it is a sparse matrix.
- Recovery of result of computation must be easy.

Challenges

- Design suitable coding matrices.
- Reformulate traditional linear algebraic methods in the erasure coding framework.
- Design efficient recovery algorithms.
- Analyze parallel performance.
- Validate tolerance to different models of fault arrivals and rates.

Fault Oblivious Computation

The concept of fault oblivious parallel execution, based on Erasure Coding, works as follows:

- Augment the input to a parallel program.
- Execute on the augmented input in a faulty environment, oblivious to faults, and generate an augmented output.
- Compute the true output based on the augmented output from the faulty execution.

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Erasure Coded Linear System Solver

Given a linear system $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ ($\boldsymbol{A}_{n\times n}$ is *SPD*), with true solution $\mathbf{x}^*$, and coding matrix $\boldsymbol{E}_{n\times k}$, we construct the augmented system $\tilde{\boldsymbol{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

$$\underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{AE} \\ \boldsymbol{E}^T\boldsymbol{A} & \boldsymbol{E}^T\boldsymbol{AE} \end{bmatrix}}_{\tilde{\boldsymbol{A}}} \underbrace{\begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}}_{\tilde{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{b} \\ \boldsymbol{E}^T\mathbf{b} \end{bmatrix}}_{\tilde{\mathbf{b}}}$$

$\tilde{\mathbf{x}} = \begin{bmatrix}\mathbf{x}^*; 0\end{bmatrix}$ is a solution to the augmented system.

Erasure Coded Linear System Solver

Properties of $\tilde{\boldsymbol{A}}$:

- If $\boldsymbol{A}$ is *SPD*, then $\tilde{\boldsymbol{A}}$ is *SPSD*;
- The null space basis of $\tilde{\boldsymbol{A}}$ is $\begin{bmatrix} \boldsymbol{E} \\ -\boldsymbol{I}_k \end{bmatrix}$;
- Any solution of the augmented system can be written as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^* \\ 0 \end{bmatrix} + a \begin{bmatrix} \boldsymbol{E} \\ -\boldsymbol{I}_k \end{bmatrix}$$

Theorem

Based on the properties of $\tilde{A}$, we can recover the true solution as follows:

$$\begin{bmatrix} \mathbf{x}^* \\ 0 \end{bmatrix} = \tilde{\mathbf{x}} + \begin{bmatrix} \boldsymbol{E} \\ -\boldsymbol{I}_k \end{bmatrix} \mathbf{r} \tag{1}$$

Erasure Coded Linear System Solver

In the presence of faults, we can rewrite the augmented system in the following form:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{Z}_1 \\ \boldsymbol{A}_{12}{}^T & \boldsymbol{A}_{22} & \boldsymbol{Z}_2 \\ \boldsymbol{Z}_1{}^T & \boldsymbol{Z}_2{}^T & \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{f} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \boldsymbol{E}^T b \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{Z}_1 \\ \boldsymbol{Z}_1^T & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \boldsymbol{E}^T b \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}_{12} \\ \boldsymbol{Z}_2^T \end{bmatrix} \mathbf{f}$$

Theorem

If $[c; r]$ is a solution to the reorganized system, then $\tilde{\mathbf{x}} = [c; f; r]$ is a solution to the augmented system.

Sufficient Condition on the Encoding Matrix

THEOREM

Let E^T be the $k \times n$ encoding matrix. If E^T has Kruskal rank k, then for any f such that $\text{card}(f) \leq k$ there exist c and r such that a solution to the encoded system is $[c^T, f^T, r^T]^T$. Furthermore, any c and r output by the fault oblivious computation satisfy recovery conditions.

Kruskal rank k the largest k such that any k columns are linearly independent.

Coding Matrix

Conditions on matrix E:

- There is always a solution to the augmented system for faults happening on any set of rows, as long as total number of faults $\leq k$.
- Given any solution computed with faulty components, we can extract and recover a solution for the original system.

Desiderata of E:

- Satisfy properties of Kruskal rank, which means every subset of k rows of matrix E is linearly independent.
- Be as sparse as possible to minimize the fill in the augmented matrix.

Coding Matrix

Definition

An $n \times k$ matrix E satisfies the recovery-at-random property if a random subset of k rows (selected uniformly with replacement) is rank k with probability approaching 1.

Proposed Coding Matrix E

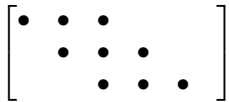
$$\begin{bmatrix} \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ \bullet & 0 & 0 & 0 & \bullet & \bullet \\ \bullet & \bullet & 0 & 0 & 0 & \bullet \end{bmatrix}$$

Coding Matrix

Proposition

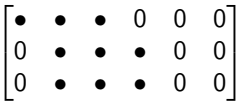
Let p be the number of nonzeros per row in E and Let E' be a submatrix of E formed by selecting any p rows of matrix E. The matrix E'^T has rank p.

All rows have distinct non-zero structure.



All rows have same non-zero structure.

Some rows from case1 and some from case2.



Coding Matrix

Theorem

The probability that a randomly chosen set of k rows from the matrix E are linearly dependent is less than $\left(\frac{e}{p+1}\right)^{p+1}$.

Proof: A sufficient condition for k rows to be linearly dependent is that some selection of $p+1$ rows from these k rows have the same non-zero structure. There are k distinct non-zeros structures for the matrix E.

$$\binom{k}{p+1}\left(\frac{1}{k}\right)^{p+1} \leq \left(\frac{e}{p+1}\right)^{p+1}.$$

As p increases, this probability rapidly approaches 0, which means that the matrix E is recovery-at-random.

Coding Matrix

To keep the coding matrix and the associated augmented matrix sparse, p should as small as possible.

Theorem

The expected number of rows from among k randomly selected rows of matrix E that have same nonzero structure is $O\left(\frac{\ln k}{\ln \ln k}\right)$.

