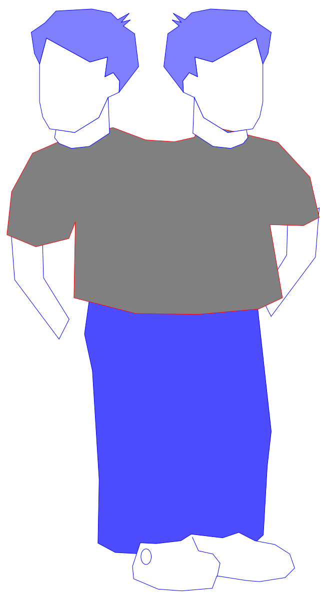
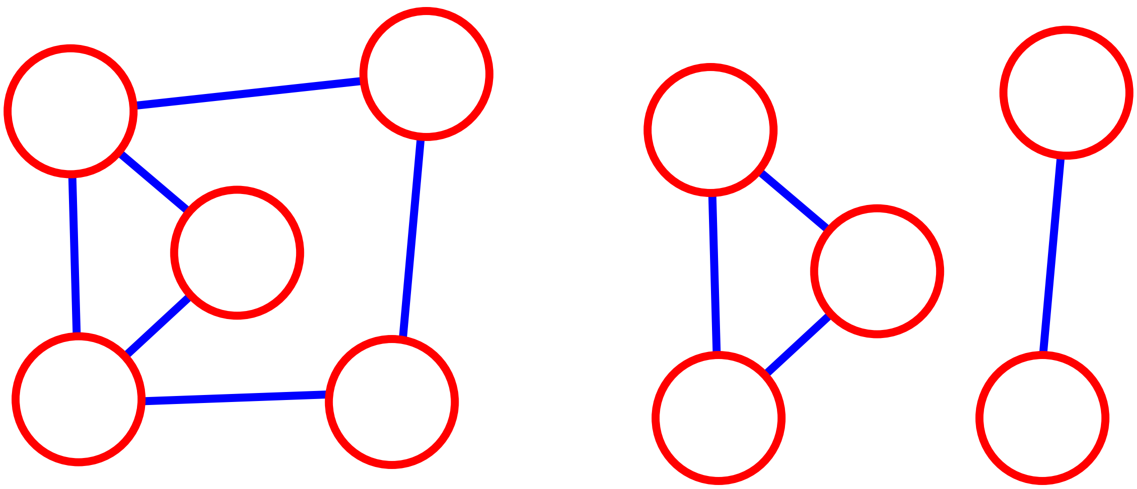
Connectivity and Biconnectivity



Connected Components

Connected Graph: any two vertices connected by a path



connected

not connected

Connected Component: maximal connected subgraph of a graph

Equivalence Relations

A ***relation*** on a set S is a set R of ordered pairs of elements of S defined by some property

Example:

- S = {1,2,3,4}
- R= {(i,j) ∈ S × S such that i < j}
  = {(1,2),(1,3),(1,4),(2,3),(2,4),{3,4)}

An ***equivalence relation*** is a relation with the following properties:

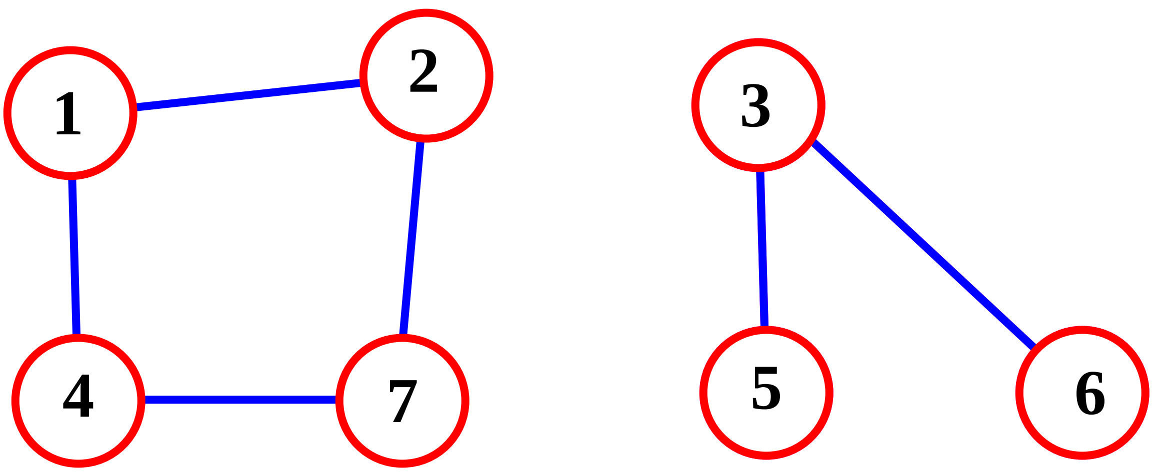
- (x,x) ∈ R, ∀ x ∈ S (***reflexive***)
- (x,y) ∈ R ⇒ (y,x) ∈ R (***symmetric***)
- (x,y), (y,z) ∈ R ⇒ (x,z) ∈ R (***transitive***)

