PDE

Partial Differential Equations

1 Introduction

Partial differential equations are at the foundation of much of computational science. Most physical phenomena depend in complex ways on space and time. Examples include fluid flow, heat transfer, nuclear and chemical and reactions and population dynamics. Computational scientists often seek to gain understanding of such phenomena by casting fundamental principles, such as conservation of mass, momentum and energy in the form of mathematical models of the underlying physical phenomena. Usually a mathematical model requires more than one independent variable to characterize the state of the physical system. For example, to describe a general fluid flow usually requires that the physical variables of interest, say pressure, density and velocity, be dependent on time and three space variables. If a mathematical model involves more than one independent and if at least one of the physical variables of interest is nonconstant with respect to space or time, then the mathematical model will involve a partial differential equation (PDE).

This chapter is not intended to be a complete discussion of partial differential equations. Instead, its aim is to serve as an introduction to a minimal amount of terminology from the field of PDEs, followed by some examples of issues that are likely to confront a computational scientist. Thus, the emphasis will be placed on

- Notation and Terminology
- Introduction to the Finite Difference Method
- Selected Numerical Algorithms for Solving Finite Difference Equations
- Performance Programming and Algorithm to Architecture Mapping

2 Basic Definitions

In mathematical terms, a partial differential equation (PDE) is any equation involving a function of more than one independent variable and at least one partial derivative of that function. The order of a PDE is the order of the highest order derivative that appears in the PDE. The principal part of a PDE is the collection of terms in the PDE containing derivatives of order equal to the order of the PDE. The following example illustrates these definitions and introduces the two most common notations for expressing partial derivatives.

Example 1 *An illustration of a PDE with subscript notation.*

If $u = u(x, y)$ is a function of the two independent variables x and y, then

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = 0 \tag{1}$$

is a PDE of first order whose principal part is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}. \tag{2}$$

Using subscript notation a more compact way to express this PDE is

$$u_x + u_y + u = 0, \tag{3}$$

in which case we would say that the principal part is $u_x + u_y$.

A PDE in u is classified as linear if all of the terms involving u and any of its derivatives can be expressed as a linear combination in which the coefficients of the u-terms are independent of u. In a linear PDE, the coefficients can depend at most on the independent variables.

Example 2 *An illustration of a linear and a nonlinear PDE.*

If $u = u(x, y)$ is a function of the two independent variables x and y, then

$$u_{xx} + 3u_{xy} + u_{yy} + u_x - u = e^{x-y} \tag{4}$$

is a linear (constant coefficient) PDE.

The PDE

$$\sin(xy)u_{xx} + 3x^2u_{xy} + u_{yy} + u_x - u = 0 \tag{5}$$

is a linear (variable coefficient) PDE.

The PDE

$$u_{xx} + 3u_{xy} + u_{yy} + u_x^2 - u = e^{x-y} \tag{6}$$

is nonlinear.

The distinction between linear and nonlinear PDEs is extremely important in computational science. Many linear PDE problems can be solved exactly using techniques such as separation of variables, superposition, Fourier series, Laplace transform and Fourier transform. Exact solutions are valuable in a computational setting because they can be used to assist the computational scientist in the often difficult exercise of code validation. Generally, nonlinear PDEs do not yield to analytical solution approaches. Since most leading edge work in computational science involves nonlinear PDEs, a great deal of effort is directed toward obtaining numerical solutions. Whenever possible, computational scientists draw from the field of linear PDEs for guidance and insight in developing numerical methods for the more difficult nonlinear PDEs.

3 Classification of Linear PDEs in Two Independent Variables

In addition to the distinction between linear and nonlinear PDEs, it is important for the computational scientist to know that there are different classes of PDEs. Just as different solution techniques are called for in the linear versus the nonlinear case, different numerical methods are required for the different classes of PDEs, whether the PDE is linear or nonlinear. The need for this specialization in numerical approach is rooted in the physics from which the different classes of PDEs arise. By analogy with the conic sections (*ellipse, parabola* and *hyperbola*) partial differential equations have been classified as elliptic, parabolic and hyperbolic. Just as an ellipse is a smooth, rounded object, solutions to elliptic equations tend to be quite smooth. Elliptic equations generally arise from a physical problem that involves a diffusion process that has reached equilibrium, a steady state temperature distribution, for example. The hyperbola is the disconnected conic section. By analogy, hyperbolic equations are able to support solutions with discontinuities, for example a shock wave. Hyperbolic PDEs usually arise in connection with mechanical oscillators, such as a vibrating string, or in convection driven transport problems. Mathematically, parabolic PDEs serve as a transition from the hyperbolic PDEs to the elliptic PDEs. Physically, parabolic PDEs tend to arise in time dependent diffusion problems, such as the transient flow of heat in accordance with Fourier's law of heat conduction.

