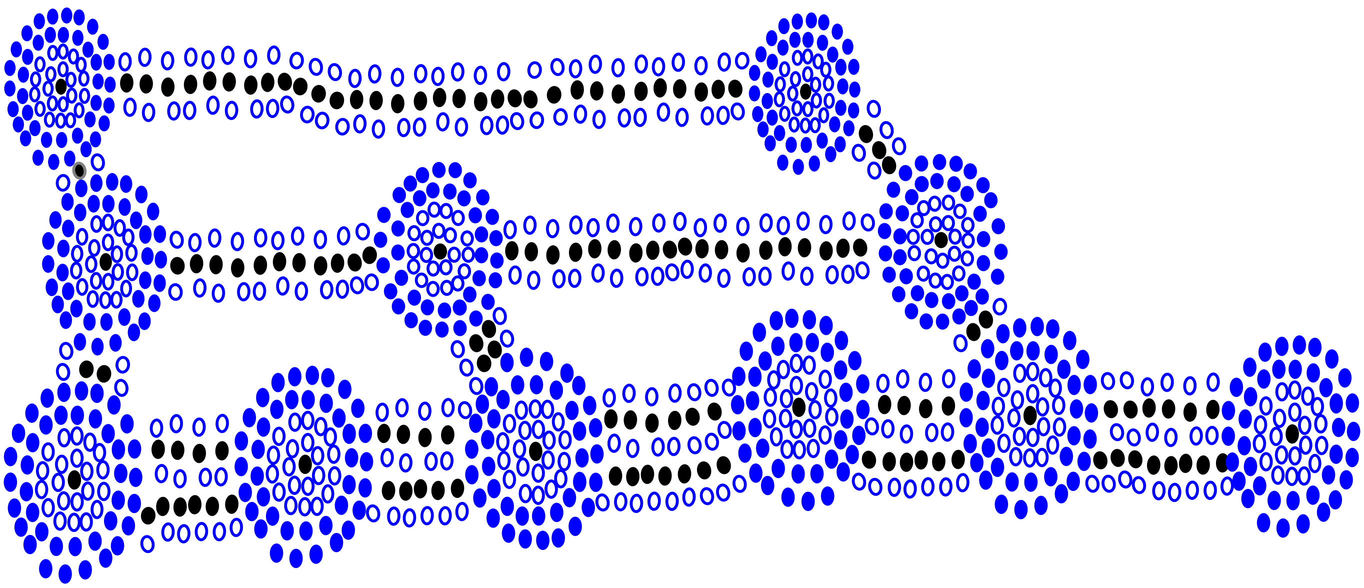
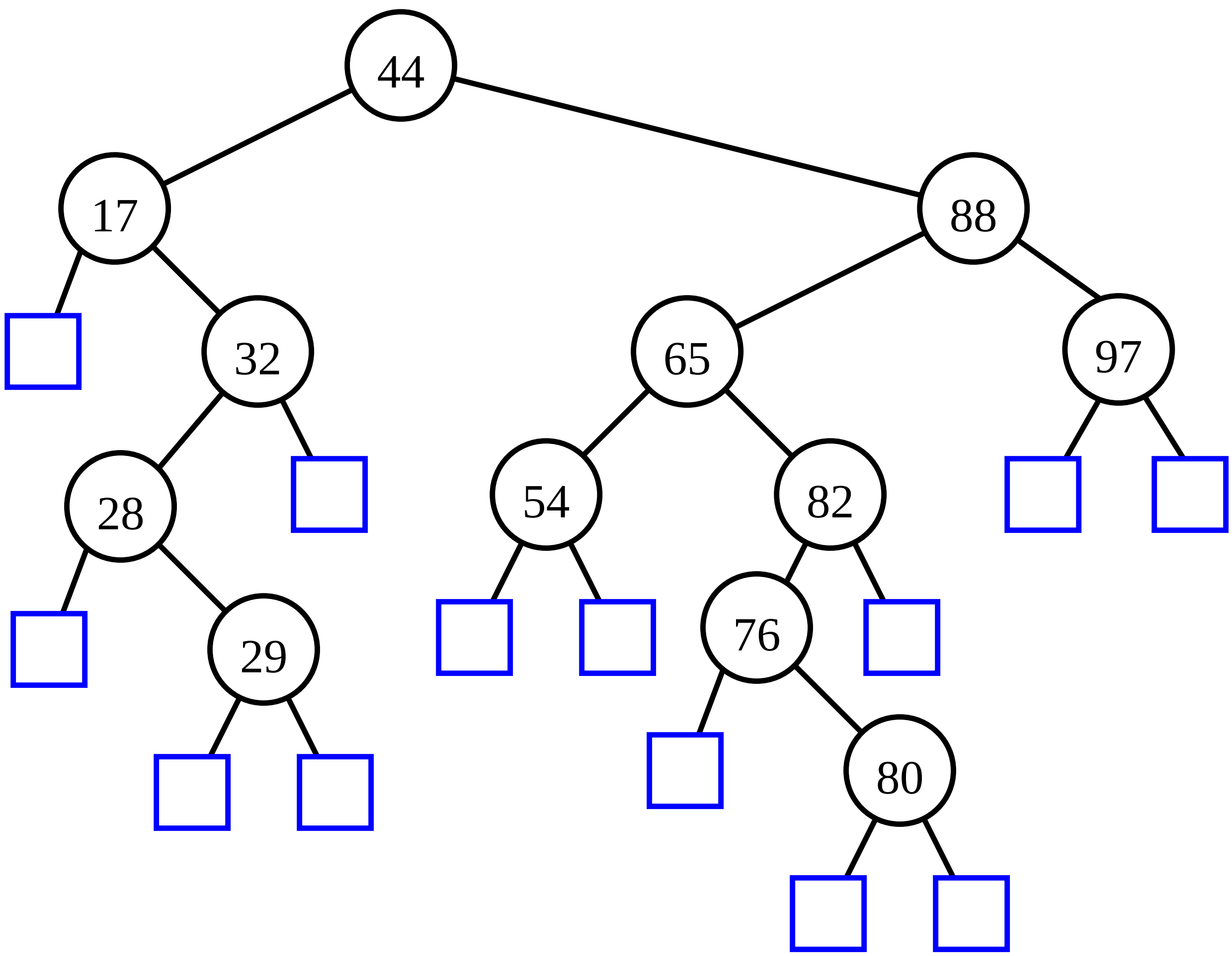
AVL TREES

- Binary Search Trees
- AVL Trees



Binary Search Trees

- A binary search tree is a binary tree T such that
  - each internal node stores an item (k, e) of a dictionary.
  - keys stored at nodes in the left subtree of v are less than or equal to k.
  - Keys stored at nodes in the right subtree of v are greater than or equal to k.
  - External nodes do not hold elements but serve as place holders.



Search

- The binary search tree *T* is a decision tree, where the question asked at an internal node *v* is whether the search key *k* is less than, equal to, or greater than the key stored at *v*.