The relation C on the set of vertices of a graph:

- (***u***,***v***) ∈ C ⇔ ***u*** and ***v*** are in the same connected component

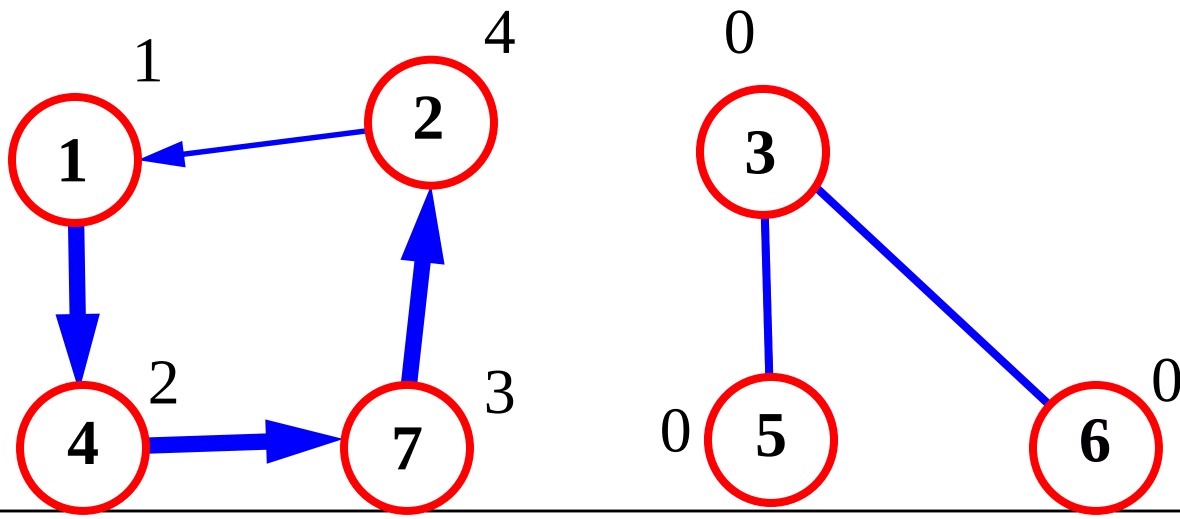
is an equivalence relation.

DFS on a Disconnected Graph



After dfs(1) terminates:

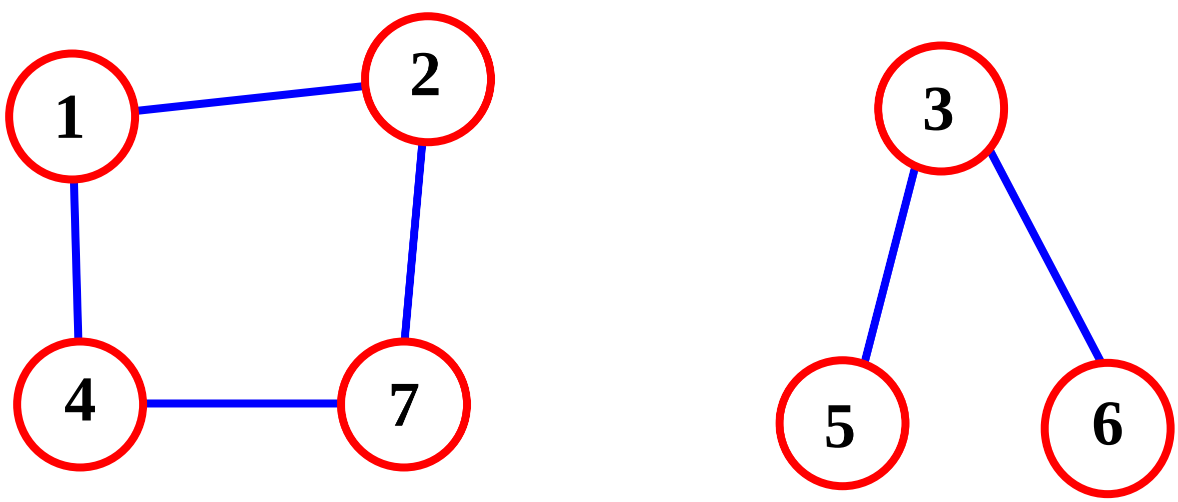
k	1	2	3	4	5	6	7
val[k]	1	4	0	2	0	0	3



DFS of a Disconnected Graph

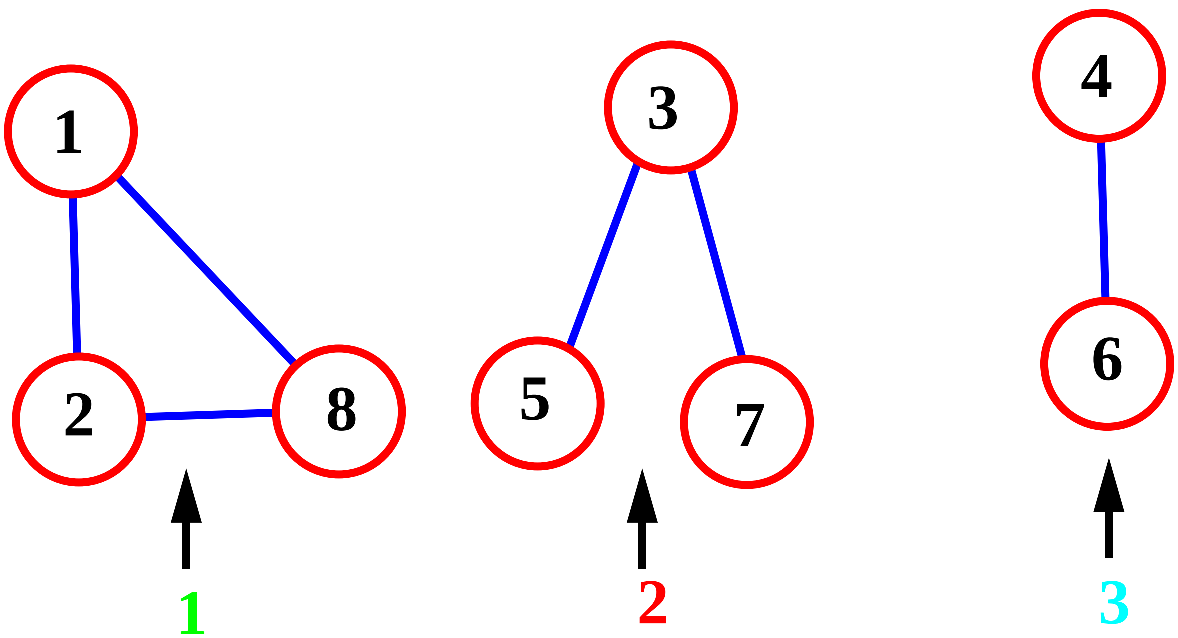
- Recursive DFS procedure visits all vertices of a connected component.
- A **for** loop is added to visit all the graph

```
for  all k from 1 to N
     if val[k] = 0
        dfs(k)
```



Representing Connected Components

Array comp [1..N]
comp[k] = i if vertex k is in
i-th connectedcomponent



vertex k	1	2	3	4	5	6	7	8
comp[k]	1	1	2	3	2	3	2	1

New DFS Algorithm

Inside DFS:

replace

```
  id = id + 1;
val [k] = id;
```

with

```
comp[k] = id;
```

Outside DFS:

```
for all k from 1 to N          for each vertex
    if comp [k] = 0            if not in comp
        id = id + 1;           new component
        dfs(k);
```

DFS Algorithm for Connected Components

Pseudocoded dfs (int k);

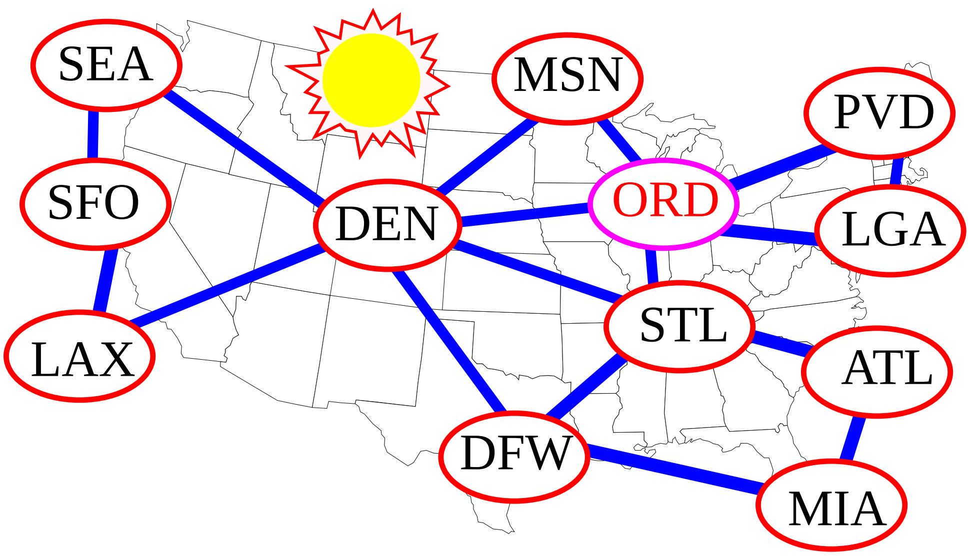
```
comp[k] = vertex.id;
vertex  = adj[k];

Vertex vertex
while (vertex != null)
     if (val[vertex.num] == 0)
          dfs (vertex.num);
          vertex = vertex.next;
. . .

for all k from 1 to N
     if (comp[k] == 0)
               id = id + 1;
               dfs (k);
```

TIME COMPLEXITY: O (N + M)