In the linear PDE of second order in two variables,

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{7}$$

if u_{xx} is formally replaced by α^2, u_{xy} by $\alpha\beta$, u_{yy} by β^2, u_x by α and u_y by β, then associated with equation (7) is a polynomial of degree two in α and β

$$P(\alpha, \beta) = a\alpha^2 + 2b\alpha\beta + c\beta^2 + d\alpha + e\beta + j. \tag{8}$$

The mathematical nature of the solutions of equation (7) are largely determined by the algebraic properties of the polynomial $P(\alpha, \beta)$. In turn, the computational strategy that one selects to numerically solve (7) is strongly influenced by the mathematical nature of the solution. Thus, before embarking on a quantitative analysis of a partial differential equation of the form (7), it is important that a computational scientist have an idea of the qualitative nature of the solution. Much of this qualitative understanding of the solution can be obtained via the following classification scheme. P(a,b) and along with it, the PDE (7) is classified as hyperbolic, parabolic, or elliptic according as its discriminant, $b^2 - ac$, is positive, zero, or negative. Note that the type of equation (7) is determined solely by its principal part (the terms involving the highest-order derivatives of u) and that the type will generally change with position in the xy-plane unless a,b, and c are constants.

Example 3 *A brief introduction to an elliptic, parabolic, and hyperbolic equations.*

Laplace's equation,

$$u_{xx} + u_{yy} = 0, \tag{9}$$

is elliptic since the discriminant, $b^2 - ac = 0^2 - 1 \bullet 1 = -1$, is negative. Laplace's equation occurs in numerous physically based simulation models and is usually associated with a diffusive or dispersive process in which the state variable, $u(x, y)$ is in an equilibrium condition. For example, $u(x, y)$ could represent an equilibrium temperature in a two dimensional thermodynamic model based on Fick's Law. Of interest to the computational scientist is the fact that solutions of Laplace's equation, and elliptic equations in general, can support large gradients only in response to external stresses manifested as a source/sink term (g in equation (7)) or as an abrupt change in type of or value of a boundary condition. Almost invariably the computational analysis of an elliptic equation reduces to a linear algebra problem of solving a system of diagonally dominant linear equations. Armed with this knowledge, the computational scientist has apriori knowledge of the types of algorithms and architectures that may provide an efficient numerical solution of an elliptic equation of the form (7).

The *diffusion equation*,

$$u_t - u_{xx} = 0, \tag{10}$$

for $u(x, t)$ is parabolic since the discriminant, $b^2 - ac = 0^2 - (-1) \bullet (0) = 0$. The diffusion equation arises in diverse settings, but most often in connection with a transient flow problem in which the flow is down gradient of some state variable u. In the setting of heat flow, the diffusion equation (sometimes called the heat equation) could be used to model a thermodynamics problem in which transient heat flow is occurring in one space dimension.

Similar to the elliptic case, parabolic equations generally have very smooth solutions. However, parabolic equations often exhibit solutions with evolving regions of high gradient. Most numerical methods for dealing with parabolic equations involve approximating the solution at successive time steps, with each approximation requiring the solution of a system of linear equations. For these types of computational problems, it is often useful to employ some matrix factorization method in conjunction with a dynamic gridding algorithm. Multispace generalizations of this example problem can be solved efficiently on vector architectures using ADI methods or on parallel architectures with some divide and conquer strategies.

The *one dimensional wave equation*,

$$u_{xx} - u_{tt} = 0, \tag{11}$$

has discriminant $b^2 - ac = 0^2 - (1)(1) = 1$ so it is classified as hyperbolic. This type of equation arises in many fields ranging from elasticity and acoustics to atmospheric science and hydraulics. Of interest to the computational scientist is the knowledge that solutions to linear hyperbolic equations can be only as smooth as their boundary and initial conditions are. Moreover, any sharp fronts or peaks in the solution are persistent and can reflect off of boundaries. For a nonlinear hyperbolic PDE, even smooth boundary and initial conditions can give rise to nonsmooth or even discontinuous solutions. Of the three types of PDEs discussed in this example, hyperbolic equations are generally the most challenging to the computational scientist. Since explicit time stepping methods are usually called for to numerically solve hyperbolic PDEs, the computational scientist must be aware of important algorithm stability issues. Explicit algorithms give excellent performance rates on vector and SIMD architectures.