- Pseudocode:

Algorithm TreeSeach(*k*, *v*):

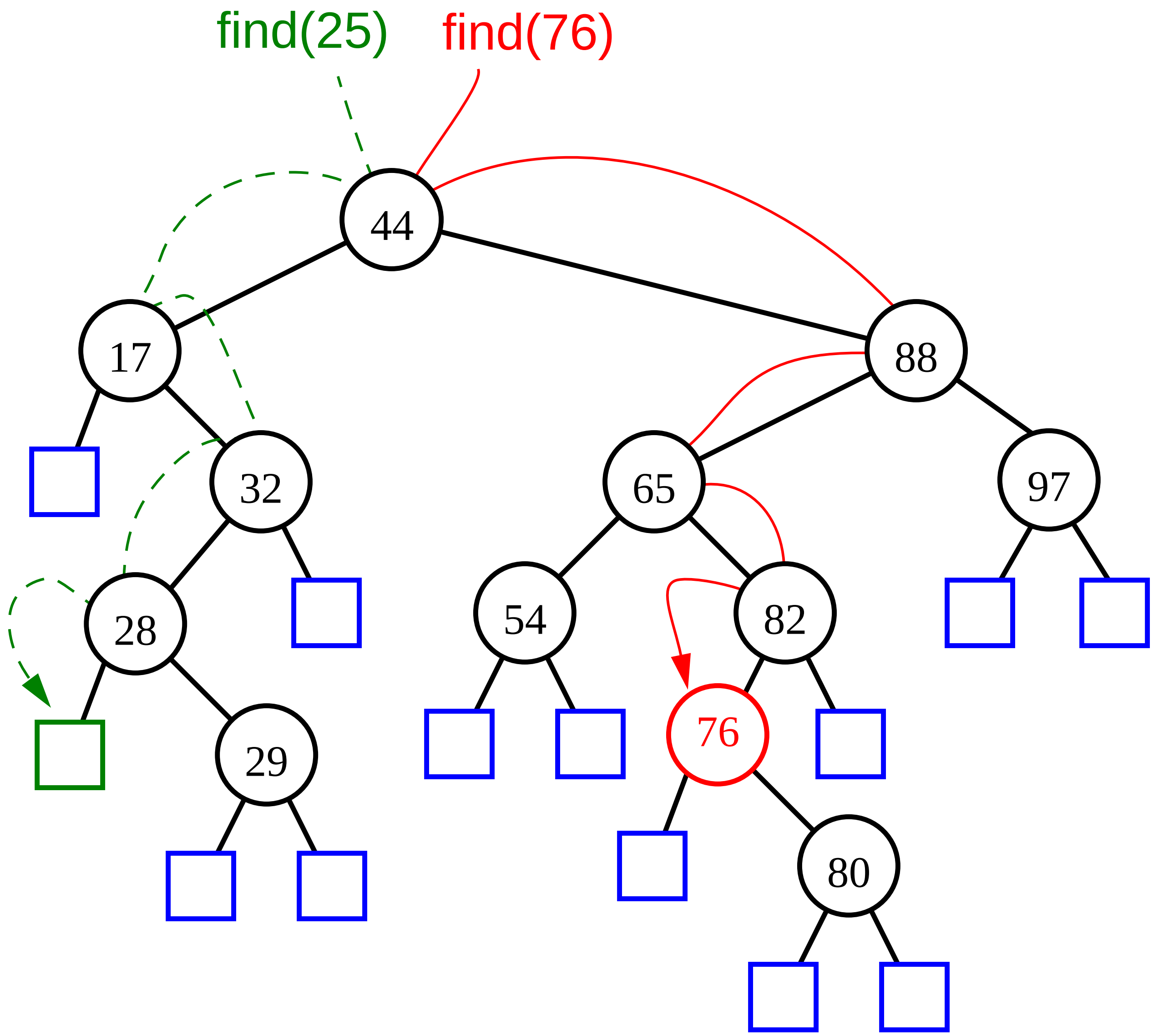
Input: A search key *k* and a node *v* of a binary search tree *T*.

Ouput: A node w of the subtree *T*(*v*) of *T* rooted at *v*, such that either w is an internal node storing key *k* or w is the external node encountered in the inorder traversal of *T*(*v*) after all the inter nal nodes with keys smaller than *k* and before all the internal nodes with keys greater than *k*.

if *v* is an external node **then**
  return *v*
if *k* = key(*v*) **then**
  return *v*
else if *k* < key(*v*) **then**
  return TreeSearch(*k*, *T*.leftChild(*v*))
else
  { *k* > key(*v*) }
  return TreeSearch(*k*, *T*.rightChild(*v*))

Search (cont.)

- A picture:



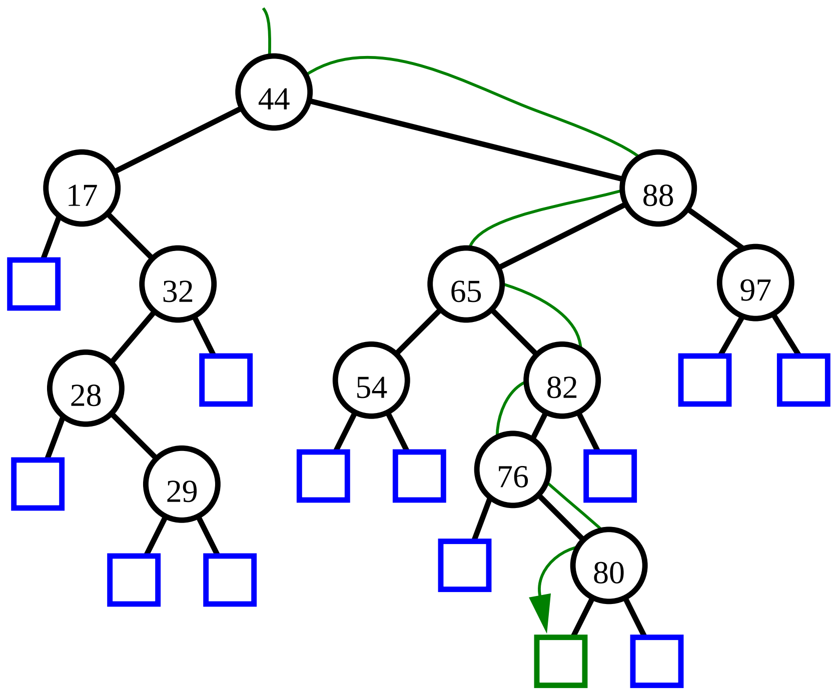
Insertion in a Binary Search Tree

- Start by calling TreeSearch(k, T.root()) on T. Let w be the node returned by TreeSearch
- If w is external, we know no item with key k is stored in T. We call expandExternal(w) on T and have w store the item (k, e)
- If w is internal, we know another item with key k is stored at w. We call TreeSearch(k, rightChild(w)) and recursively apply this alorithm to the node returned by TreeSearch.