Define a random variable M to be the number of rows that have the same non-zero structure when we select k rows uniformly at random from the matrix E.

$$Pr(M = t) = \binom{k}{1}\binom{k}{t}\left(\frac{1}{k}\right)^t\left(1 - \frac{1}{k}\right)^{k-t} \leq k\left(\frac{e}{t}\right)^t$$

Coding Matrix

The expected number of rows $E(M)$ is given by:

$$
\begin{aligned}
E(M) &= \sum_{t=1}^{k} t \cdot Pr(M = t) \\
&= \sum_{t=1}^{c\frac{\ln k}{\ln\ln k}} t \cdot Pr(M = t) + \sum_{t=\frac{c\ln k}{\ln\ln k}}^{k} t \cdot Pr(M = t) \\
&\leq \sum_{t=1}^{\frac{c\ln k}{\ln\ln k}} \frac{c\ln k}{\ln\ln k} \cdot Pr(M = t) + \sum_{t=\frac{c\ln k}{\ln\ln k}}^{k} k \cdot Pr(M = t) \\
&\leq \frac{c\ln k}{\ln\ln k} + k \cdot \frac{1}{k^{c/2-1}} \\
&= O(\frac{\ln k}{\ln\ln k})
\end{aligned}
\tag{2}
$$

Parallel Implementation

Since A is SPD and $\tilde{A}$ is SPSD, we can apply CG to $\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

Algorithm 1 Fault Oblivious CG with a Two-term Recurrence

1: Let x_0 be the initial guess and $r_0 = b - Ax_0$, $\beta_0 = 0$.
2: **for** $t = 0, 1, \ldots$ until convergence **do**
3: **if** Fault detected **then** $\beta_t = 0$ **else** $\beta_t = \dfrac{\|r_t\|_2^2}{\|r_{t-1}\|_2^2}$
4: $p_t = r_t + \beta_t p_{t-1}$
5: $q_t = Ap_t$
6: $\alpha_t = \dfrac{\|r_t\|_2^2}{\langle q_t, p_t \rangle}$
7: $x_{t+1} = x_t + \alpha_t p_t$
8: $r_{t+1} = r_t - \alpha_t q_t$

Parallel Implementation

Assume that each viable process can detect the breakdown of its neighbor processes.

- Inner products $\langle r_t, r_t \rangle$ and $\langle q_t, p_t \rangle$.

$$\begin{aligned} \langle \mathbf{r}_t, \mathbf{r}_t \rangle &= \langle (\mathbf{r}_t)_{[n+k]\setminus F_t}, (\mathbf{r}_t)_{[n+k]\setminus F_t} \rangle \\ \langle \mathbf{q}_t, \mathbf{p}_t \rangle &= \langle (\mathbf{q}_t)_{[n+k]\setminus F_t}, (\mathbf{p}_t)_{[n+k]\setminus F_t} \rangle \end{aligned} \tag{3}$$

- Matrix-vector multiplication $\mathbf{q}_t = \boldsymbol{A}\mathbf{p}_t$.

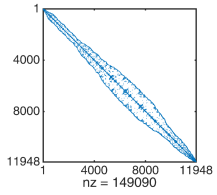
$$\boldsymbol{A}_{I_i,:}\mathbf{p}_t = \boldsymbol{A}_{I_i,[n+k]\setminus F_t}(\mathbf{p}_t)_{[n+k]\setminus F_t} \tag{4}$$

- When a fault is detected, we truncate the update $\mathbf{p}_t = \mathbf{r}_t + \beta_t \mathbf{p}_{t-1}$ to be

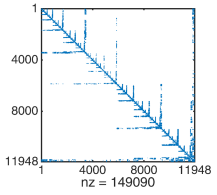
$$\mathbf{p}_t = \mathbf{r}_t. \tag{5}$$

This corresponds to a reset of the Krylov process.

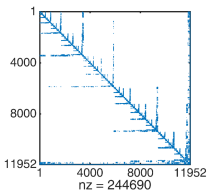
Reordering and Partitioning



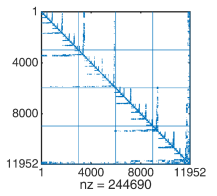
(a) original matrix



(b) original reordered matrix



(c) augmented matrix



(d) augment reordered matrix

Experimental Data

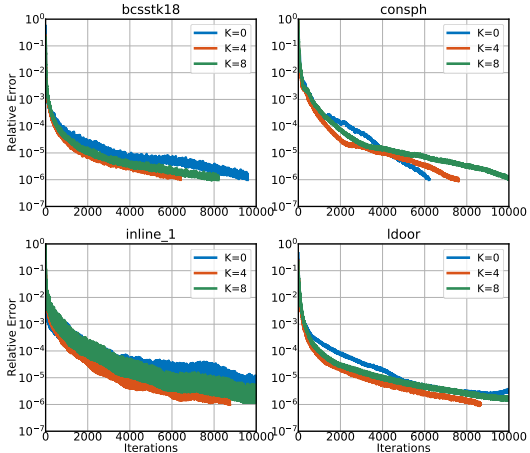
We select matrices from the University of Florida Matrix Collection for our tests.