Cutvertices

Cutvertex (separation vertex):
its removal disconnects the graph

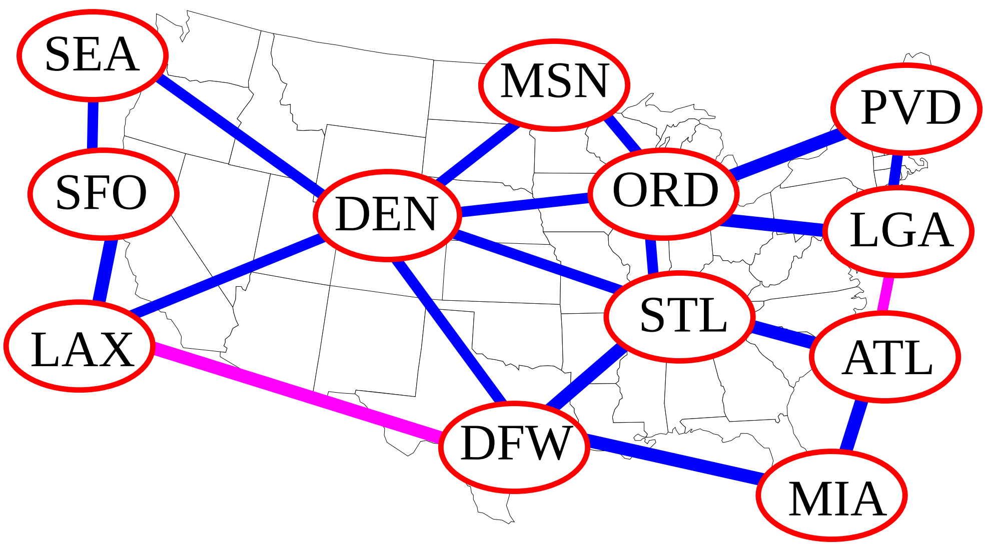
If the Chicago airport is closed, then there is no way to get from Providence to beautiful Denver, Colorado!



- Cutvertex: ORD

Biconnectivity

Biconnected graph: has no cutvertices

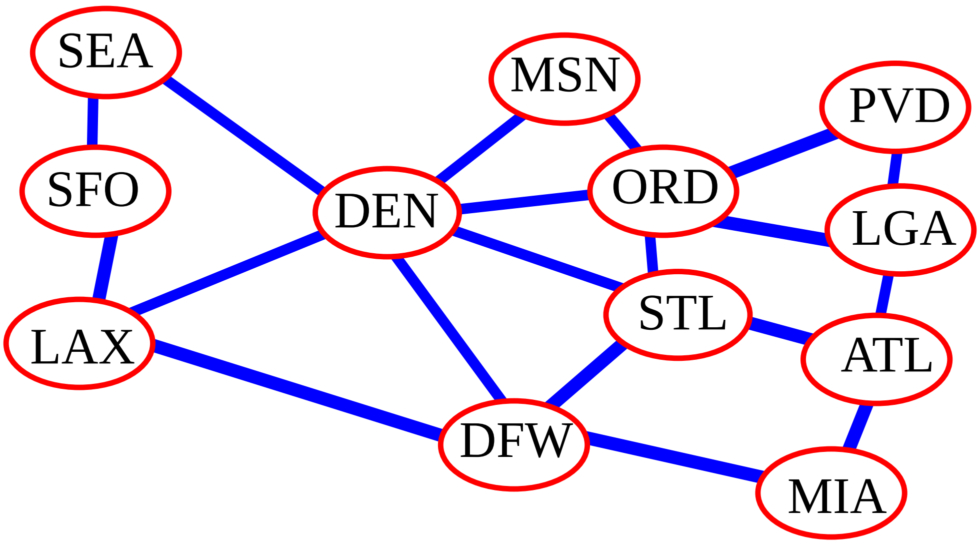


New flights:

LGA-ATL and DFW-LAX

make the graph biconnected.