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4 Equations with n Independent Variables

Many problems encountered in computational science involve several space variables and possibly a time variable. As indicated in Example 3, it is important for the computational scientist to be aware of the type of equation under consideration. Although the clear trichotomy of types of section 2 is not maintained in this setting, it is still possible to identify equations of elliptic, parabolic and hyperbolic types. The remarks of section 2 regarding algorithms and architectures for problems involving two variables apply equally well to their n-variable counterparts.

A general linear PDE of order two in n variables has the form

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = d. \tag{12}$$

If $u_{x_i x_j} = u_{x_j x_i}$, then the principal part of equation (12) can always be arranged so that $a_{ij} = a_{ji}$; thus, the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ can be assumed symmetric. In linear algebra it is shown that every real, symmetric $n \times n$ matrix has n real eigenvalues. These eigenvalues are the (possibly repeated) zeros of an nth-degree polynomial in λ, $\det(\mathbf{A} - \lambda\mathbf{I})$, where $\mathbf{I}$ is the $n \times n$ identity matrix. Let P denote the number of positive eigenvalues, and Z the number of zero eigenvalues (i.e., the multiplicity of the eigenvalue zero), of the matrix $\mathbf{A}$. Then equation (12)is:

hyperbolic if $Z = 0$ and $P = 1$ or $Z = 0$ and $P = n - 1$

parabolic if $Z > 0$ (equivalently, if $\det \mathbf{A} = 0$)

elliptic if $Z = 0$ and $P = n$ or $Z = 0$ and $P = 0$

ultrahyperbolic if $Z = 0$ and $1 < P < n - 1$

If any of the a_{ij} is nonconstant, the type of equation (12) can vary with position.

Example 4 *An illustration of the matrix of a PDE.*

For the PDE $3u_{x_1x_1} + u_{x_2x_2} + 4u_{x_2x_3} + 4u_{x_3x_3} = 0$ the matrix $\mathbf{A}$ is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}. \tag{13}$$

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5 Classification Of First Order Systems

In addition to the second order equations of the type discussed in sections 2–3, systems of first order equations are also frequently encountered in computational science.

Example 5 *An illustration of systems of first order partial differential equations.*

The current $i = i(x, t)$ and voltage $v = v(x, t)$ at position x and time t in a transmission line satisfy the first order equations

$$\frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} = -Gv, \qquad \frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} = -Ri \tag{14}$$

where R, L, C and G denote, respectively, resistance, inductance, capacitance and leakage conductance per unit length of transmission line.

(b) The first order system

$$
\begin{aligned}
(\rho v)_x + \rho_t &= 0 \\
v v_x + v_t &= -\frac{1}{\rho} Px \\
v p_x + p &= -\gamma p v_x
\end{aligned}
$$

governs the one dimensional flow of an ideal gas with velocity $v = v(x,t)$, density $\rho = \rho(x,t)$ and pressure $p = p(x,t)$. γ is a physical constant determined by the specific heat of the gas.

Problems such as these present computational scientists with systems of first order partial differential equations. The general *quasilinear system* of n first order partial differential equations in two independent variables has the form

$$
\sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial t} = c_i \quad i = 1, 2, \ldots, n \quad , \tag{15}
$$

where a_{ij} , b_{ij} and c_{ij} may depend on $x, t, u_1, u_2, \ldots, u_n$. If each a_{ij} and b_{ij} is independent of $u_1, u_2, \ldots, u_n$, the system (15) is called *almost linear*. If, in addition, each c_i depends linearly on $u_1, u_2, \ldots, u_n$,

$$
c_i = \sum_{j=1}^{n} r_{ij} u_j + S_i, \tag{16}
$$

with r_{ij} and S functions of at most x and t, the system is said to be *linear*. If $c_i = 0$ for $i = 1, 2, \ldots, n$, the system is called *homogeneous*. If C, G, R and L depend at most on x and t the transmission line equations are linear. The ideal gas equations are quasilinear.

