CS514 Fall '00

Numerical Analysis

(Sketched) Solution of Homework 3

1. Questions from text in Chapter 3

Problem 1:

$$\begin{array}{llll}
0 & f_0 & & \\
\frac{1}{8} & f_1 & 8(f_1 - f_0) & \\
\frac{1}{4} & f_2 & 8(f_2 - f_1) & 32(f_2 - 2f_1 - f_0) \\
\frac{1}{2} & f_3 & 4(f_3 - f_2) & \frac{32}{3}(f_3 - 3f_2 + 2f_1) \quad \frac{64}{3}(f_3 - 6f_2 + 8f_1 - 3f_0)
\end{array}$$

Therefore, $f'(0) = \frac{1}{3}(2f_3 - 24f_2 + 64f_1 - 42f_0) + e$, and

$$|e| \le \frac{1}{8}\frac{1}{4}\frac{1}{2}\frac{M_4}{4!} = \frac{M_4}{1536}$$

.

Problem 6:
(a)

$$I(h) = \frac{1}{3}h^3 + \frac{2}{7}h^{7/2},$$

$$T(h) = \frac{1}{2}h^3 + \frac{1}{2}h^{7/2}.$$

Then,

$$E(h) = O(h^3) \quad \text{as} \quad h \to 0.$$

(b)

$$I(h) = \frac{1}{3}h^3 + \frac{2}{3}h^{3/2},$$

$$T(h) = \frac{1}{2}h^3 + \frac{1}{2}h^{3/2}.$$

Then,

$$E(h) = O(h^{3/2}) \quad \text{as} \quad h \to 0.$$

The error is larger by $1\frac{1}{2}$ order compared to (a). f in (a) is in $C^2[0,h]$; but f in (b) is in $C[0,h]$.

Problem 7:

(a) By Taylor's Theorem, letting $x_k + \frac{1}{2}h$ be written as x',

$$f(x) = f(x') + (x - x')f'(x') + \frac{1}{2}(x - x')^2 f''(\xi(x)),$$

$$x_k \le x \le x_k + h,$$

where $\xi \in [x_k, x_k + h]$. Integrating gives,

$$\int_{x_k}^{x_k+h} f(x)dx = hf(x') + f'(x') \int_{x_k}^{x_k+h} (x - x')dx$$
$$+\frac{1}{2}\int_{x_k}^{x_k+h} (x - x')^2 f''(\xi(x))dx.$$

The second term on the right hand side is zero and apply Mean Value Theorem, yields,

$$\int_{x_k}^{x_k+h} f(x)dx = hf(x') + \frac{1}{2}f''(\xi(x)) \int_{x_k}^{x_k+h} (x - x')^2 dx.$$

Calculate

$$\frac{1}{2}f''(\xi(x)) \int_{x_k}^{x_k+h} (x - x')^2 dx = \frac{h^3}{12}.$$

Hence,

$$\int_{x_k}^{x_k+h} f(x)dx = hf(x')f(x') + \frac{h^3}{24}f''(\xi_k), x_k \le \xi_k \le x_k + h.$$

(b) Sum all subintervals,

$$\int_a^b f(x)dx = h\sum_{k=0}^{n-1} f(x') + \frac{h^2}{24}\frac{b-a}{n}\sum_{k=0}^{n-1} f''(\xi_k),$$

that is,

$$\int_a^b f(x)dx = h\sum_{k=0}^{n-1} f(x') + \frac{h^2}{24}(b - a)f''(\xi), a \le \xi \le b.$$

Problem 8:

(a)

$$\begin{array}{llllll}
1 & y_{-1} & & & \\
0 & y_0 & y_0 - y_{-1} & & \\
0 & y_0 & y_0' & y_0' - y_0 + y_{-1} & \\
1 & y_1 & y_1 - y_0 & y_1 - y_0 - y_0' & \frac{1}{2}(y_1 - y_{-1} - 2y_0')
\end{array}$$

$$p_3(y;t) = y_{-1} + (t + 1)(y_0 - y_{-1}) + (t + 1)t(y_0' - y_0 + y_{-1})$$
$$+\frac{1}{2}(t + 1)t^2(y_1 - y_{-1} - 2y_0').$$