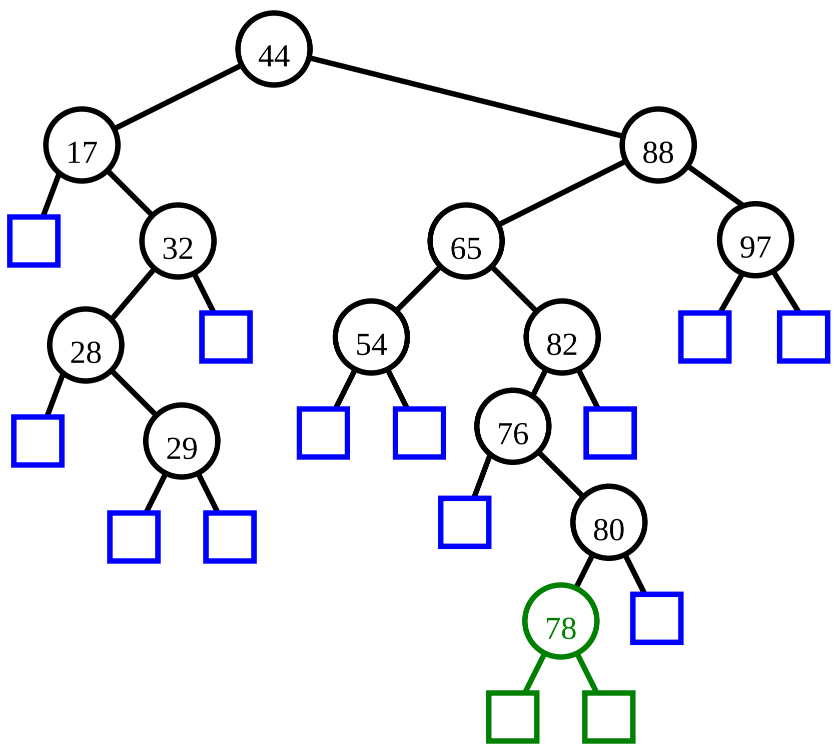
Insertion in a Binary Search Tree (cont.)

- Insertion of an element with key 78:

a)

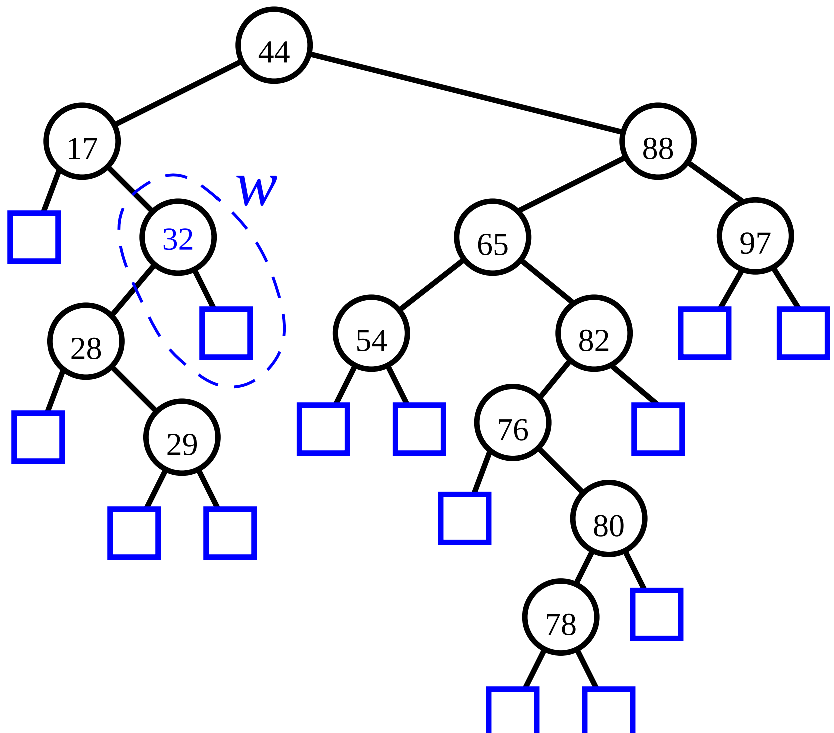


b)



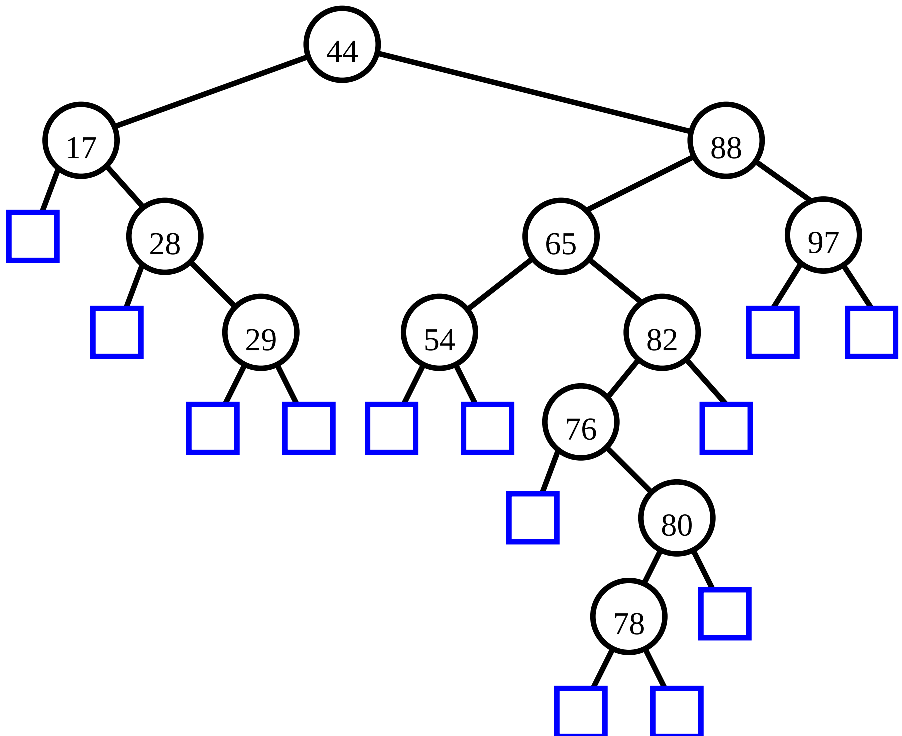
Removal from a Binary Search Tree

- Removal where the key to remove is stored at a node (w) with an external child:



(a)

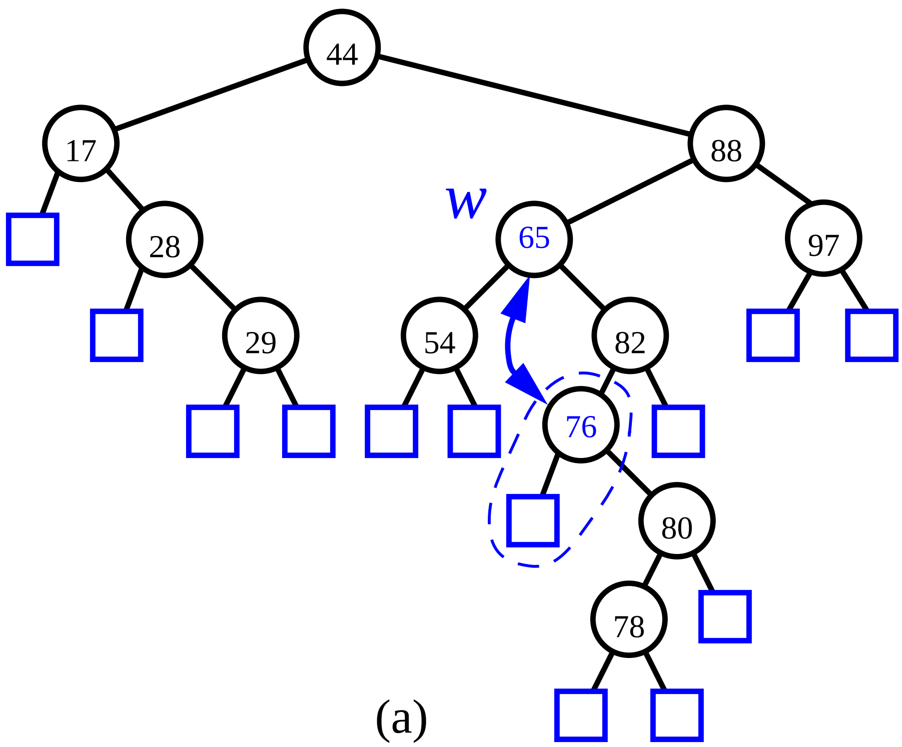
Removal from a Binary Search Tree (cont.)