In terms of the $n \times n$ matrices $\mathbf{A} = [a_{ij}], \mathbf{B} = [b_{ij}]$ and the column vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T, \mathbf{c} = [c_1, c_2, \ldots, c_n]^T$, the system of equations (15) can be written as

$$
\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_t = \mathbf{c} \tag{17}
$$

Example 6 *Matrix expression of transmission line equations.*

To express the transmission line equations in the matrix notation of equation (17), introduce the notation

$$
\mathbf{u} = \begin{bmatrix} i \\ v \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & C \\ L & 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -Gv \\ -Ri \end{bmatrix}. \tag{18}
$$

In most applications the matrix $\mathbf{B}$ is nonsingular. In all that follows we assume this to be the case; therefore, we take $\det(\mathbf{B}) \neq 0$. Associated with the system (11) is a *characteristic polynomial* defined by

$$
F(\lambda) = \det(\mathbf{A} - \lambda \mathbf{B}). \tag{19}
$$

Since $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and $\det(\mathbf{B}) \neq 0$, the polynomial F has degree n.

If $F(\lambda)$ has n distinct real zeros, we classify the first order system (17) as *hyperbolic*. The system is also called hyperbolic if $F(\lambda)$ has n real zeros and the generalized eigenvalue problem $(\mathbf{A} - \lambda\mathbf{B})\mathbf{u} = 0$ has n linearly independent solutions. If $F(\lambda)$ has no real zeros, then (17) is called *elliptic*. If $F(\lambda)$ has n real zeros, but $(\mathbf{A} - \lambda\mathbf{B})\mathbf{u} = 0$ does not have n linearly independent solutions, then the system (11) is classified as *parabolic*. An exhaustive classification cannot be carried out when $F(\lambda)$ has both real and complex zeros.

Example 7 *An illustration that the transmission line equations are hyperbolic.*

To classify the transmission line equations, we define the characteristic polynomial

$$F(\lambda) = \det(\mathbf{A} - \lambda\mathbf{B}) = \det\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & C \\ L & 0 \end{bmatrix}\right] = 1 - CL\lambda^2. \tag{20}$$

Since the parameters C and L are positive, $F(\lambda)$ has the distinct real roots $\lambda_1 = \sqrt{1/(CL)}, \lambda_2 = \sqrt{1/(CL)}$. This shows that the transmission line equations are hyperbolic.

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6 Well Posed PDE Problems

In the previous sections we saw some examples of partial differential equations. We now consider some important issues regarding the formulation and solvability of PDE problems. A solution to a PDE can be described as simply a function that reduces that PDE to an identity on some region of the independent variables. In general, a PDE alone, without any auxiliary boundary or initial conditions, will either have an infinity of solutions, or have no solution. Thus, in formulating a PDE problem there are three components: (i) the PDE; (ii) the region of space-time on which the PDE is required to be satisfied; (iii) the auxiliary boundary and initial conditions to be met.

For a PDE based mathematical model of a physical system to give useful results, it is generally necessary to formulate that model as what mathematicians call a well posed PDE problem. A PDE problem is said to be well posed if

1. a solution to the problem exists
2. the solution is unique, and
3. the solution depends continuously on the problem data.

(In a PDE problem the problem data consists of the coefficients in the PDE; the functions appearing in boundary and initial conditions; and the region on which the PDE is required to hold.)

If one of these conditions is not satisfied, the PDE problem is said to be ill-posed. In practice, the question of whether a PDE problem is well posed can be difficult to settle. Roughly speaking the following guidelines apply:

- The auxiliary conditions imposed must not be too many or a solution will not exist.
- The auxiliary conditions imposed must not be too few or the solution will not be unique.
- The kind of auxiliary conditions must be correctly matched to the type of the PDE or the solution will not depend continuously on the data.

More specific guidelines can be stated for second order linear PDE problems.

- Well posed elliptic PDE problems usually take the form of a boundary value problem (BVP) with the PDE required to hold on the interior of some region and the solution required to satisfy a single boundary condition (BC) at each point on the boundary of the region. Typical boundary conditions are:
  - Dirichlet BC - the solution value is specified on the boundary
  - Neumann BC - the normal derivative of the solution is specified on the boundary
  - Robin BC - a linear combination of the solution and its normal derivative is specified on the boundary.