`bcsstk18`	$11,948$	$149,090$
`consph`	$83,334$	$6,010,480$
`inline_1`	$503,712$	$36,816,170$
`ldoor`	$952,203$	$42,493,817$

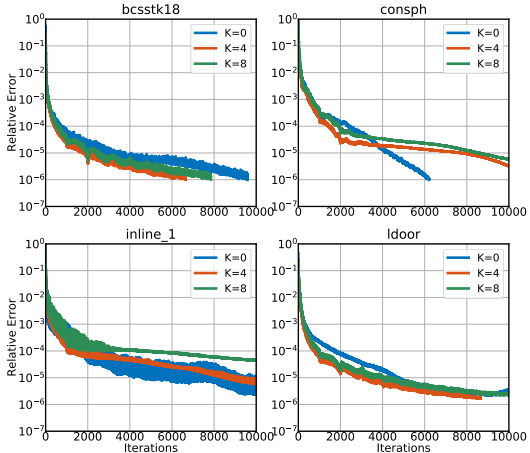
Experiment Setup

- The right-hand-side vector **b** is first normalized (which means $\|b\|_2 = 1$). The relative error $rtol = \frac{\|Ax - b\|_2}{\|b\|_2}$ equals the residual norm $\|r\|_2 = \|Ax - b\|_2$.
- The termination condition is set to $\|r\|_2 < 10^{-6}$ for all matrices, and the maximum number of iterations for CG is set to 10000.
- For the instantaneous fault arrival model, faults arrive at the 1000-th iteration.
- For the exponential fault arrival model, the fault rate is set as 10^{-3}, which implies the average number of steps between two consecutive faults is 1000.

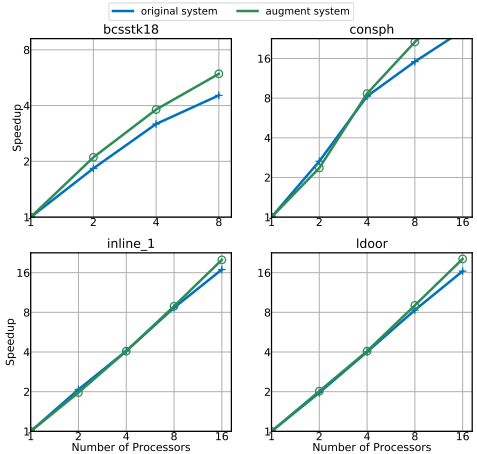
Convergence--Fault-Free Mode



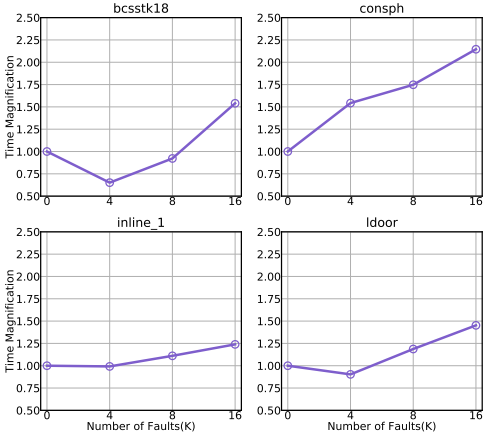
Convergence--Faulty Execution



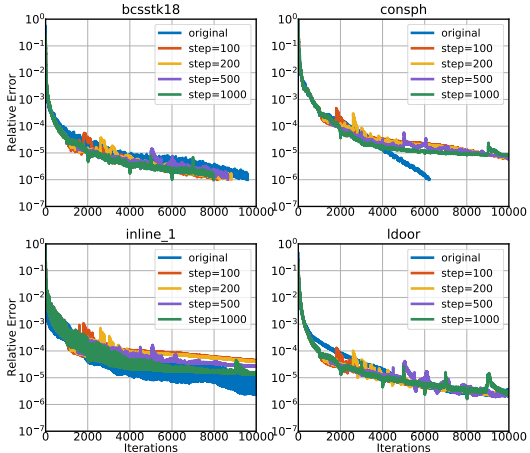
Speedup


original system
augment system
bcsstk18
consph
inline_1
ldoor
Speedup
Number of Processors

Time Overhead



Convergence under Different Fault Rates ($K = 8$)



Different Fault Models

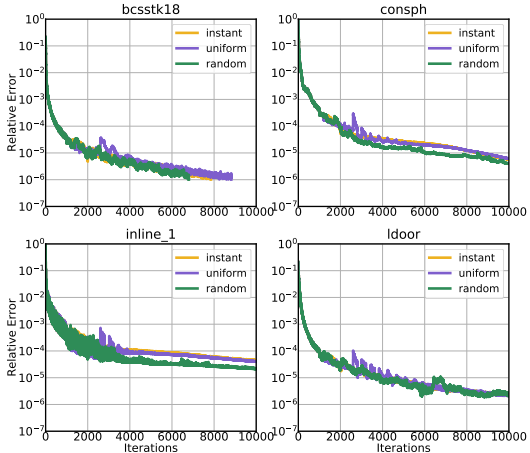
The exponential distribution is the most commonly used random fault arrival model. It assumes the time to failure to be exponentially distributed.

The probability distribution function (PDF) of the time (τ) to failure is given by:

$$P_e(t < \tau) = 1 - e^{-r_e \tau}$$

Here r_e is the failure rate.

Convergence under Different Fault Arrival Models ($K = 8$)



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Adaptive Fault Tolerant Linear System Solver

The Distributed Fault Tolerant Linear System Solver runs the augmented system from the beginning and can tolerate as many faults as the size of augmentation block during the execution.

- System size is augmented ($n \rightarrow n + k$).
- System property changes (SPD $\rightarrow$ SPSD).
- Computational overhead paid at each iteration.

Adaptive Fault Tolerant Linear System Solver

Adaptive Fault Tolerant Linear System Solver runs on the original system until a fault occurs. The erased blocks are compensated for by the addition of an identical number of rows (and columns) selected from the pre-computed coding blocks $[\boldsymbol{E}^T\boldsymbol{A}, \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}]$.

- System size is the same (always n).
- System property is maintained (always SPD).
- Computational Overhead is negligible.

