Chebychev Polynomials:

Consider the following:
\[ \cos nt = T_n(\cos t) \]
Where \( T_n \) is an \( n \)-degree polynomial.

This can be easily shown by the recurrence relation:
\[ \cos (k+1)t = 2 \cos t \cos kt - \cos (k-1)t \]
By induction, you can show that if \( \cos kt \) is a polynomial of degree \( k \) in \( \cos t \) and so is \( \cos (k-1)t \), then \( \cos (k+1)t \) is a polynomial of degree \( k+1 \) in \( \cos t \).

This corresponds to a recurrence relation:
\[ T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \]
(here \( x = \cos t \)).

Chebychev Polynomials:

\( T_0 = 1, \ T_1(x) = x \)
\( T_2(x) = 2x^2 - 1 \)
\( T_3(x) = 4x^3 - 3x \)
and so on.

These polynomials are satisfied for arbitrary real or complex \( x \).

The zeros of \( T_n(x) \) are simply given at \( \cos nt = 0 \), or
\[ x_k^{(n)} = \cos \left( \frac{(k-1)\pi}{n} \right), \quad k = 1, 2, \ldots, n \]
The zeros of the polynomial are all real, distinct, and contained on a unit circle. The corresponding \( x \) values are projections to the \( x \) axis of these points.

Chebychev Polynomials:

\( T_n \) oscillates and achieves extreme values:
\[ y_k^{(n)} = \cos y_k^{(n)}, \quad y_k^{(n)} = \frac{k\pi}{n}, k = 0, 1, \ldots, n \]

Theorem: For an arbitrary monic polynomial \( p_n \) of degree \( n \), there holds:
\[ \max_{-1 \leq x \leq 1} |p_n(x)| \leq \max_{-1 \leq x \leq 1} |T_n(x)| = \frac{1}{2^{n-1}} \]
The proof of this is by contradiction and counting the number of changes of sign of the difference between \( p \) and \( T \). Since both are monic polynomials, the difference must be degree \( n-1 \) or less. Therefore it could not have \( n \) roots. Thus the difference must be zero.