Then,

$$\int_{-1}^{1} p_3(y;t)dt$$

$$= 2y_{-1} + 2(y_0 - y_{-1}) + \frac{2}{3}(y_0' - y_0 + y_{-1}) + \frac{1}{3}(y_1 - y_{-1} - 2y_0')$$

$$= \frac{1}{3}(y_{-1} + 4y_0 + y_1).$$

(b)

$$E^S(y) = \int_{-1}^{1} (t+1)t^2(t-1)\frac{y^{(4)}(\tau(t))}{4!}dt$$

$$= \frac{-1}{90}y^{(4)}(\tau), -1 \le \tau \le 1.$$

(c) Let n be even, $h = (b-a)/n$, $x_k = a + kh$, and $f_k = f(x_k)$. Then,

$$\int_{x_k}^{x_{k+2}} f(x)dx = h\int_{-1}^{1} f(x_{k+1} + th)dt.$$

Let $y(t) = f(x_{k+1} + th)$ in (a) and (b), we obtain

$$\int_{x_k}^{x_{k+2}} f(x)dx = h\{\frac{1}{3}(f_k + 4f_{k+1} + f_{k+2}) - \frac{1}{90}h^4 f^{(4)}(\xi_k)\},$$

$$\xi_k = x_{k+1} + \tau h, -1 \le \tau \le 1.$$

Sum all even k from 0 to n-2,

$$\int_a^b f(x)dx = \sum_{k=0}^{n-2} \int_{x_k}^{x_{k+2}} f(x)dx,$$

k is even. After calculating, we will get

$$\int_a^b f(x)dx = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n) + E_n^S(f),$$

where

$$E_n^S(f) = \frac{a-b}{180}h^4 f^{(4)}(\xi), a \le \xi \le b.$$

Problem 10:

$$f = e^{-x^2}, f'' = 2(2x^2-1)e^{-x^2}, f^{(4)} = 4(4x^4 - 12x^2 + 3)e^{-x^2}.$$

so,

$$|f''| \le 2, |f^{(4)}| \le 12 \text{ on } [0,1].$$

(a)

$$E_n^T = -\frac{1}{12n^2} f''(\xi), 0 \le \xi \le 1,$$

we get,

$$|E_n^T| \le \frac{1}{6n^2} \le \frac{1}{2}10^{-6} \quad \text{if} \quad n \ge \frac{1}{3}10^6.$$

That is,

$$n \ge \frac{10^3}{\sqrt{3}} \approx 578.$$

(b)

$$E_n^S = -\frac{1}{180n^4} f^{(4)}(\xi), 0 \le \xi \le 1,$$

we get,

$$|E_n^S| \le \frac{1}{180n^4} 12 = \frac{1}{15n^4} \le \frac{1}{2}10^{-6}$$

if

$$n^4 \ge \frac{2}{15}10^6.$$

That is,

$$n \ge [\frac{200}{15}]^{1/4} 10 \approx 20.$$

Problem 35 The area of a unit disk can be computed as

$$A = 2\int_{-1}^{1} (1-t^2)^{1/2} dt = 2\int_{-1}^{1} (1-t^2)(1-t^2)^{-1/2} dt,$$

hence can be evaluated exactly by 2 point Gauss-Chebyshev quadrature rule applied to $f(t) = 1 - t^2$:

$$A = 2\frac{\pi}{2}(1 - t_1^2 + 1 - t_2^2) = \pi(2 - t_1^2 - t_2^2).$$

But,

$$t_1 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and

$$t_2 = \cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}}.$$

Thus,

$$A = \pi(2 - 1/2 - 1/2) = \pi.$$

2. Machine Assignment 1 in Chapter 3.

   After computing, we get

$$e_1(h) = \pi - \frac{(f_1 - f_0)}{h}, \quad e_2(h) = 2\pi^2 - \frac{f_1 - 2f_0 + f_{-1}}{h^2},$$

