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CS514 Fall '00

Numerical Analysis

(Sketched) Solution of Homework 2

1. Questions from text in Chapter 2

Problem 2:

(a) $\|f - \hat{f}\|_\infty = \max_{1 \le i \le N} |f(t_i) - c\| = \max\{\|c-1, y-c\|\}$. Minimize this and get $\hat{f} = \frac{y+1}{2}$.

(b) $\|f - f'\|_2 = [(N-1)(c-1)^2 + (c-y)^2]$. When $c = \frac{N-1+y}{N}$, it has a minimum. So, $f' = c = \frac{N-1+y}{N}$.

(c) When N approaches to ∞, $\hat{f}$ still is $\frac{y+1}{2}$ but $f' \to 1$.

Problem 7:

(i) $\pi_1 = \frac{1}{1+t}$ and $\pi_2 = \frac{1}{(1+t)^2}$. Then, $a_{11} = (\pi_1, \pi_1)$, $a_{12} = (\pi_1, \pi_2) = a_{21}$, and $a_{22} = (\pi_2, \pi_2)$. Also, $b_1 = (\pi_1, f)$ and $b_2 = (\pi_2, f)$. Note, $(u, v) = \int uv$. After calculations, $a_{11} = \frac{1}{2}$, $a_{12} = a21 = \frac{3}{8}$, $a_{22} = \frac{7}{24}$, $b_1 = 0.4634$, and $b_2 = 0.3526$. Note, the hint will be needed when calculating b_i's. Then, find A^{-1} and $\|A^{-1}\|_infty$. $\text{cond}_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 147$

(ii) $t = 0, f(0) - \psi(0) = -0.0486$; $t = 1, f(1) - \psi(1) \approx 0.0156$; and $t = 2, f(2) - \psi(2) \approx -0.0347$;

Problem 10:

(a) $(\pi_i, \pi_j) = \int \pi_i \pi_j d\lambda$. Since equally weight, $(\pi_i, \pi_j) = \int_0^\infty e^{-it} e^{-jt} dt = \frac{1}{i+j}$. Also, $(\pi_i, f) = \int_0^\infty e^{-it} dt = \frac{1-e^{-i}}{i}$. Let $a_{ij} = (\pi_i, \pi_j)$ and $b_i = (\pi_i, f)$. Then the normal equations are as:

$$\begin{pmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \cdots & \frac{1}{1+n} \\ \frac{1}{2+1} & \frac{1}{2+2} & \cdots & \frac{1}{2+n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{n+n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = \begin{pmatrix} \frac{1-e^{-1}}{1} \\ \frac{1-e^{-2}}{2} \\ \cdots \\ \frac{1-e^{-n}}{n} \end{pmatrix}.$$

The Hilbert matrix is $h_{ij} = \frac{1}{i+j-1}$.

(b) Plots.

Problem 17:

(a) $B_0^n(t) = (1-0)^n = 1$. Note that, $t = 0$ is the zero of multiplicity j of $B_j^n(t)$. Therefore, the claim is proved.

(b) At $t = 1$, we can have similar property. $B_n^n(t) = i^n(1-1)^0 = 1$, and for $j = 0, 1, \cdots, n$, $\frac{d^r}{dt^r} B_j^n(t)|_{t=1} = 0$, $r = 0, 1, \cdots, n-j-1$; $\frac{d^{n-j}}{dtn-j} B_j^n(t)|_{t=1} \neq 0$. SInce $B_j^n(t) = B_{n-j}^n(1-t)$, one can derive properties at $t = 1$ by setting $B_{n-j}^n(t)$ at $t = 0$.

(c) plots.

(d) If not, there exist constants $c_0, \cdots, c_n$ which are not all 0 such that $0 = \sum_{i=0}^n c_i B_i^n(t)$. Then, $0 = \sum_{i=0}^n c_i B_i^n(t) = c_0 + [\sum_{i=0}^1 c_i \begin{pmatrix} n \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}]t + \cdots + [\sum_{i=0}^n c_i \begin{pmatrix} n \\ n \end{pmatrix} \begin{pmatrix} n \\ n \end{pmatrix}]t^n$. This implies $c_0 = c_1 = \cdots = c_n = 0$. A contradiction.

(e) $\sum_0^n B_j^n(t) = (t+1-t)^n = 1$.

Problem 18: Since $\{\pi_j\}_{j=1}^n$ is linear dependent, $\exists i$ such that $\pi_i = \sum_{j=1, j\neq i}^n c_j \pi_j$. Then, $a_{1i} = (\pi_1, \pi_i) = \int \pi_1 \pi_i = \sum_{j=1, j\neq i}^n c_j a_{1j}$, $a_{2i} = (\pi_2, \pi_i) = \sum_{j=1, j\neq i}^n c_j a_{2j}$, $\cdots$, $a_{ni} = (\pi_n, \pi_i) = \sum_{j=1, j\neq i}^n c_j a_{nj}$. Then, ith column of matrix A is a linear combination of the other columns of A. Therefore, A is singular.