Properties of Biconnected Graphs

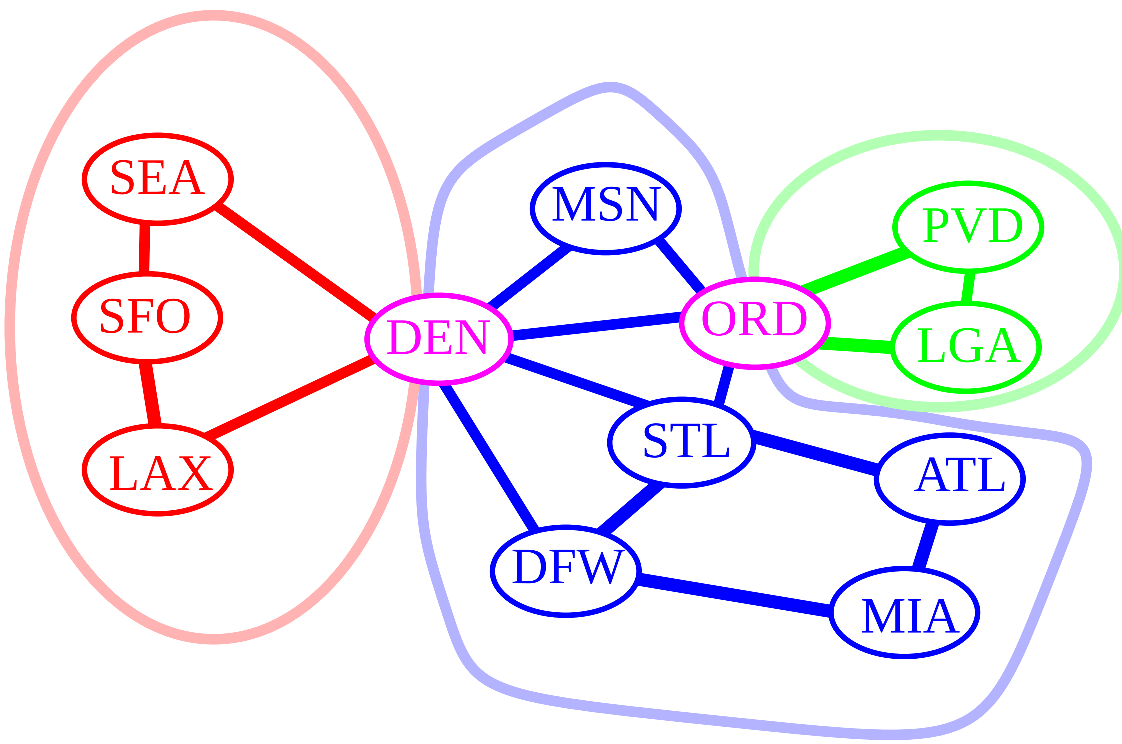


- There are two disjoint paths between any two vertices.
- There is a simple cycle through any two vertices.

By convention, two nodes connected by an edge form a biconnected graph, but this does not verify the above properties.

Biconnected Components

Biconnected component (block):
maximal biconnected subgraph



Biconnected components are edge-disjoint but share cutvertices.

Finding Cutvertices:
Brute Force Algorithm

for each vertex v
remove v;
test resulting graph for connectivity;
put back v;

Time Complexity:

- **N** connectivity tests
- each taking time O (**N** + **M**)

Total time:

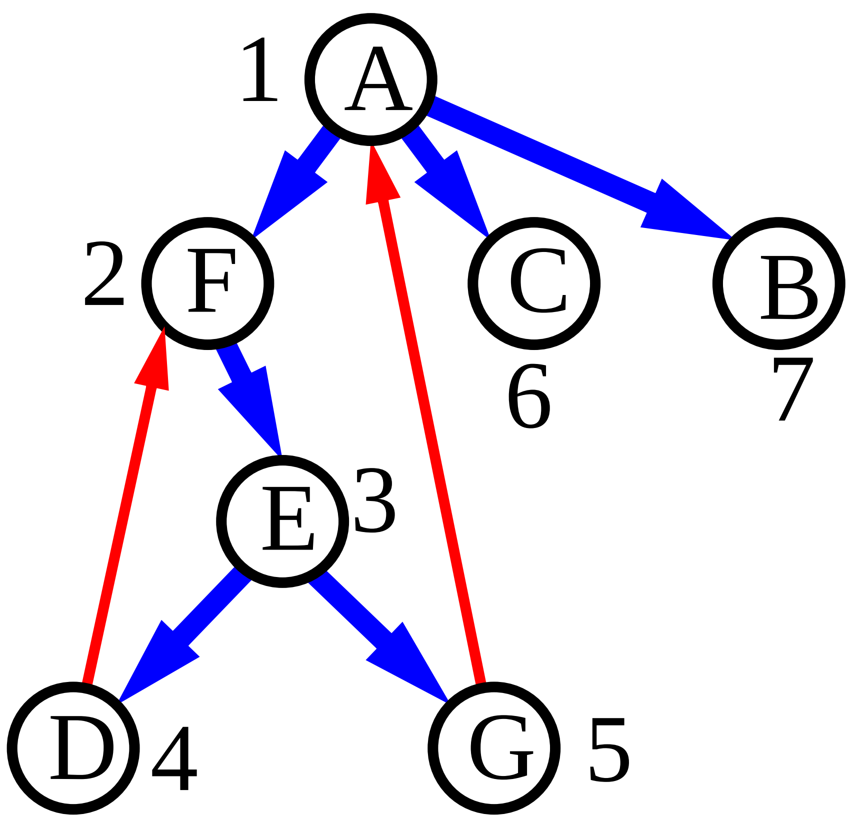
- $O(N^2 + NM)$

DFS Numbering

We recall that depth-first-search partitions the edges into tree edges and back edges

- (u,v) tree edge ⇔ val [u] < val [v]

- (u,v) back edge ⇔ val[u] > val[v]