  The kind of boundary condition can vary from point to point on the boundary, but at any given point only one BC can be specified. Physically a Dirichlet BC usually corresponds to setting the value of a field variable, such as temperature; a Neumann BC usually specifies a flux condition on the boundary; and a Robin BC typically represents a radiation condition. When the region on which the PDE problem is posed is unbounded, one or more of the above boundary conditions is usually replaced by a growth condition that limits the behavior of the solution "at infinity".

- Well posed parabolic PDE problems usually involve one or more spatial variables and a time variable as well. Parabolic PDE models often arise in connection with evolutionary systems in which the flux of some material quantity is "down gradient" with respect to a field variable. Typically, a well posed parabolic problem requires the same boundary conditions on the spatial variables as in the case of elliptic problems. In addition an initial condition specifying the state of the system at time $t = 0$ is required. Thus, a well posed second order parabolic PDE problem usually takes the form of and initial boundary value problem (IBVP).

- Well posed, second order, hyperbolic PDE problems also require the same boundary conditions as elliptic problems. Usually second order, hyperbolic PDE model arise in connection with physical problems involving wave motion, vibration or oscillation. In these problems, two initial conditions at time $t = 0$ are required (one to describe the initial state of the system and another to describe the initial velocity).

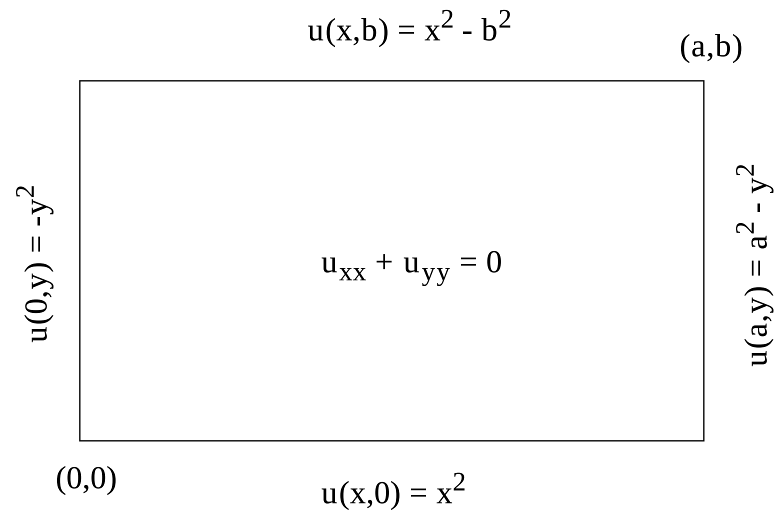

Figure 1: Laplace's equation on the rectangular region $0 < x < a$, $0 < y < b$ subject to the Dirichlet boundary conditions described in Example 8.

A discussion of the well posedness of PDE problems involving systems of first order equations requires an understanding of the characteristic curves associated with such systems. Systems of first order equations are very important in the field of computational science, but are not dealt with here, since the remainder of this chapter focus on second order PDEs. To conclude this section, several examples of well posed and ill posed second order PDE problems are presented.

Example 8 *An illustration of Laplace's equation.*

Laplace's equation on the rectangular region $0 < x < a, 0 < y < b$, subject to the Dirichlet boundary conditions

$$\begin{array}{ll} u(x,0) = x^2 & u(x,b) = x^2 - b^2 \\ u(0,y) = -y^2 & u(a,y) = a^2 - y^2 \end{array} \tag{21}$$

is well posed. For the case of these example boundary conditions, one can show that the unique solution to this BVP is $u(x,y) = x^2 - y^2$. If any one of the four boundary conditions is deleted, then the problem becomes ill-posed, because is would then admit multiple solutions. If a second, independent Dirichlet condition were added on any part of the boundary, the problem would again be ill-posed, in this case due to lack of existence of a solution. More generally, if two, independent boundary conditions are imposed on any part of the boundary of the region, then the problem will fail to have a solution.