$$e_3(h) = 6\pi^3 - \frac{f_2 - 3f_1 + 3f_0 - f_{-1}}{h^3},$$

$$e_4(h) = 24\pi^4 - \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4}.$$

   The program is as following:

```
%
% Machine assignment 1 in Chapter 3, the problem description is not clear
% fixed and solve it.
% index for -2 -1 0 1 2 in f is 1 2 3 4 5
%
format long;
i=[-2 -1 0 1 2];
PI = 4*atan(1);
E1 = PI;
E2 = 2*pi^2;
E3 = 6*pi^3;
E4 = 24*pi^4;
k = 0:10;
h=2.^(-k-2);
f = 1 ./ (1 - PI * (i'*h));
P = [ ((f(4,:)-f(3,:)) ./ h) ;
      ((f(4,:)-2*f(3,:)+f(2,:)) ./ (h.^2));
      ((f(5,:)-3*f(4,:)+3*f(3,:)-f(2,:)) ./ (h.^3));
      ((f(5,:)-4*f(4,:)+6*f(3,:)-4*f(2,:)+f(1,:)) ./ (h.^4)) ];
E = [E1 - ((f(4,:)-f(3,:)) ./ h) ;
     E2 - ((f(4,:)-2*f(3,:)+f(2,:)) ./ (h.^2));
     E3 - ((f(5,:)-3*f(4,:)+3*f(3,:)-f(2,:)) ./ (h.^3));
     E4 - ((f(5,:)-4*f(4,:)+6*f(3,:)-4*f(2,:)+f(1,:)) ./ (h.^4)) ];
R=[];
```

```
for j=2:11
R = [R abs(E(:,j)./ E(:,j-1))];
end

fid1 = fopen('MA3_1','w');
fprintf(fid1,'         exact derivates    \n');
fprintf(fid1,' %8.3f  %8.3f  %8.3f  %8.3f \n', E1, E2, E3, E4);
fprintf(fid1,'        approximate derivates    \n');
fprintf(fid1,' %8.3f  %8.3f  %8.3f  %8.3f \n', P);
fprintf(fid1,'         approximate errors \n');
fprintf(fid1,' %8.3f  %8.3f  %8.3f  %8.3f \n', E);
fprintf(fid1,'      r1        r2        r3        r4 \n');
fprintf(fid1,' %8.3f  %8.3f  %8.3f  %8.3f \n', R);
fclose(fid1);
```

The result is as

```
         exact derivates
    3.142    19.739   186.038  2337.818
        approximate derivates
   14.639    51.518  -850.651  -4158.085
    5.173    23.338  1024.958  7214.081
    3.909    20.531   318.619  2874.910
    3.484    19.931   233.744  2455.227
    3.304    19.787   206.788  2366.272
    3.221    19.751   195.759  2344.877
    3.181    19.742   190.747  2339.580
    3.161    19.740   188.356  2338.258
    3.151    19.739   187.188  2337.930
    3.146    19.739   186.611  2337.840
    3.144    19.739   186.324  2337.781
         approximate errors
  -11.498   -31.779  1036.688  6495.903
   -2.031    -3.599  -838.921  -4876.262
```

-0.768	-0.792	-132.581	-537.092
-0.342	-0.192	-47.706	-117.409
-0.162	-0.048	-20.751	-28.454
-0.079	-0.012	-9.721	-7.059
-0.039	-0.003	-4.710	-1.761
-0.019	-0.001	-2.318	-0.440
-0.010	-0.000	-1.150	-0.111
-0.005	-0.000	-0.573	-0.022
-0.002	-0.000	-0.286	0.037

r1	r2	r3	r4
0.177	0.113	0.809	0.751
0.378	0.220	0.158	0.110
0.446	0.243	0.360	0.219
0.474	0.248	0.435	0.242
0.487	0.250	0.468	0.248
0.494	0.250	0.484	0.250
0.497	0.250	0.492	0.250
0.498	0.250	0.496	0.253
0.499	0.250	0.498	0.194
0.500	0.250	0.499	1.705

One can realize that the convergence order is $O(h)$ for $n = 1$ and $n = 3$, the ration tends to $\frac{1}{2}$. However, for $n = 2$ and $n = 4$, the order is $O(h^2)$ and the ratio tends to be $\frac{1}{4}$. Rounding error occurs in the last few values of $n = 4$ which corrupts the limit of the ration.