Adaptive Fault Tolerant Linear System Solver

The initial solution of the original system can be written as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} x_c \\ x_f \end{bmatrix} = \begin{bmatrix} b_c \\ b_f \end{bmatrix} \tag{6}$$

The augmented system can now be written as:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{Z}_1 \\ \boldsymbol{A}_{12}{}^T & \boldsymbol{A}_{22} & \boldsymbol{Z}_2 \\ \boldsymbol{Z}_1{}^T & \boldsymbol{Z}_2{}^T & \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_f \\ \mathbf{x}_r \end{bmatrix} = \begin{bmatrix} \mathbf{b}_c \\ \mathbf{b}_f \\ \boldsymbol{E}^T b \end{bmatrix} \tag{7}$$

After erasures, we solve the new system:

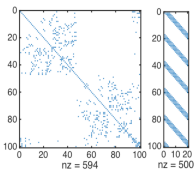
$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{Z}_1 \\ \boldsymbol{Z}_1{}^T & \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_r \end{bmatrix} = \begin{bmatrix} \mathbf{b}_c \\ \boldsymbol{E}^T b \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}_{12} \\ \boldsymbol{Z}_2{}^T \end{bmatrix} \mathbf{x}_f \tag{8}$$

Adaptive Fault Tolerant Linear System Solver

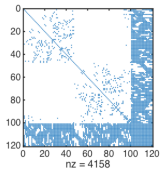
Algorithm 2 Adaptive Fault Oblivious CG

(Reliably) Compute and save the entries $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}, \boldsymbol{E}^T\mathbf{b}$ for matrix $\boldsymbol{E}$
$\boldsymbol{A}^{(\text{cur})} = \boldsymbol{A}$
$\mathbf{b}^{(\text{cur})} = \mathbf{b}$
$\mathbf{x}_0 =$ the initial guess
$\mathbf{r}_0 = \mathbf{b}^{(\text{cur})} - \boldsymbol{A}^{(\text{cur})}\mathbf{x}_0$
$\beta_0 = 0$
for $t = 1, \ldots$ until convergence **do**
  if Fault detected **then**
    $\mathbf{p}_t = \mathbf{r}_{t-1}$
  else
    $\mathbf{p}_t = \mathbf{r}_{t-1} + \dfrac{\|\mathbf{r}_{t-1}\|_2^2}{\|\mathbf{r}_{t-2}\|_2^2} \cdot \mathbf{p}_{t-1}$
  $\mathbf{q}_t = \boldsymbol{A}^{(\text{cur})}\mathbf{p}_t$
  $\alpha_t = \dfrac{\|\mathbf{r}_{t-1}\|_2^2}{\langle \mathbf{q}_t, \mathbf{p}_t \rangle}$
  $\mathbf{x}_t = \mathbf{x}_{t-1} + \alpha_t \mathbf{p}_t$
  $\mathbf{r}_t = \mathbf{r}_{t-1} - \alpha_t \mathbf{q}_t$
  if Faults detected **then**
    $\boldsymbol{A}^{(\text{cur})} = \begin{bmatrix} \boldsymbol{A}_{11} & \tilde{\boldsymbol{Z}}_1 \\ \tilde{\boldsymbol{Z}}_1^T & \tilde{\boldsymbol{E}}^T\boldsymbol{A}\tilde{\boldsymbol{E}} \end{bmatrix}$
    $\mathbf{b}^{(\text{cur})} = \begin{bmatrix} \mathbf{b}_c - \boldsymbol{A}_{12}\mathbf{x}_f \\ \tilde{\boldsymbol{E}}^T\mathbf{b} - \tilde{\boldsymbol{Z}}_2^T\mathbf{x}_f \end{bmatrix}$
    $\mathbf{x}_t = \begin{bmatrix} \mathbf{x}_c \\ 0 \end{bmatrix}$
    $\mathbf{r}_t = \mathbf{b}^{(\text{cur})} - \boldsymbol{A}^{(\text{cur})}\mathbf{x}_t$

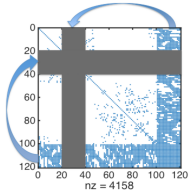
Reordering and Partition



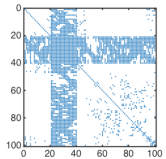
(a) Input matrix $\boldsymbol{A}$ (left) and Coding matrix $\boldsymbol{E}$ (right)



(b) Augmented System (Input matrix with coding blocks)



(c) Compensation of erasures in input matrix from coding blocks



(d) Compensated matrix (of same size) after erasures

Experimental Data

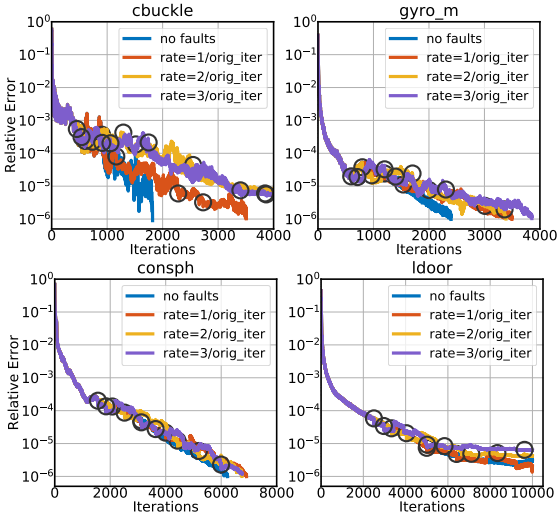
We select matrices from the University of Florida Matrix Collection for our tests – `cbuckle` and `gyro_m` are used to validate the convergence of adaptive fault tolerant linear solver; `consph` and `ldoor` are used to validate parallel scalability and robustness to different fault arrival models.