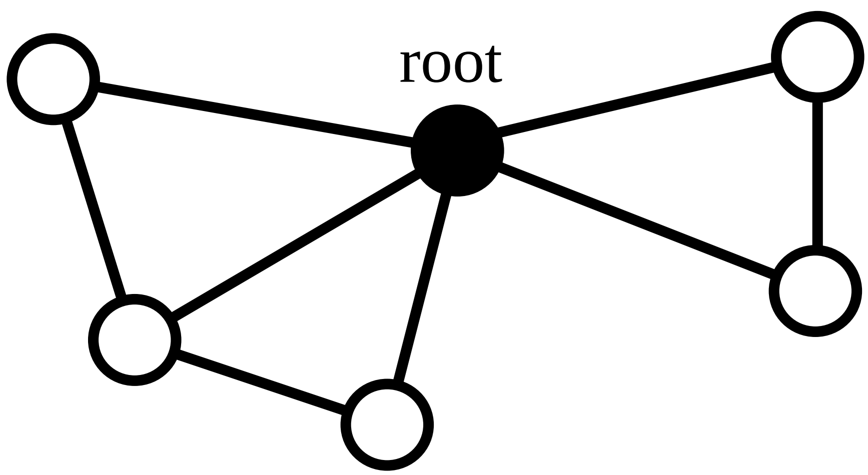


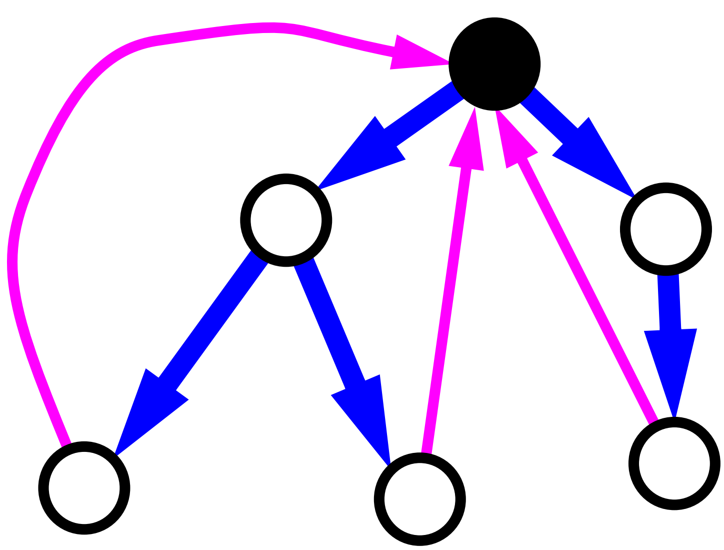
We shall characterize cutvertices using the DFS numbering and two properties ...

Root Property

The root of the DFS tree is a cutvertex if it has two or more outgoing tree edges.

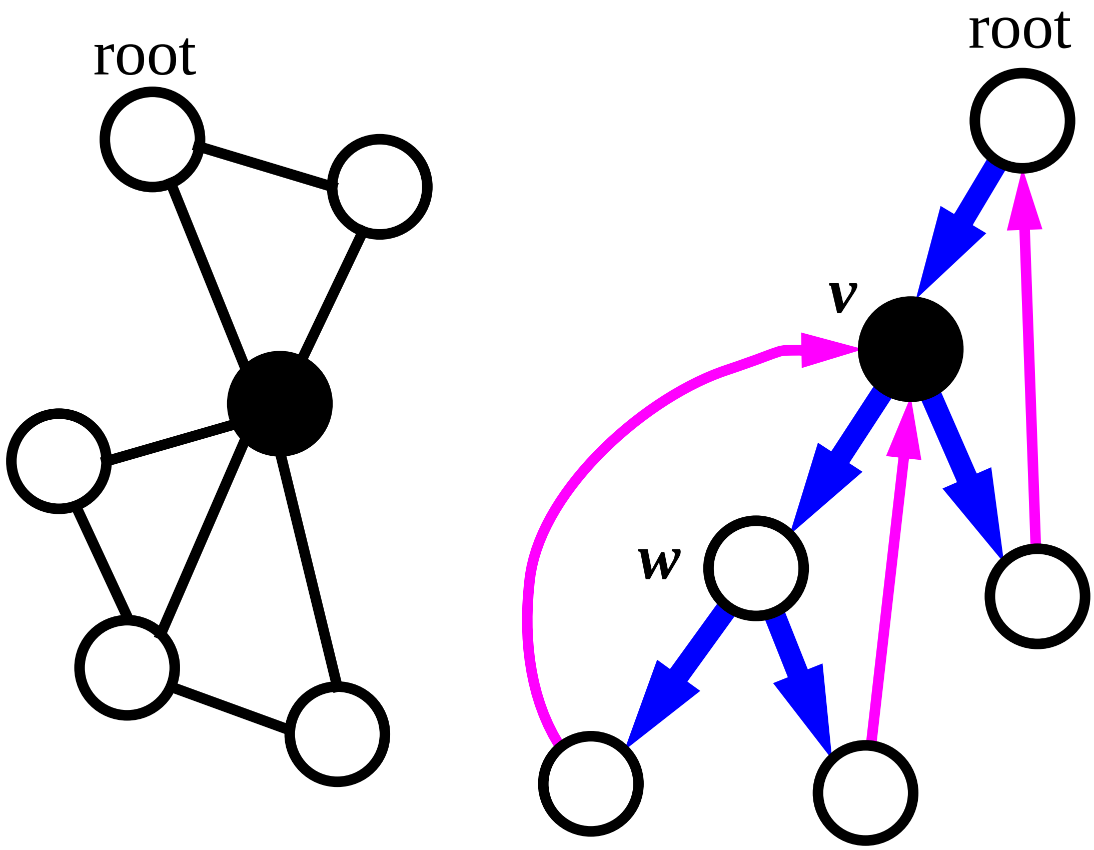
- no cross/horizontal edges
- must retrace back up
- stays within subtree to root, must go through root to other subtree