(b)

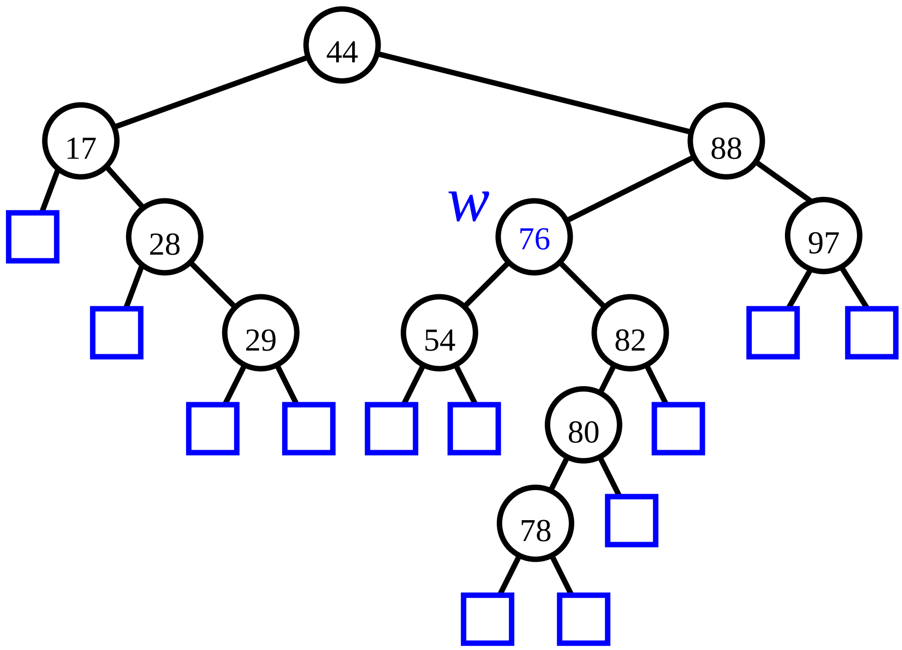
Removal from a Binary Search Tree (cont.)

- Removal where the key to remove is stroed at a node whose children are both internal:



(a)

Removal from a Binary Search Tree (cont.)



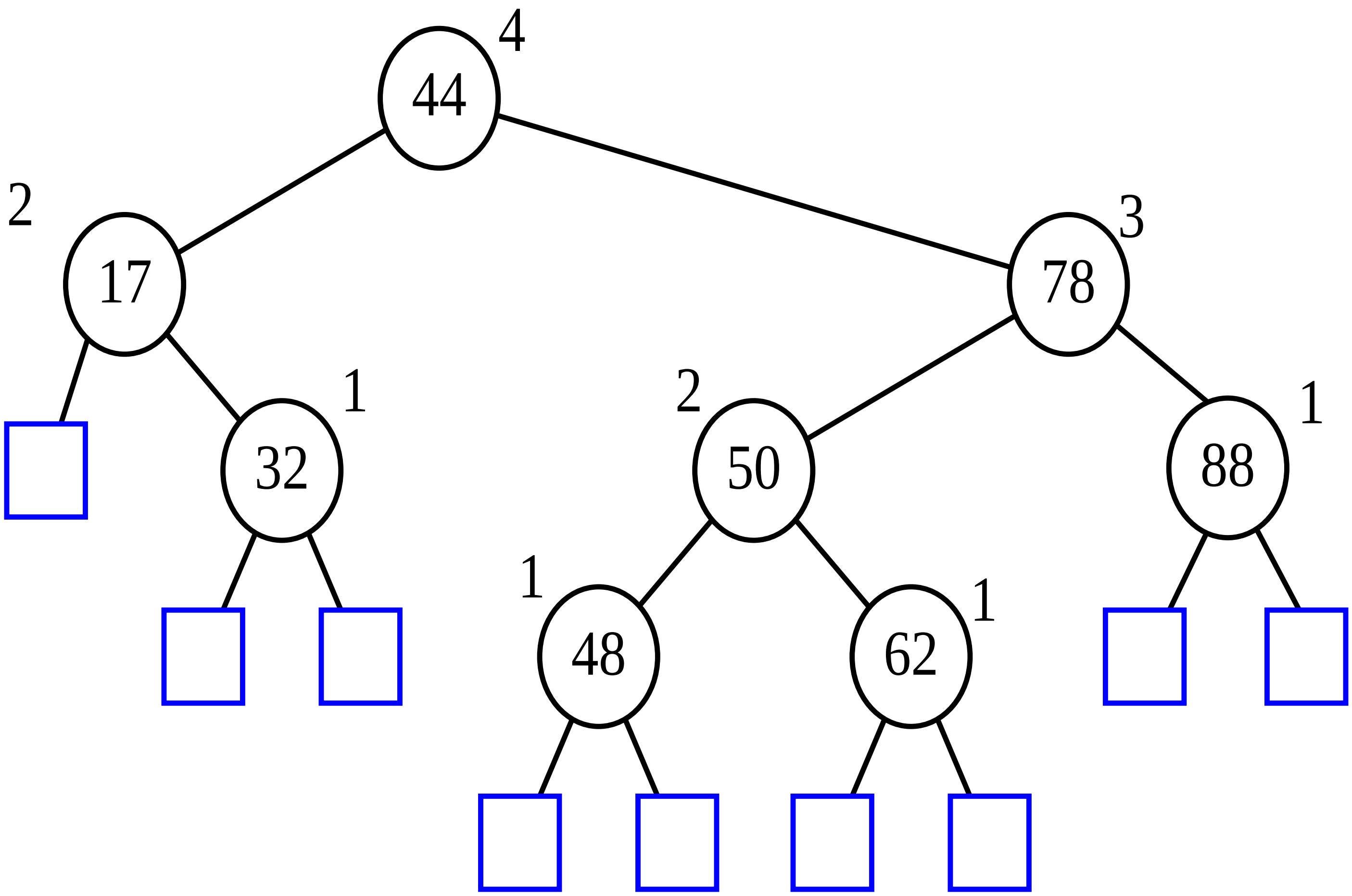
(b)

Time Complexity

- Searching, insertion, and removal in a binary search tree is $\boldsymbol{O}(h)$, where h is the height of the tree.
- However, in the worst-case search, insertion, and removal time is $\boldsymbol{O}(n)$, if the height of the tree is equal to n. Thus in some cases searching, insertion, and removal is no better than in a sequence.
- Thus, to prevent the worst case, we need to develop a rebalancing scheme to bound the height of the tree to log n.