`cbuckle`	$13,681$	$676,515$
`gyro_m`	$17,361$	$340,431$
`consph`	$83,334$	$6,010,480$
`ldoor`	$952,203$	$42,493,817$

Experiment Setup

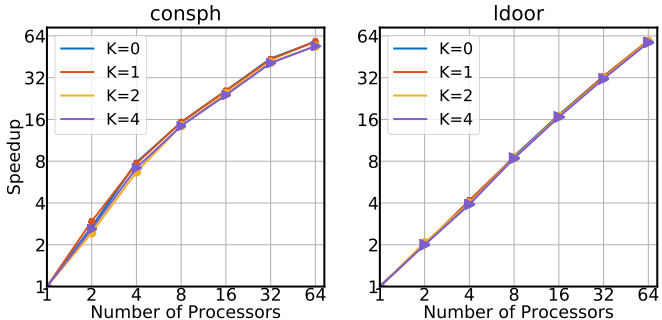
- The right hand side $\mathbf{b}$ is normalized ($||\mathbf{b}||_2 = 1$). The relative residual $rtol = \frac{||\boldsymbol{A}\mathbf{x}-\mathbf{b}||_2}{||\mathbf{b}||_2}$ (equals to $||\mathbf{r}||_2 = ||\boldsymbol{A}\mathbf{x} - \mathbf{b}||_2$) is calculated.
- $||\mathbf{r}||_2$ is monitored at each iteration and the termination condition is set as $||\mathbf{r}||_2 < 10^{-6}$ and the maximum number of iterations of CG is set to 10000 for all matrices.
- For parallel performance, the matrices are first reordered using Metis.
- For exponential fault arrival model, different fault rates(r_e) ranging from $\frac{1}{\texttt{orig\_iter}}$ to $\frac{3}{\texttt{orig\_iter}}$ are tested.
- In our tests, we set the first fault to happen at $\frac{\texttt{orig\_iter}}{1+\texttt{orig\_iter}/r_e}$.

Convergence



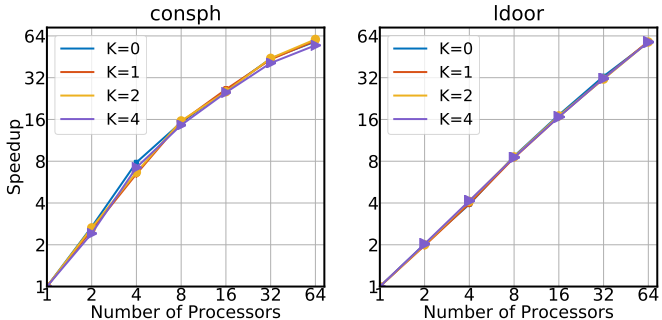
Speedup

Parallel performance of adaptive linear solver under exponential fault arrival model.



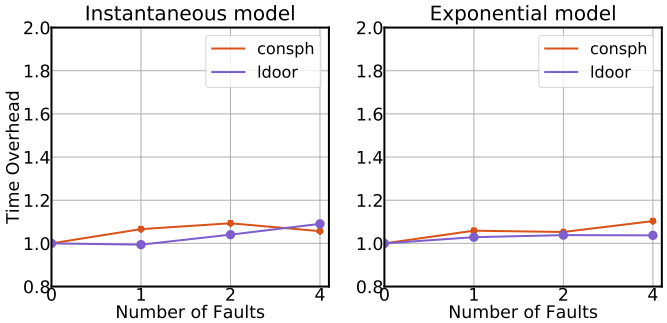
Speedup

Performance of adaptive linear solver under instantaneous fault arrival model.



Overhead

The time overhead of adaptive linear solver.



Background and Motivation

Distributed Fault Tolerant Linear System Solver

Adaptive Fault Tolerant Linear System Solver

Erasure Coded Eigensolver

Concluding Remarks

Reformulation

Theorem

Given an eigenvalue problem

$$Ax^\star = \lambda^\star x^\star, \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$. We construct a generalized eigenvalue problem:

$$\underbrace{\begin{bmatrix} A & AE \\ E^T & E^TAE \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x \\ r \end{bmatrix}}_{\tilde{x}} = \lambda \underbrace{\begin{bmatrix} I & E \\ E^T & E^TE \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} x \\ r \end{bmatrix}}_{\tilde{x}}, \tag{10}$$

where $E \in \mathbb{R}^{n\times k}$ is a coding matrix, then $x^\star = x + Er$ and $\lambda^\star = \lambda$.

Equivalence of Eigensystems

We can write the generalized eigenvalue system as:

$$\begin{bmatrix} A_{11} & A_{12} & Z_1 \\ A_{12}^T & A_{22} & Z_2 \\ Z_1^T & Z_2^T & R \end{bmatrix} \begin{bmatrix} c \\ f \\ r \end{bmatrix} = \lambda \begin{bmatrix} B_{11} & B_{12} & Q_1 \\ B_{12}^T & B_{22} & Q_2 \\ Q_1^T & Q_2^T & S \end{bmatrix} \begin{bmatrix} c \\ f \\ r \end{bmatrix} \tag{11}$$

where

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} E, \qquad \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} E, \quad \text{and}$$

$$R = E^T A E, \qquad S = E^T B E.$$

Equivalence of Eigensystems

The generalized eigenvalue system will become the following purified $n \times n$ system when faults happen:

$$\begin{bmatrix} A_{11} & Z_1 \\ Z_1^T & R \end{bmatrix} \begin{bmatrix} c \\ r \end{bmatrix} = \lambda \begin{bmatrix} B_{11} & Y_1 \\ Y_1^T & S \end{bmatrix} \begin{bmatrix} c \\ r \end{bmatrix} - \begin{bmatrix} \lambda B_{12} - A_{12} \\ \lambda Y_2^T - Z_2^T \end{bmatrix} f$$

Theorem

If $[c; r]$ is the solution of the purified system, then $[c; f; r]$ is the solution of the generalized eigenvalue system.