Example 9 *An illustration of an elliptic PDE.*

To illustrate that boundary value problems, not initial value problems, are the appropriate setting for elliptic PDE problems, we present the following example due to Hadamard. To view this problem as an initial value problem, one should think of y as a time variable. Consider the initial value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad -\infty < x < \infty, y > 0 \\ u(x,0) &= f(x) \qquad -\infty < x < \infty \\ u_x(x,0) &= g(x) \qquad -\infty < x < \infty \end{aligned}$$

For $f = f_1(x) = 0$ and $g = g_1(x) = 0$, it is clear that the corresponding solution to the above initial value problem is $u_1(x,y) = 0$. For the case $f = f_2(x) = 0$ and $g = g_2(x) = n^{-1}\sin(nx)$, it is easy to verify that the corresponding solution is

$$u_2(x,y) = n^{-2}\sinh(ny)\sin(nx). \tag{22}$$

Observe that the functions f_1 and f_2 are identical and that

$$\lim_{n\to\infty} |g_1(x) - g_2(x)| \tag{23}$$

uniformly in x. Thus, we see that the data of the two problems, f_1, g_1 and f_2, g_2, can be made arbitrarily close. But, if we compare the two solutions at $x = \pi/2$, then we obtain

$$|u_1(\frac{\pi}{2}, y) - u_2(\frac{\pi}{2}, y)| = \frac{1}{n^2\sinh(ny)} = \frac{e^{ny} - e^{-ny}}{2n^2}. \tag{24}$$

For y positive, e^{ny} approaches infinity faster than n^2, as n goes to infinity. Therefore, we conclude that

$$\lim_{n\to\infty} |u_1(\frac{\pi}{2}, y) - u_2(\frac{\pi}{2}, y)| = \infty \tag{25}$$

illustrating that as the data for the two problems becomes more alike, the solutions become increasingly different. This is what is meant by failure of the solution to depend continuously on the problem data.

Example 10 *An illustration of an IBVP for the diffusion equation.*

The following IBVP for the diffusion equation in one space variable is an example of a well posed parabolic PDE problem for $u = u(x,t)$.

$$\begin{aligned} u_t - \kappa u_{xx} &= 0 \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \qquad 0 < x < L \\ u(0,t) &= 0 \qquad t > 0 \\ u(L,t) &= 0 \qquad t > 0 \end{aligned}$$

One physical interpretation of this problem is that $u(x,t)$ is the temperature at position x and time t in a one dimensional heat conducting medium (say a metal rod, for example) with thermal diffusivity κ. The initial condition, $u(x,0) = f(x)$, specifies the temperature in the rod at the assigned time $t = 0$. The boundary conditions, $u(0,t) = 0$ and $u(L,t) = 0$ state that the ends of the rod are held at temperature zero for all time.

Simple problems such as this make excellent validation tools for the computational scientist. Since the exact solution to this IBVP can be shown (by separation of variables and Fourier series methods) to be

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{L})^2} \sin\frac{n\pi x}{L}, \tag{26}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{27}$$

One can use this exact solution to test the results of a computer code.

Example 11 *A classic example of an ill-posed parabolic PDE.*

The classic example of an ill-posed parabolic PDE problem is the "backward-in-time heat equation".

$$\begin{aligned} u_t - \kappa u_{xx} &= 0 \qquad 0 < x < L, 0 < t < T \\ u(x,T) &= f(x) \qquad 0 < x < L \\ u(0,t) &= 0 \qquad 0 < t < T \\ u(L,t) &= 0 \qquad 0 < t < T \end{aligned}$$

Here, if we think of $u(x,t)$ as the temperature in a one dimensional heat conduction rod, the condition $u(x,T) = f(x)$ can be thought of as giving the temperature distribution at some specific time $t = T$. The PDE problem calls for using this information, together with the heat balance equation and the boundary conditions to predict the temperature distribution at some earlier time, sat $t = 0$. It can be shown (see Schaum's Outline of PDE, solved problem 4.9) that if $f(x)$ is not infinitely continuously differentiable, then no solution to the problem exists. If $f(x)$ is infinitely continuously differentiable, then it is shown that the solution on $0 < t < T$ does not depend continuously on the data, namely $f(x)$.

Example 12 *An illustration of a second order hyperbolic PDE problems.*

For second order hyperbolic PDE problems, the vibrating string is most frequently used as an example of a well posed problem. Think of $u(x,t)$ as representing the vertical displacement at position x and time t of an ideal string which in static equilibrium occupies the horizontal