AVL Tree

- An AVL Tree is a binary search tree such that for every internal node *v* of *T*, the heights of the children of *v* can differ by at most 1.
- An example of an AVL tree where the heights are shown next to the nodes:



Height of an AVL Tree

- **Proposition**: The height of an AVL tree *T* storing *n* keys is $\boldsymbol{O}(\log n)$.
- **Justification**: The easiest way to approach this problem is to try to find the minimum number of internal nodes of an AVL tree of height *h*: $n(h)$.
- We see that $n(1) = 1$ and $n(2) = 2$
- for n 3, an AVL tree of height *h* with $n(h)$ minimal contains the root node, one AVL subtree of height n-1 and the other AVL subtree of height n-2.
- *i*.e. $n(h) = 1 + n(h\text{-}1) + n(h\text{-}2)$
- Knowing $n(h\text{-}1) > n(h\text{-}2)$, we get $n(h) > 2\text{n}(h\text{-}2)$
  - $n(h) > 2\text{n}(h\text{-}2)$
  - $n(h) > 4\text{n}(h\text{-}4)$

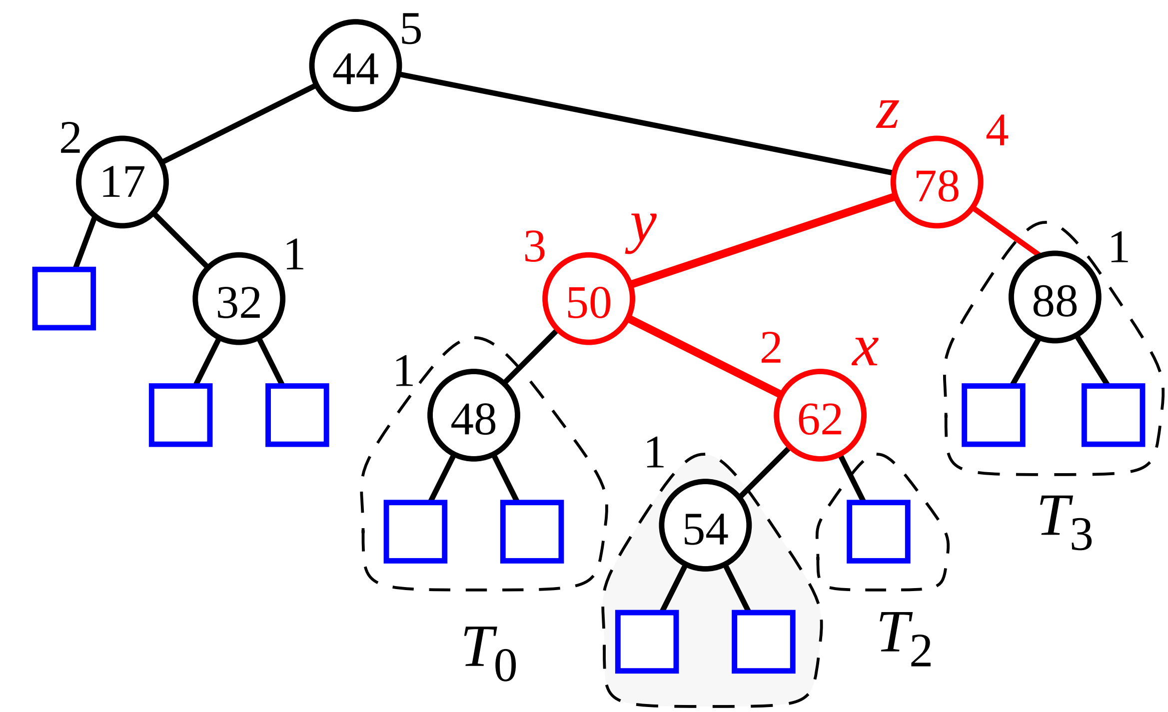
    ...
  - $n(h) > 2^i n(h\text{-}2i)$
- Solving the base case we get: $n(h)$ $2^{h/2\text{-}1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is $\boldsymbol{O}(\log n)$

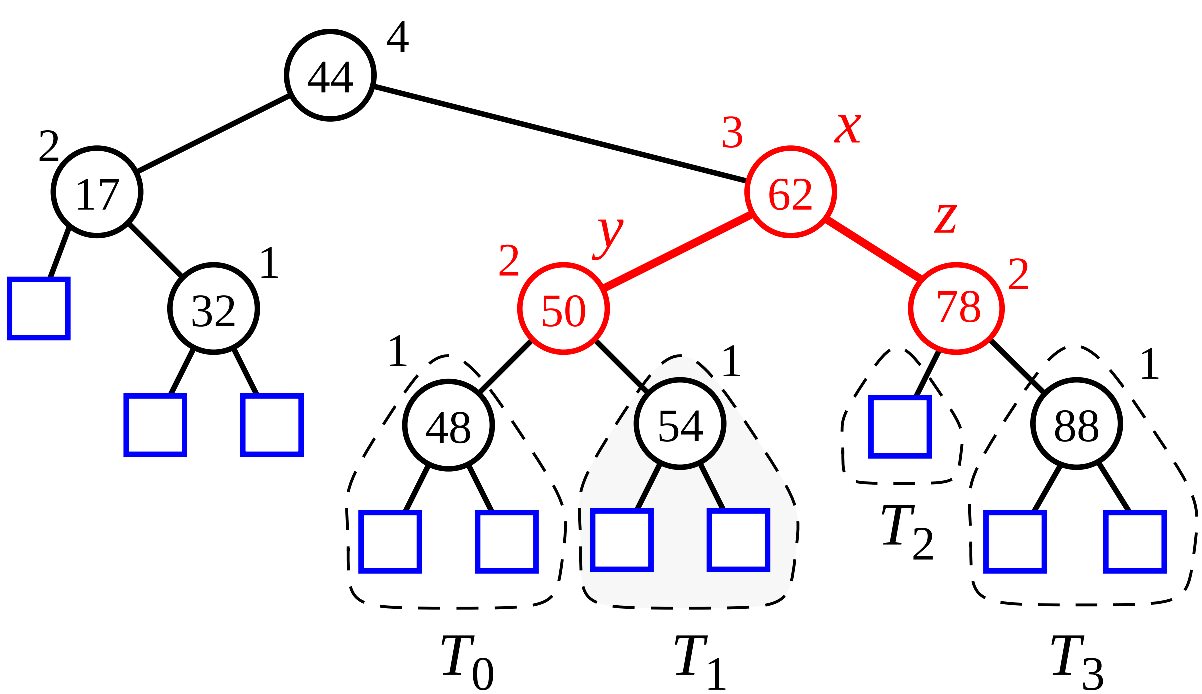
Insertion

- A binary search tree T is called balanced if for every node v, the height of v's children differ by at most one.
- Inserting a node into an AVL tree involves performing an expandExternal(w) on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced, we travel up the tree from the newly created node until we find the first node x such that its grandparent z is unbalanced node.
- Since z became unbalanced by an insertion in the subtree rooted at its child y,
height(y) = height(sibling(y)) + 2
- To rebalance the subtree rooted at z, we must perform a *restructuring*
  - we rename x, y, and z to a, b, and c based on the order of the nodes in an in-order traversal.
  - z is replaced by b, whose children are now a and c whose children, in turn, consist of the four other subtrees formerly children of x, y, and z.