Equivalence of Eigensystems

The purified system gives us

$$A_{11}c + Z_1 r - \lambda B_{11}c - \lambda Q_1 r = \lambda B_{12} f - \lambda A_{12} f \tag{12}$$

$$Z_1^T c + Rr - \lambda Q_1^T - \lambda S r = \lambda Q_2^T f - \lambda Z_2^T f \tag{13}$$

Equation (13) $-E_1^T \times$ Equation (12) yields

$$E_2^T A_{12} c + E_2^T Z_2 r - \lambda E_2^T B_{12} c - \lambda E_2^T Q_2 r = E_2^T (\lambda B_{22} - A_{22}) f \tag{14}$$

Premultiplying Equation (14) by E_2^{-T} gives

$$A_{12}c + A_{22}f + Z_2 r = \lambda B_{12}c + \lambda B_{22} f + \lambda Q_2 r$$

which is the second equation in the Equation (11).

Perturbation

Since $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ are SPSD, the potential eigenvectors may fall into their null space. We add a perturbation to the augmented systems to avoid this problem.

$$\tilde{\boldsymbol{A}}_p = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{AE} \\ \boldsymbol{E}^T\boldsymbol{A} & \epsilon \boldsymbol{I}_k + \boldsymbol{E}^T\boldsymbol{AE} \end{bmatrix}, \quad \tilde{\boldsymbol{B}}_p = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{E} \\ \boldsymbol{E}^T & \epsilon \boldsymbol{I}_k + \boldsymbol{E}^T\boldsymbol{E} \end{bmatrix}$$

The perturbation is added to the lower-right $k \times k$ block ($\epsilon = 10^{-6}$ used here).

- $\tilde{\boldsymbol{A}}_p$ and $\tilde{\boldsymbol{B}}_p$ are *SPD* and TraceMin can be used to solve the generalized eigenvalue problem.
- Purification (once only) will be done once the trace is small enough.

Purification

For the system $\tilde{\boldsymbol{A}}_p\tilde{\mathbf{x}} = \lambda\tilde{\boldsymbol{B}}_p\tilde{\mathbf{x}}$, we obtain the approx (μ, u)(approximate true eigenpairs $\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$) after TraceMin iterations. Hence, we have:

$$\boldsymbol{A}(\mathbf{u} + \delta\mathbf{u}) = (\mu + \delta\mu)(\mathbf{u} + \delta\mathbf{u}) \tag{15}$$

Also,

$$\mathbf{u}^T\delta\mu = 0 \tag{16}$$

Combining Equation (15) and Equation (16), we will get the linear system:

$$\begin{bmatrix} \boldsymbol{A} - \mu\boldsymbol{I}_n & -\mathbf{u} \\ -\mathbf{u}^T & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta\mu \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{A}\mathbf{u} - \mu\mathbf{u}) \\ 0 \end{bmatrix} \tag{17}$$

Based on $\delta\mathbf{u}$ and $\delta\mu$, we can update the approximation of the eigenpairs and continue the TraceMin procedure.

Fault Oblivious TraceMin

Algorithm 3 Fault Oblivious Trace Minimization

1: Choose an $n \times s$ random matrix V_1 of full rank such that ${V_1}^T \tilde{B}_p V_1 = I(s = 2p)$.
2: **for** $t = 0, 1, \ldots$ until convergence **do**
3: Compute $W_t = \tilde{A}_p V_t$ and the interaction matrix $H_t = {V_t}^T W_t$
4: Compute the eigenpairs of (Y_t, Θ_t) for H_t.
5: Do the purification if the purification condition is satisfied.
6: Sort the eigenvalue in ascending order and rearrange eigenvectors.
7: Compute the corresponding Ritz Vectors $X_t = V_t Y_t$.
8: Compute the residue $R_t = \tilde{A}_p X_t - \tilde{B}_p X_t \Theta_t$.
9: Test for Convergence.
10: Solve the following linear system approximately via the CG to get Δ_t.

$$\begin{bmatrix} \tilde{A}_p & \tilde{B}_p X_t \\ {X_t}^T \tilde{B}_p & 0 \end{bmatrix} \begin{bmatrix} \Delta_t \\ L_t \end{bmatrix} = \begin{bmatrix} \tilde{A}_p X_t \\ 0 \end{bmatrix}$$

11: $\tilde{B}_p$-orthonormalize $X_t - \Delta_t$ into V_{t+1}.
12: **return** Θ.

Implementation

The operations affected by faults in a distributed environment are the aggregation operations – the matrix-matrix and matrix-vector multiplication.

- The Matrix-Matrix Operation:

$$(\tilde{\boldsymbol{A}}_p, \tilde{\boldsymbol{V}}_t) = \left((\tilde{\boldsymbol{A}}_p)_{[n+k]\setminus F_t}, (\tilde{\boldsymbol{V}}_t)_{[n+k]\setminus F_t}\right)$$
$$(\tilde{\boldsymbol{A}}_p, \tilde{\boldsymbol{X}}_t) = \left((\tilde{\boldsymbol{A}}_p)_{[n+k]\setminus F_t}, (\tilde{\boldsymbol{V}}_t)_{[n+k]\setminus F_t}\right)$$
$$(\tilde{\boldsymbol{B}}_p, \tilde{\boldsymbol{X}}_t) = \left((\tilde{\boldsymbol{B}}_p)_{[n+k]\setminus F_t}, (\tilde{\boldsymbol{X}}_t)_{[n+k]\setminus F_t}\right)$$

- The Matrix-Vector Operation in Step 10:

$$(\tilde{\boldsymbol{A}}_p, \Delta_t) = \left((\tilde{\boldsymbol{A}}_p)_{[n+k]\setminus F_t}, (\Delta_t)_{[n+k]\setminus F_t}\right)$$

Experimental Data

We select two matrices from the University of Florida Matrix Collection.

`minsurfo`	$40,806$	$203,622$	Optimization Problem
`s3dkq4m2`	$90,449$	$4,427,725$	Structural Problem

Experiment Setup

- For validation of convergence, we monitor $\frac{||r_1||_2}{\lambda_1}$.The stopping criteria is set as 10^{-08} for all the matrices.
- Define $t_i = \frac{aug_{\lambda_i} - orig_{\lambda_i}}{orig_{\lambda_i}}$ and construct $t = [t_1, t_2, \cdots, t_{10}]$. Then final relative error $rtol = ||t||_2$.
- Augmented blocks with different sizes are added to the original system. $K = 0$ corresponds to the original system; $K = d$ corresponds to an augmented block size of d, and d faults happen during the execution ($d = 1, 8, 16$).