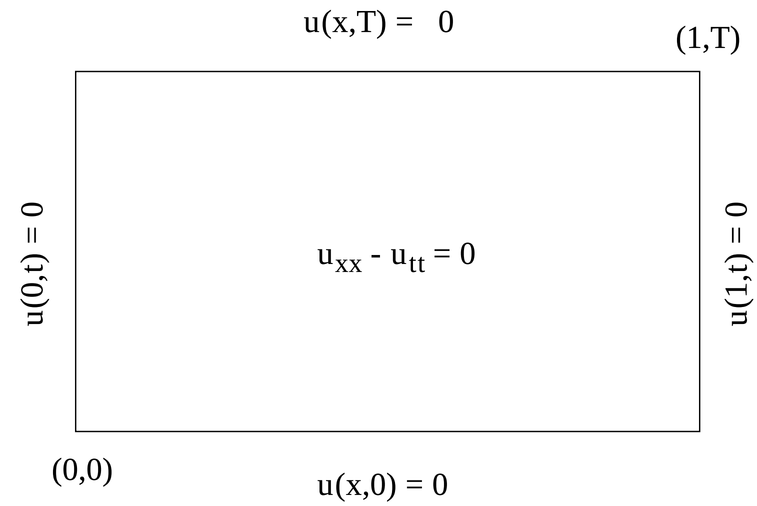

Figure 2: Ill-posed, second order hyperbolic PDE problem described in Example 13.

line joining $x = 0$ and $x = L$. Then the following IBVP models the movement of the string subject to an initial displacement given by $f(x)$ and an initial velocity given by $g(x)$.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \qquad 0 < x < L \\ u_t(x,0) &= g(x) \qquad 0 < x < L \\ u(0,t) &= 0 \qquad t > 0 \\ u(L,t) &= 0 \qquad t > 0 \end{aligned}$$

Example 13 *An example of an ill-posed, second order hyperbolic PDE problem.*

A dramatic example of an ill-posed, second order hyperbolic PDE problem is given by the following BVP for the one dimensional wave equation. It can be shown that if T is irrational, then the only solution of this BVP for the wave equation is u identically zero; whereas if T is rational, the problem has infinitely many nontrivial solution. Thus the solution fails to depend continuously on the data - namely on the size of the region on which the problem is stated.

7 Exercises

Exercise 1 *Describe where the PDE is hyperbolic, parabolic, or elliptic.*

Describe the regions where the PDE is hyperbolic (h); parabolic (p) and elliptic (e):
(a) $u_{xx} - u_{xy} - 2u_{yy} = 0$
(b) $2u_{xx} + 4u_{xy} + 3u_{yy} + 7u = 0$
(c) $yu_{xx} - 2u_{xy} + e^x u_{yy} - u = 3$

Exercise 2 *Classfication of the PDE.*

Classify the the PDE $3u_{x_1x_1} + u_{x_2x_2} + 4u_{x_2x_3} + 4u_{x_3x_3} = 0$ with the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}. \tag{28}$$

Exercise 3 *The formulation of a PDE in regard to your area of specialization.*

Formulate a well posed PDE problem that pertains to your area of specialization (e.g. physics, engineering, atmoshpheric science, etc.) and discuss the physics connected with the PDE problem.

Exercise 4 *The construction of a program to solve an equation, the storage of the data, the use of some visualization software.*

With $a = b = 1$, write a program to evaluate the solution $u(x, y) = x^2 - y^2$ on a 400×400 grid of points. Store these solution values in the HDF format of a Scientific Data Set (SDS). Use the visualization package NCSA XImage to display the solution as a two dimensional raster map.

Exercise 5 *The determination of the initial and boundary conditions of the wave equation, and plotting the results.*

A solution to the wave equation $u_{xx} - u_{tt} = 0$ is given by $\sin(x)\sin(t)$ for $0 < x < p, t > 0$. What are the initial and boundary conditions that this solution obeys. Use the graphics package xmgr to plot (on the same graph) the solution at times $t = 0, \pi/2, \pi$.

Exercise 6 *Plot the solution of the diffusion equation.*

The diffusion equation $u_t - u_{xx} = 0$ has solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}. \tag{29}$$

Use xmgr to plot this solution for $-5 < x < 5$ for $t = .1$, $t = 1.1$ and $t = 2.1$.

Exercise 7 *A consideration of practical application leads to Laplace's equation, which reduces to an ODE.*

Consider the steady state temperature in a circular gun barrel. Assuming radial symmetry, and no variation in temperature along the length of the barrel, leads to Laplace's equation for temperature as a function of the radial distance r from the center of the barrel. With temperature given by $u = u(r)$ for $r_1 < r < r_2$, the Laplace PDE reduces to the ODE $(ru')' = 0$.