Chebychev Polynomials:

Consider the following:

$\cos nt = T_n(\cos t)$

Where T_n is an n-degree polynomial.

This can be easily shown by the recurrence relation:

$\cos (k+1)t = 2 \cos t \cos kt - \cos (k-1)t$

By induction, you can show that if cos kt is a polynomial of degree k in cos t and so is cos (k-1)t, then cos (k+1)t is a polynomial of degree k+1 in cos t.

This corresponds to a recurrence relation:

$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$
(here x = cos t).

Chebychev Polynomials:

$T_0 = 1$, $T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$

and so on.

These polynomials are satisfied for arbitrary real or complex x.

The zeros of Tn(x) are simply given at cos nt = 0, or

$$x_k^{(n)} = \cos\left(t_k^{(n)}\right),\ t_k^{(n)} = \frac{2k-1}{2n}\pi, k = 1, 2, \ldots, n$$

The zeros of the polynomial are all real, distinct, and contained on a unit circle. The corresponding x values are projections to the x axis of these points.

Chebychev Polynomials:

T_n oscillates and achieves extreme values:

$$y_k^{(n)} = \cos \eta_k^{(n)}, \eta_k^{(n)} = \frac{k\pi}{n}, k = 0, 1, \ldots, n$$

Theorem: For an arbitrary monic polynomial p_n of degree n, there holds:

$$max_{(-1 \le x \le 1)}|p_n(x)|\ \ max_{(-1 \le x \le 1)}|\tilde{T}_n(x)| = \frac{1}{2^{n-1}}$$

The proof of this is by contradiction and counting the number of changes of sign of the difference between p and T. Since both are monic polynomials, the difference must be degree n-1 or less. Therefore it could not have n roots. Thus the difference must be zero.