3. Problem 3 on HW sheet.

(a) trapezoid rule:

$$\int_0^1 e^{-x}dx = 0.1(\frac{1}{2}e^0 + \sum_{i=1}^{9} e^{\frac{i}{10}} + \frac{1}{2}e^{-1}) \approx 0.63265.$$

$$\int_0^1 e^{-x}dx = 0.63212$$

Then the error is -0.00053.

(b) gaussian quadrature: Let $\pi_3(t) = t^3+p_1t^2+p_2^t+p_3$. Since it is orthogonal to 1, t , and t^2, one can solve the coefficients which $p_1 = -\frac{3}{2}$, $p_2 = \frac{3}{5}$, and $p_3 = -\frac{1}{20}$. Then, we can get 3 roots for π_3. Then, one need to find the weights w_1, w_2, and w_3. Consider $f(t) \equiv 1$, $f(t) \equiv t$, and $f(t) \equiv t^2$, then solve the linear equation system and get w_1, w_2, and w_3. The results are as

$$t_1 = 0.887298,\ t_2 = 0.5,\ t_3 = 0.112702,$$

$$w_1 = 0.277778,\ w_2 = 0.444444,\ w_3 = 0.277778.$$

And,

$$\int_0^1 e^{-x}dx \approx 0.632120$$

$$\int_0^1 e^{-x}dx = 0.6321205588$$

Then the error is -0.0000005588 (more precision applied here). Therefore, gaussian quadrature preforms better for this problem

4. Questions from text in Chapter 4.

Problem 7:

(a) Plot the function. Since there are three intersections with the line $x = 0$, there are 3 real root. Their approximated location is -2, 2, and 10.5 respectively.

(b) No.

Problem 21:
(a)

$$x_{n+1} - \alpha = \frac{1}{2}(x_n + \frac{a}{x_n}) - \alpha = \frac{x_n^2 + a - 2\alpha x_n}{2x_n}$$

$$= \frac{x_n^2 - 2\alpha x_n + \alpha^2}{2x_n} = \frac{(x_n + \alpha)^2}{2x_n}.$$

$$\Rightarrow \frac{1}{2x_n} = \frac{x_{n+1} - \alpha}{x_n - \alpha}.$$

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{x_n - \alpha} = \lim_{n\to\infty} \frac{1}{2x_n} = \frac{1}{2\alpha}.$$

(b) Let $f(x) = x^3 - a$. Then $f'(x) = 3x^2$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - a}{3x_n^2} = \frac{3x_n^3 - x_n^3 + a}{3x_n^2}$$

$$= \frac{1}{3}(2x_n + \frac{a}{x_n^2}).$$

Then,

$$x_{n+1} - \alpha = \frac{1}{3}(2x_n + \frac{a}{x_n^2}) - \alpha = \frac{2x_n^3 + a - 3\alpha x_n^2}{3x_n^2}$$

$$= \frac{2x_n^3 + \alpha^3 - 3\alpha x_n^2}{2x_n^2} = \frac{\alpha^3 - x_n^3 + 3(x_n^3 - \alpha x_n^2)}{3x_n^2}$$

$$= \frac{(\alpha^3 - x_n^3) - 3x_n^2(\alpha - x_n)}{3x_n^2} = \frac{(\alpha - x_n)^2(\alpha + 2x_n)}{3x_n^2}.$$

$$\Rightarrow \frac{x_{n+1} - \alpha}{(x_n - \alpha)^2} = \frac{\alpha + 2x_n}{3x_n^2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^2} = \lim_{n\to\infty} \frac{\alpha + 2x_n}{3x_n^2} = \frac{1}{\alpha}.$$

Problem 22:

(a)

$$d_n = x_{n+1} - x_n = \frac{1}{2}(\frac{a}{x_n} - x_n).$$

$$\Rightarrow 2d_n x_n = a - x_n^2.$$

$$\Rightarrow x_n 2 + 2dx_n - a = 0.$$

$$\Rightarrow x_n = \frac{-2d_n \pm \sqrt{4dn^2 + 4a}}{2} = -d_n \pm \sqrt{d_n^2 + a}.$$