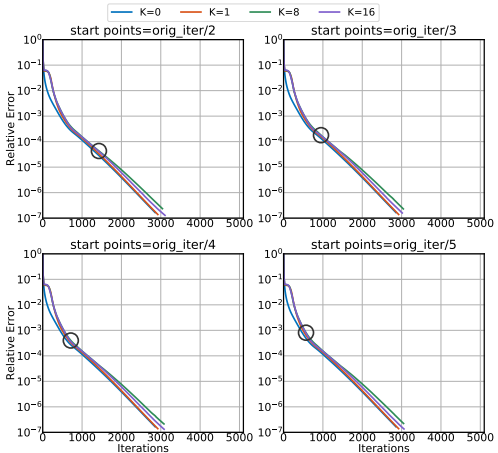
Experiment Setup

- Leverage score is used to sample a large matrix and can measure the importance of each row of A. For $A = U\Sigma V$ (SVD decomposition), the leverage score for each row is calculated as follows:

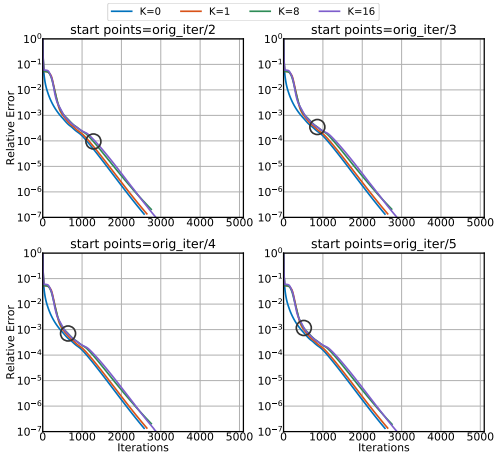
$$l(i) = \sum_{j=1}^{n} U(i,j)$$

- Two different fault arrival models, instantaneous and exponential, were tested.
- For exponential fault model, failure rates ranging from $\frac{1}{\texttt{orig\_iter}}$ to $\frac{4}{\texttt{orig\_iter}}$ were tested.

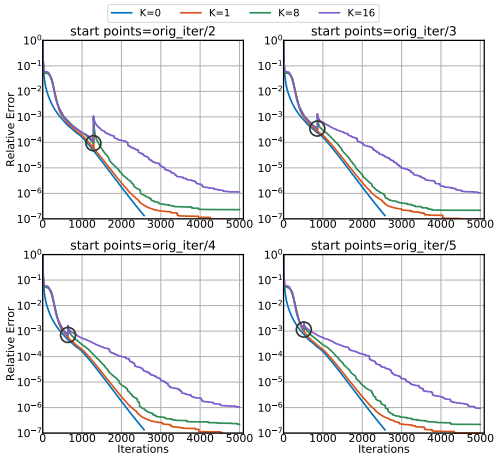
Convergence for Random Case (`minsurfo`)



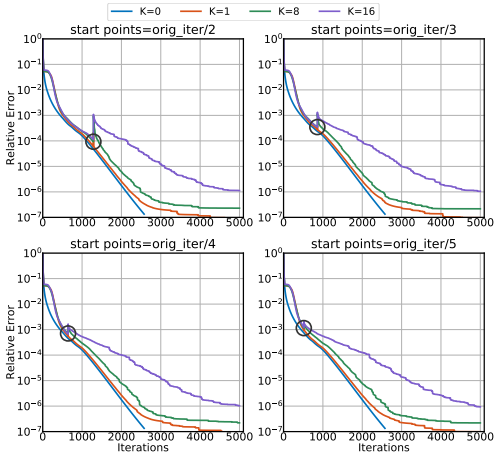
Convergence for Random Case (s3dkq4m2)



Convergence for Worst Case (`minsurfo`)



Convergence for Worst Case (s3dkq4m2)



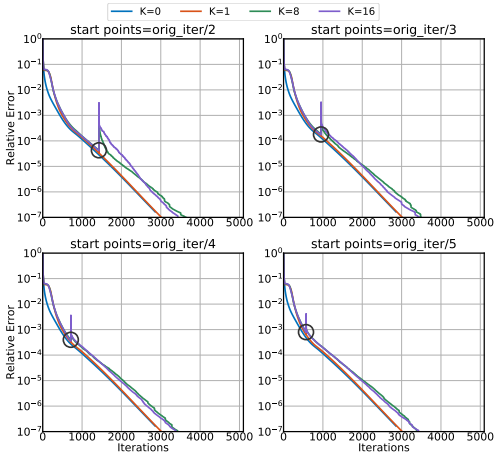
Adapative Coding Scheme

Coding blocks are periodically updated using estimates of leverage scores from prior iteration. The coding matrix $\boldsymbol{E}$ is adaptively updated as follows:

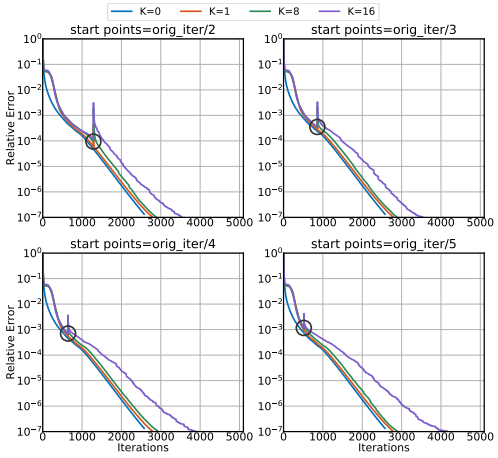
$$E(i,:) = E(i,:) * \frac{l(i)}{\bar{l}}$$

Here $E(i,:)$ is the i^{th} row of coding matrix E, $l(i)$ is the leverage score of i^{th} row and $\bar{l}$ is the average leverage score of all rows.