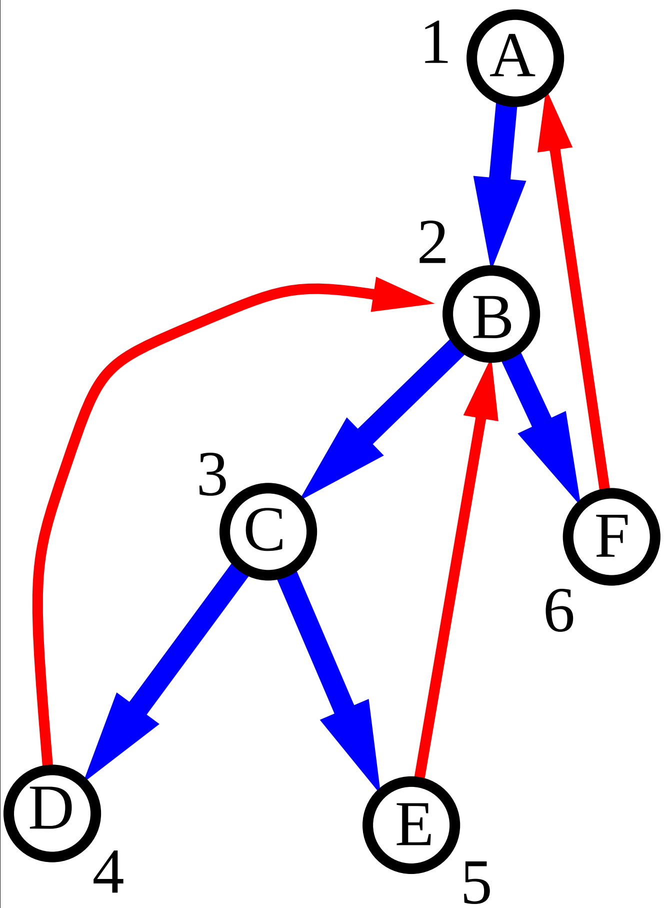
Complicated Property

A vertex v which is not the root of the DFS tree is a cutvertex if v has a child w such that no back edge starting in the subtree of w reaches an ancestor of v.



Definitions

- **low**(***v***): vertex with the lowest val (i.e., "highest" in the DFS tree) reachable from ***v*** by using a directed path that uses at most one back edge
- **Min**(***v***) = **val**(**low**(***v***))



v	low(***v***)	Min(***v***)
A	A	1
B	A	1
C	B	2
D	B	2
E	B	2
F	A	1

DFS Algorithm for Finding Cutvertices

1. Perform DFS on the graph
2. Test if root of DFS tree has two or more tree edges (***root property***)
3. For each other vertex ***v***, test if there is a tree edge (***v***,***w***) such that Min(***w***) $\geq$ val[***v***] (***complicated property***)

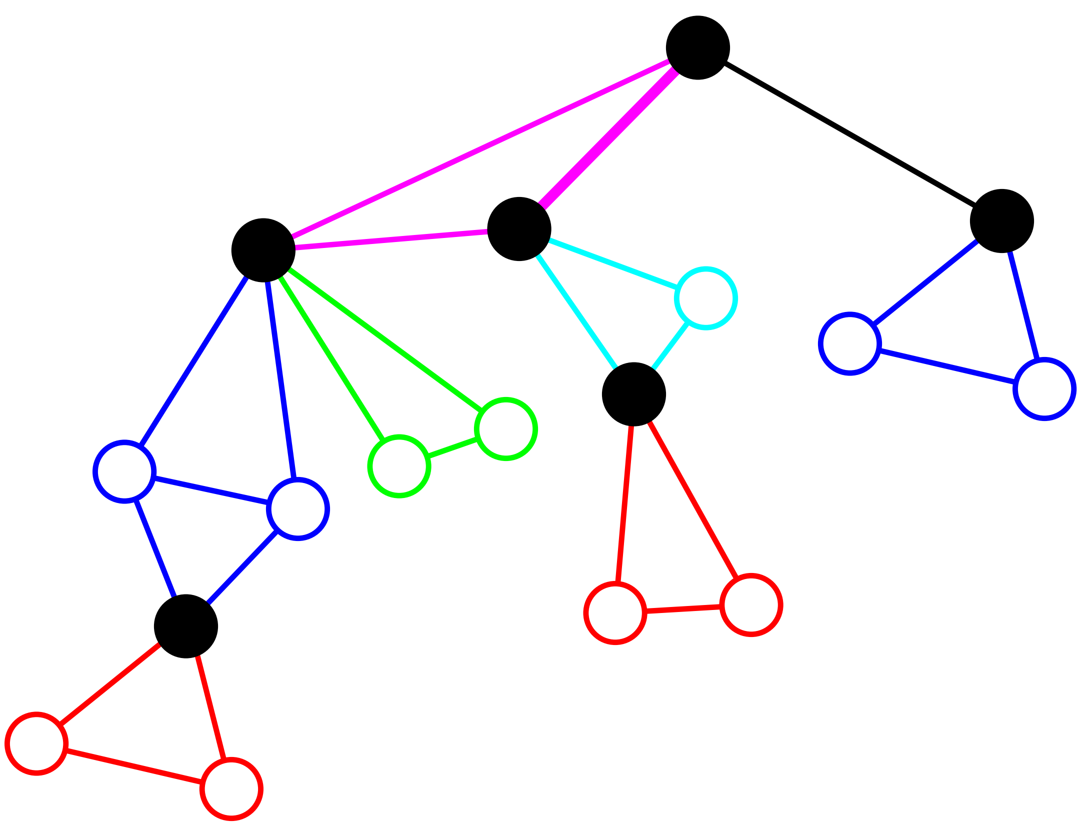
Min(v) = val(low(v)) is the minimum of:

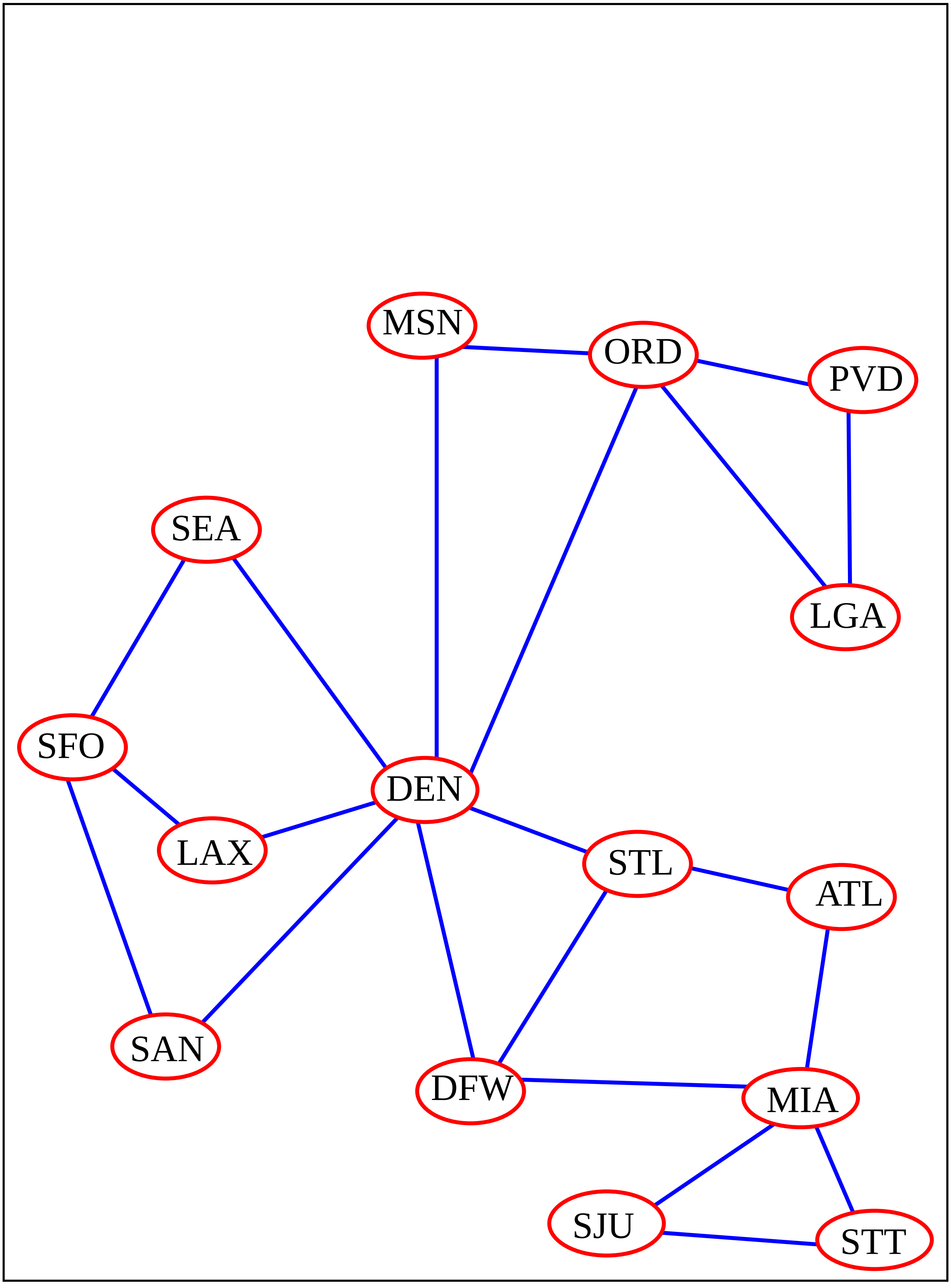
- val[v]
- minimum of Min(w) for all tree edges (v,w)
- minimum of val[z] for all back edges (v,z)

Implement this recursively and you are done!!!!

Finding the Biconnected Components

- DFS visits the vertices and edges of each biconnected component consecutively
- Use a stack to keep track of the biconnected component currently being traversed

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