Insertion (contd.)

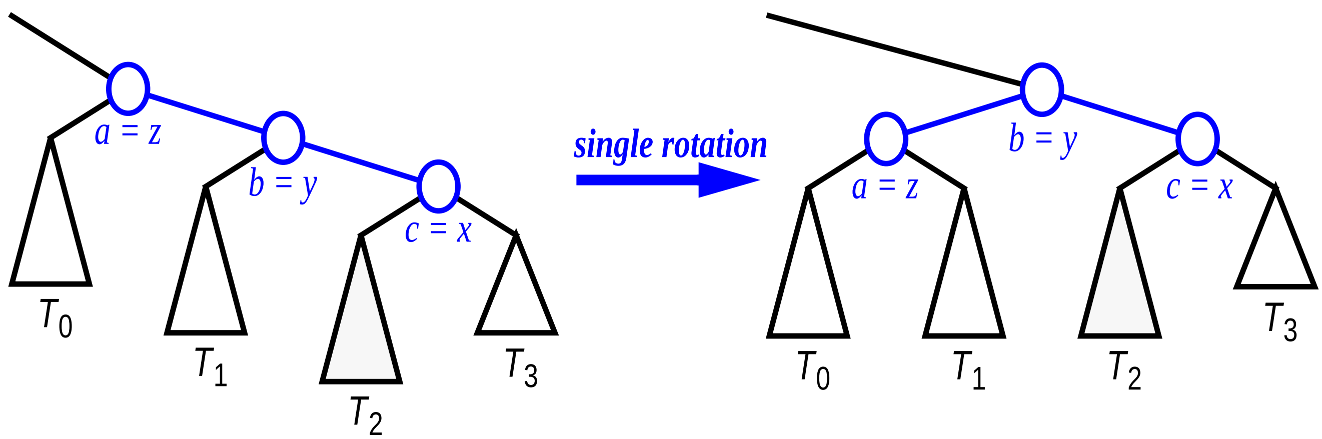
- Example of insertion into an AVL tree.

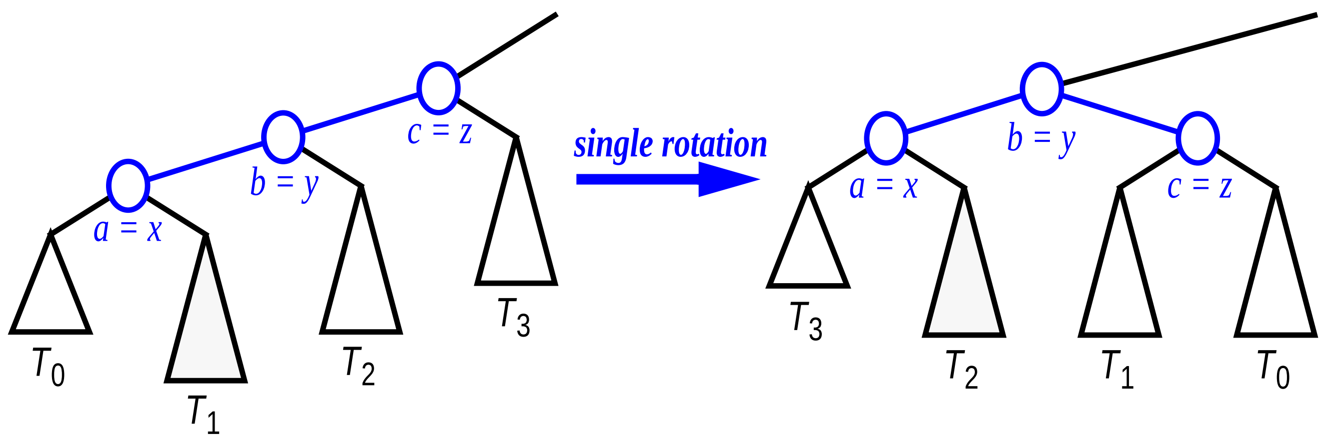




Restructuring

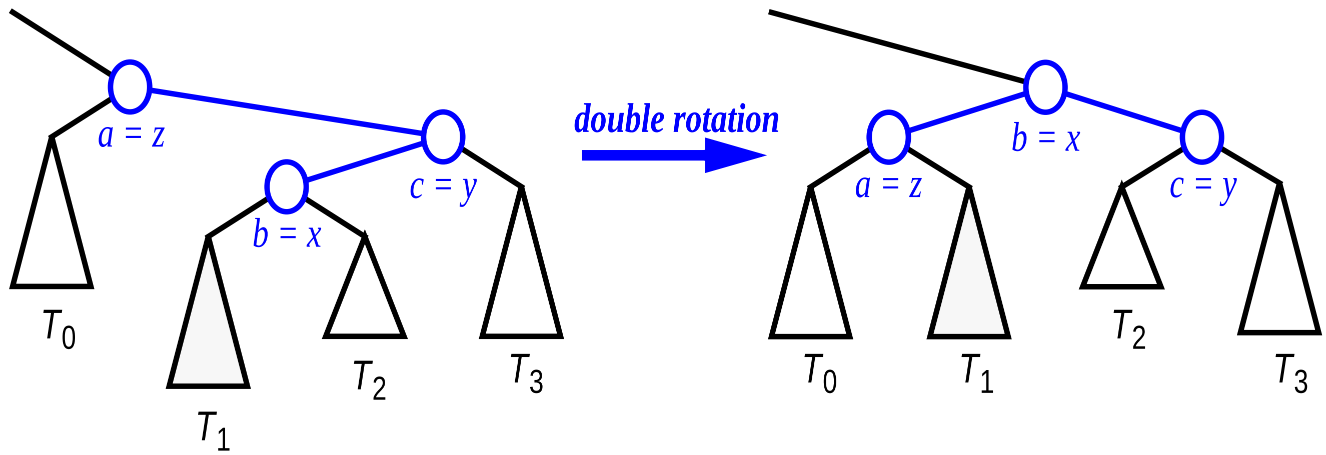
- The four ways to rotate nodes in an AVL tree, graphically represented:
  - Single Rotations:

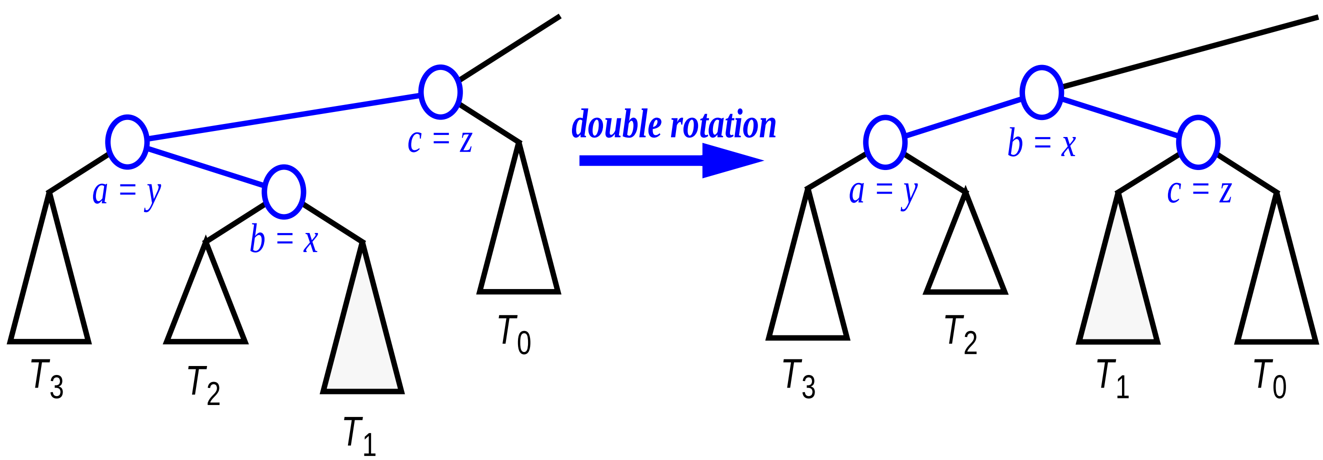




Restructuring (contd.)