We assume $x_0 > 0$, then $x_n > 0$. Therefore,

$$x_n = -d_n + \sqrt{d_n^2 + a} = \frac{a + d_n^2 - d_n^2}{d_n + \sqrt{d_n^2 + a}}.$$

$$\Rightarrow x_n = \frac{a}{d_n + \sqrt{d_n^2 + a}}.$$

(b)

$$x_n = \frac{a}{d_n + \sqrt{d_n^2 + a}} = \frac{1}{2}(x_{n-1} + \frac{a}{x_{n-1}})$$

$$= \frac{1}{2}(\frac{a}{d_{n-1} + \sqrt{d_{n-1}^2 + 1}} + (d_{n-1} + \sqrt{d_{n-1}^2 + a})).$$

$$\Rightarrow \frac{a}{d_n + \sqrt{d_n^2 + a}} = \frac{1}{2}(\frac{a(d_{n-1} - \sqrt{d_{n-1}^2 + a})}{d_{n-1}^2 - d_{n-1}^2 - a} + d_{n-1} + \sqrt{d_{n-1}^2 + a})$$

$$= \frac{1}{2}(\sqrt{d_{n-1}^2 + a} - d_{n-1} + d_{n-1} + \sqrt{d_{n-1}^2 + a}) = \sqrt{d_{n-1}^2 + a}.$$

$$\Rightarrow \frac{a}{d_n + \sqrt{d_n^2 + a}} = \sqrt{d_{n-1}^2 + a}.$$
$$\Rightarrow d_n + \sqrt{d_n^2 + a} = \frac{a}{\sqrt{d_{n-1}^2 + a}}.$$
$$\Rightarrow \sqrt{d_n^2 + a} = \frac{a}{\sqrt{d_{n-1}^2}} - d_n.$$
$$\Rightarrow d_n^2 + a = d_n^2 - \frac{2d_n a}{\sqrt{d_{n-1}^2 + a}} + \frac{a^2}{d_{n-1}^2 + a}.$$
$$\Rightarrow \frac{-2d_n a}{\sqrt{d_{n-1}^2 + a}} = \frac{d_{n-1}^2}{d_{n-1}^2 + 2}.$$
$$\Rightarrow -d_n = \frac{d_{n-1}^2}{2\sqrt{d_{n-1}^2 + a}}.$$

Therefore,

$$|d_n| = \frac{d_{n-1}^2}{2\sqrt{d_{n-1}^2 + a}}.$$

Problem 35:

(a) $\varphi(x) = \frac{2}{x}$, $x \neq 0$. Then,

$$\varphi'(\alpha) = -2\alpha^{-1} \neq 0.$$

Claim $|\frac{2}{x^2}| < 1$, then $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$. It will converge only when $x_0 = \pm\sqrt{2}$. Otherwise, it never converges.

(b) $\varphi(x) = x^2 + x - 2$. Then,

$$\varphi'(\alpha) = 2\alpha + 1 \neq 0.$$

Claim $|2x + 1| < 1$, then $x \in (-1, 0)$. It will never converge to the root.

(c) $\varphi(x) = \frac{x+2}{x+1}$, $x \neq -1$. Then,

$$\varphi'(\alpha) = \frac{-1}{(\alpha + 1)^2} \neq 0.$$

Claim $|\frac{-1}{(x+1)^2}| < 1$, then, $x \in (-\infty, -2) \cup (0, \infty)$. Since $-\sqrt{2} > -2$, it will converge when $x_0 > 0$.