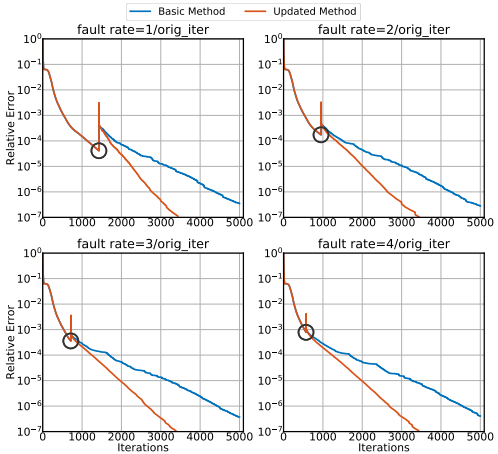
Convergence of Updating Method (`minsurfo`)



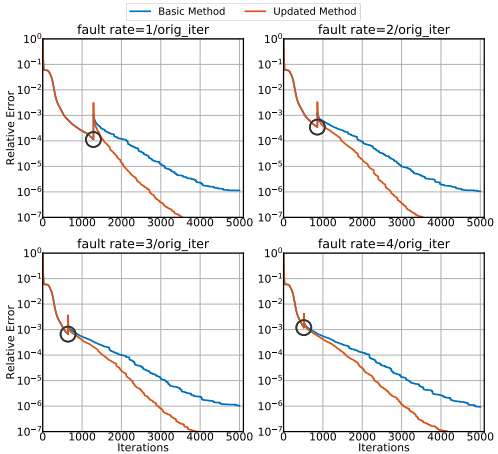
Convergence of Updating Method (s3dkq4m2)



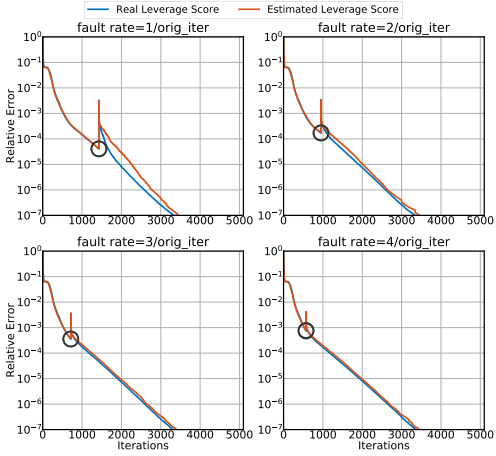
Benefit of Using Estimated Leverage Scores (`minsurfo`)



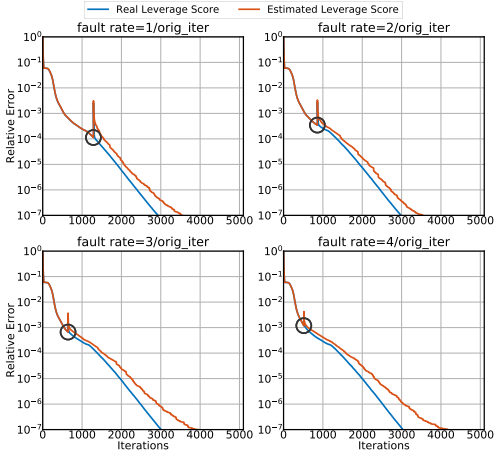
Benefit of Using Estimated Leverage Scores (s3dkq4m2)



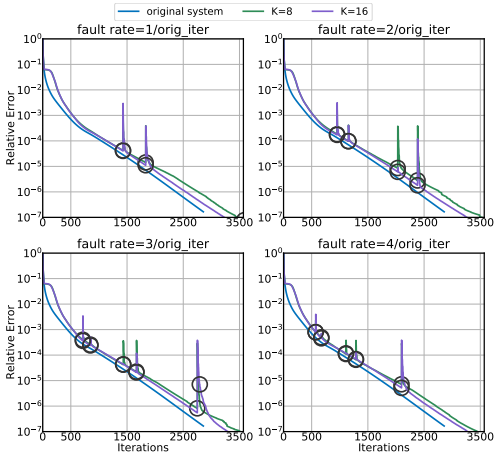
Comparison of Results from Exact and Estimated Leverage Scores (`minsurfo`)



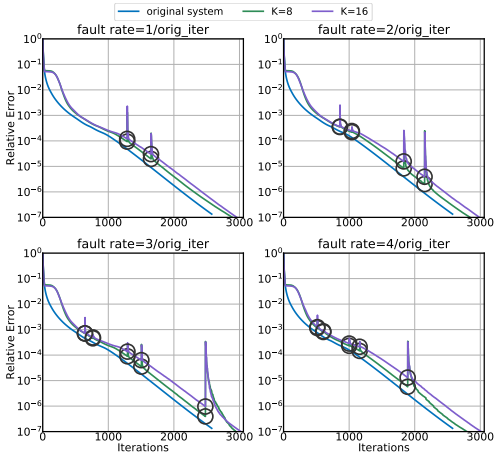
Comparison of Results from Exact and Estimated Leverage Scores (s3dkq4m2)



Different Fault Arrival Models (`minsurfo`)



Different Fault Arrival Models (s3dkq4m2)



Background and Motivation

Distributed Fault Tolerant Linear System Solver

Adaptive Fault Tolerant Linear System Solver

Erasure Coded Eigensolver

Concluding Remarks

Concluding Remarks

- Erasure coding presents a compelling new approach to fault tolerance;
- These techniques have significantly lower overhead in computation, particularly as fault rates become high;
- They can be implemented at scale with low algorithmic and parallel overhead;
- Many new core methods forthcoming.

Thanks!