Problem 30 $f(x) - p_2(f; x) = (x-x_0)(x-x_1)(x-x_2)\frac{f'''(\xi)}{6}$ and $\|f - p_2(f; .)\|_\infty \leq \frac{M_3}{9\sqrt{3}} h^3$, $M_3 = \|f'''\|\infty$. Bring $x_0 = 10$, $x_1 = 11$, and $x_2 = 12$ into the error function. Then, $\|f - p_2(f; .)\| \leq |(11.1-10)(11.1-11)(11.1-12)\frac{f'''(\xi)}{6}| \leq 3.3 \times 10^{-5}$, where $f'''(x) = 2x^{-3}$. Also, since $f(10) \leq f(x)$, for $x = 10, 11, 12$, the relative error will be $\leq \frac{3.3\times 10^{-5}}{f(10)}$.

Problem 32 $f(x) = \frac{1}{x}$, then $f'(x) = -x^{-2}$ and $f''(x) = 2x^{-3}$. $x_0 = 1$, $f(x_0) = f(1) = 1$; $x_1 = 2$,$f(x_1) = f(2) = \frac{1}{2}$. Then, $p_1(f; x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f_1(x_1)\frac{x-x_0}{x1-x_0} = \frac{1}{2}(x-1) - (x-2) = \frac{3}{2} - \frac{1}{2}x$. Then, $f(x) - p_1(f; x) = \frac{1}{x} - \frac{3}{2} - \frac{1}{2}x = (x-x_0)(x-x_1)\frac{f''(\xi(x))}{2}$. Then, $(x-1)(x-2)\frac{1}{\xi^3(x)} = \frac{1}{2x}(x-1)(x-2)$. So, $\xi(x) = \sqrt[3]{2x}$. Therefore, max $= \sqrt[3]{4}$ and min $= \sqrt[3]{2}$.

Problem 38 $f(x) = \int_5^\infty \frac{e^{-t}}{t-x} dt$. $f^{(n)}(x) = (-1)^n n! \int_5^\infty \frac{e^{-t}}{(t-x)^{n+1}} dt$. And, $\|f - p_{n-1}(f; x)\|_\infty \leq \frac{\|f^{(n)}(x)\|_\infty}{n!} \frac{1}{2^{n-1}}$. Then, $\|f - p_{n-1}(f; x)\|_\infty \leq \frac{1}{2^{n-1}} \| \int_5^\infty \frac{e^{-t}}{(t-x)^{n+1}} dt\|_\infty$. Since $-1 \leq x \leq 1$, $\|f - p_{n-1}(f; x)\|_\infty \leq \frac{1}{2^{n-1}} \int_5^\infty \frac{e^{-t}}{(t-1)^{n+1}} dt$. Also, since $t \geq 5$, $\|f - p_{n-1}(f; x)\|_\infty \leq \frac{1}{2^{n-1}} \int_5^\infty \frac{e^{-t}}{(5-1)^{n+1}} dt \leq \frac{1}{2^{3n+1} e^5}$.

Problem 46 Let $T_n = \cos(n\theta)$. Since $x = T(x)$, $\cos\theta = \cos n\theta$, where $0 \leq \theta \leq \pi$. Therefore, $\cos\theta - \cos n\theta = 0$. Solve this equation and discuss and count the possible toor for this equation.

Problem 51

(i) $[0, 1, 1, 1, 2, 2]f = 30$

(ii) From (eq. 2.68), $[0,1,1,1,2,2]f = \frac{1}{5!}f^{(5)}(\xi)$. Also, $f^{(5)}(x) = 7 \times 6 \times 5 \times 4 \times 3x^2$. Hence, $\xi = \sqrt{\frac{10}{7}}$.

Problem 56

(a) $p_3(f;x) = 5 + (-2)(x-0) + (x-0)(x-1) + \frac{1}{4}(x-0)(x-1)(x-3) = \frac{1}{4}x^3 - \frac{9}{4}x + 5$.

(b) Set $p_3'(f;x) = 0$ and then get the min location of $x = \sqrt{3}$.

Problem 58

(a) $p(x) = 1 + (x-0) + (-2)(x-0)(x-1) + \frac{2}{3}(x-1)^2(x-0) + \frac{-5}{36}(x-3)(x-1)^2(x-0) = 1 + \frac{49}{12}x + \frac{-155}{36}x^2 + \frac{49}{36}x^3 + \frac{-5}{36}x^4$. Then $p(2) = \frac{11}{18}$.

(b) $|f-p| = |(x-0)(x-1)^2(x-3)^2\frac{f^{(5)}(\xi)}{5!}|$ $|f-p| \leq |(x-0)(x-1)^2(x-3)^2\frac{M}{5!}|$. Since $x = 2$, $|f-p| \leq \frac{M}{20}$.