- double rotations:





Restructuring (contd.)

- In Pseudo-Code:

Algorithm restructure(x):
Input: A node x of a binary search tree T that has both a parent y and a grandparent z
Output: Tree T restructured by a rotation (either single or double) involving nodes x, y, and z.

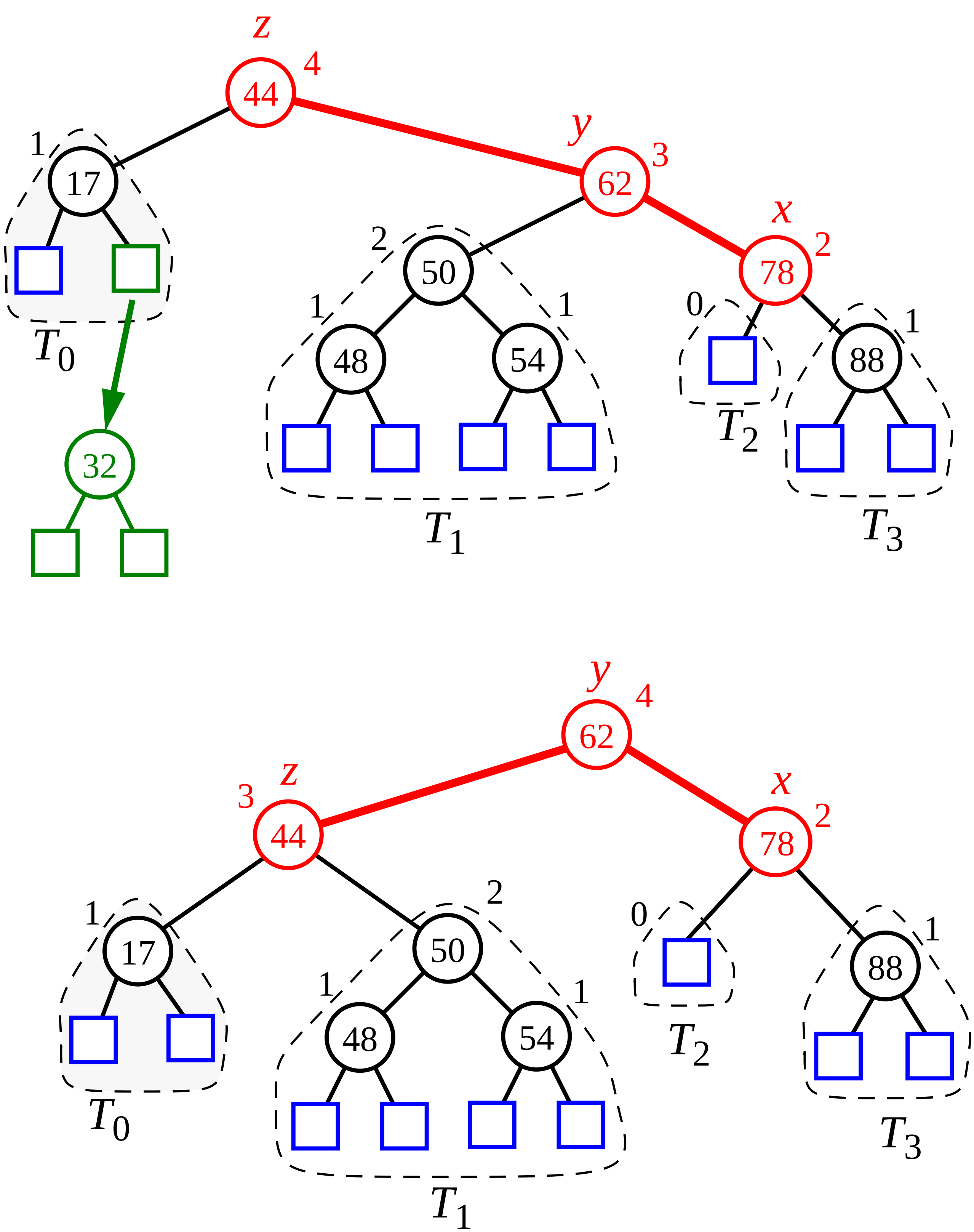
1: Let (a, b, c) be an inorder listing of the nodes x, y, and z, and let (T_0, T_1, T_2, T_3) be an inorder listing of the the four subtrees of x, y, and z not rooted at x, y, or z
2. Replace the subtree rooted at z with a new subtree rooted at b
3. Let a be the left child of b and let T_0, T_1 be the left and right subtrees of a, respectively.
4. Let c be the right child of b and let T_2, T_3 be the left and right subtrees of c, respectively.

Removal

- We can easily see that performing a removeAboveExternal(w) can cause T to become unbalanced.
- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height.
- We can perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached.

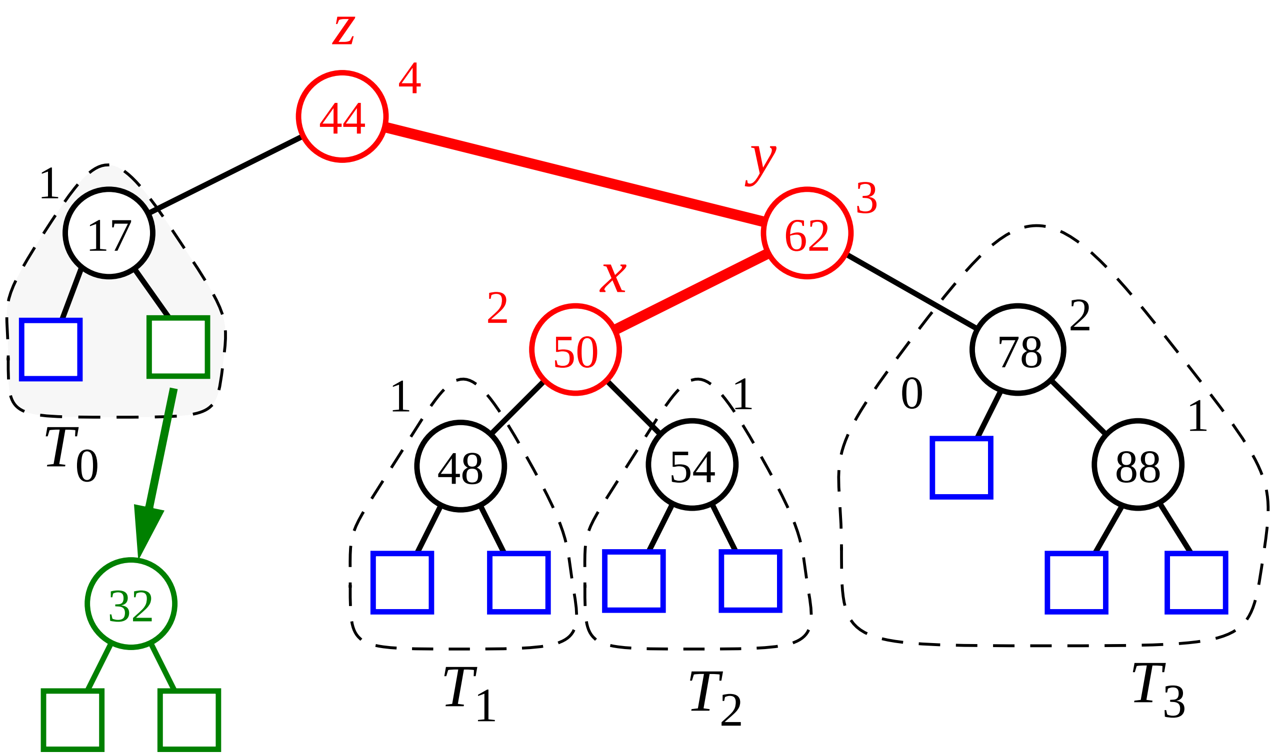
Removal (contd.)