5. Machine Assignment 1 in Chapter 4.

The program is as following:

FORTRAN(f90) part, main program

```
!
! This is for Machine Assignment 1 in Chapter 4
!
      PROGRAM ma4_1
      real e
      real from, to, h
      integer i, n

      n=200
      e = Eps()

! For part (a):
      open(20, file='ma4_1a')
      from = 0.93
      to = 1.07
      h = (to-from)/n
      write(20,2) from
      write(20,2) to
      do i=0, n
         x = from+i*h
         write(20,1)(pa(x)/e)
      enddo
      close(20)
 1    format(e16.7)
 2    format(f16.7)

! For part (b):
      open(20, file='ma4_1b')
      from = 21.7
      to = 22.2
      h = (to-from)/n
      write(20,2) from
      write(20,2) to
      do i=0, n
```

```
         x = from+i*h
         write(20,1)(pb(x)/e)
      enddo
      close(20)
      end

      real function Eps()
      real a, b, c
      a=4./3.
      b=a-1
      c=b+b+b
      Eps=abs(c-1)
      return
      end

      real function pa(x)
      real x
      pa=x**5-5*x**4+10*x**3-10*x**2+5*x-1;
      return
      end

      real function pb(x)
      real x
      pb=x**5-100*x**4+3995*x**3-79700*x**2+794004*x-3160088;
      return
      end
```

MATLAB part, for plot only

```
%
% for plot in Machine assignment 1 of Chapter 4, read file ma4_1a and ma4_1b
%

fid1 = fopen('ma4_1a', 'r');
```

```
pa = fscanf(fid1, '%g', [203]);
fclose(fid1);
fid1 = fopen('ma4_1b', 'r');
pb = fscanf(fid1, '%g', [203]);
fclose(fid1);

ha = (pa(2)-pa(1))/200;
hb = (pb(2)-pb(1))/200;
ia = pa(1):ha:pa(2);
ib = pb(1):hb:pb(2);

figure;
subplot(2,1,1);
plot(ia, pa(3:203), '-');
title('polynomial a/eps');
grid on;
subplot(2,1,2);
plot(ib, pb(3:203), '-');
title('polynomial b/eps')
grid on;
```

The plots are as below

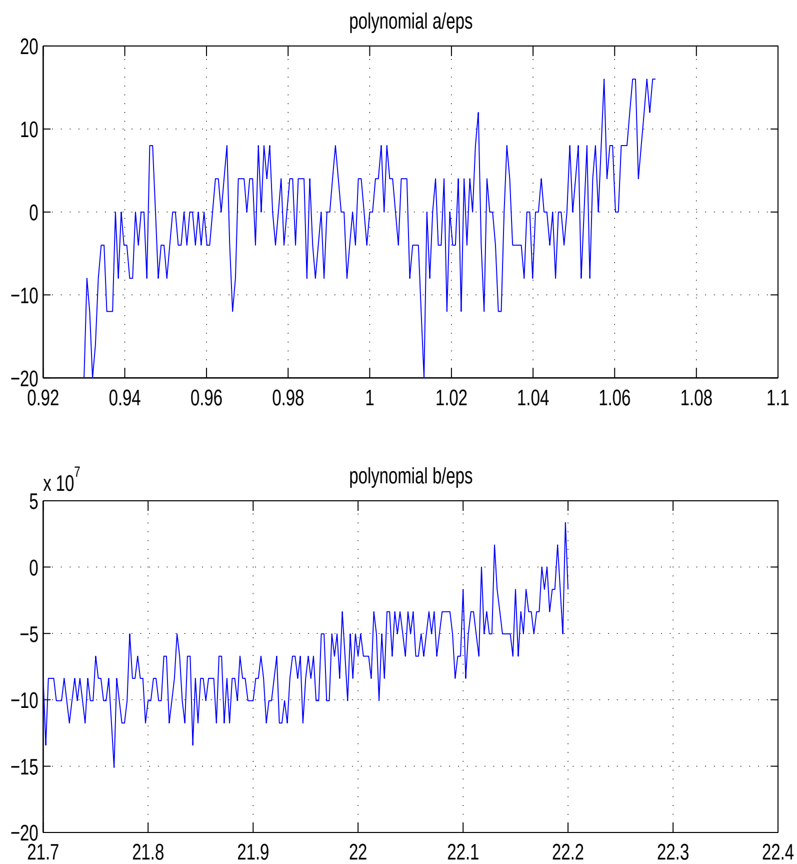

Figure 1: The plots for two functions