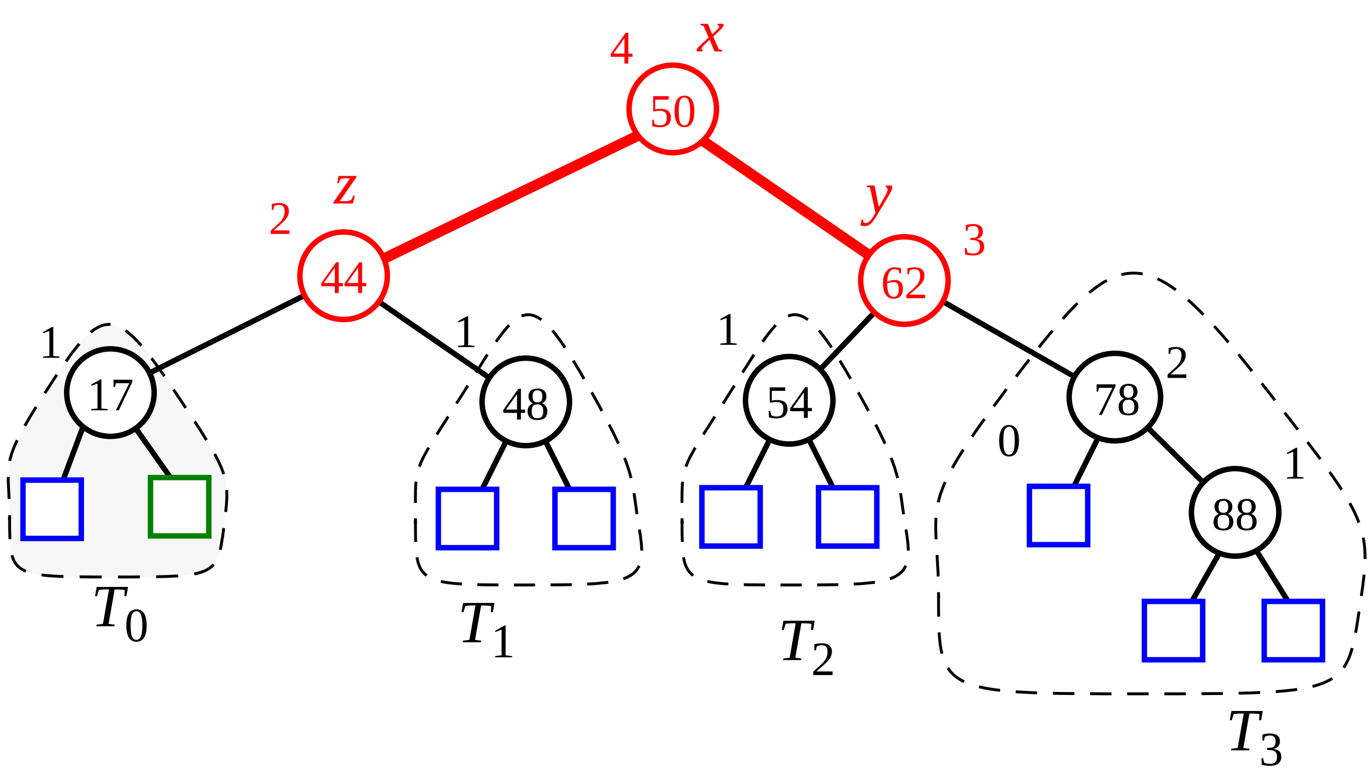
- example of deletion from an AVL tree:



Removal (contd.)

- example of deletion from an AVL tree





Implementation

- A Java-based implementation of an AVL tree requires the following node class:

```
public class AVLItem extends Item  {
  int height;

  AVLItem(Object k, Object e, int h)  {
    super(k, e);
    height = h;
  }

  public int height()  {
    return height;
  }

  public int setHeight(int h)  {
    int oldHeight = height;
    height = h;
    return oldHeight;
  }
}
```

Implementation (contd.)

```
public class SimpleAVLTree
  extends SimpleBinarySearchTree
  implements Dictionary {

public SimpleAVLTree(Comparator c) {
      super(c);
      T = new RestructurableNodeBinaryTree();
    }
    private int height(Position p) {
      if (T.isExternal(p))
          return 0;
      else
          return ((AVLItem) p.element()).height();
    }
    private void setHeight(Position p) { // called only
                                          // if p is internal
      ((AVLItem) p.element()).setHeight
          (1 + Math.max(height(T.leftChild(p)),
                         height(T.rightChild(p))));
    }
```

Implementation (contd.)

```
private boolean isBalanced(Position p) {
      // test whether node p has balance factor
      // between -1 and 1
   int bf = height(T.leftChild(p)) - height(T.rightChild(p));
   return ((-1 <= bf) && (bf <= 1));
 }


private Position tallerChild(Position p) {
       // return a child of p with height no
       // smaller than that of the other child
   if(height(T.leftChild(p)) >= height(T.rightChild(p)))
      return T.leftChild(p);
   else
      return T.rightChild(p);
  }
```

Implementation (contd.)

```
private void rebalance(Position zPos) {
//traverse the path of T from zPos to the root;
//for each node encountered recompute its
//height and perform a rotation if it is
//unbalanced
   while (!T.isRoot(zPos)) {
     zPos = T.parent(zPos);
     setHeight(zPos);
     if (!isBalanced(zPos)) { // perform a rotation
       Position xPos =  tallerChild(tallerChild(zPos));
       zPos = ((RestructurableNodeBinaryTree)
                   T).restructure(xPos);
       setHeight(T.leftChild(zPos));
       setHeight(T.rightChild(zPos));
       setHeight(zPos);
     }
   }
}
```

Implementation (contd.)

```
public void insertItem(Object key, Object element)
    throws InvalidKeyException {
  super.insertItem(key, element);// may throw an
                                 // InvalidKeyException
  Position zPos = actionPos; // start at the
                             // insertion position
  T.replace(zPos, new AVLItem(key, element, 1));
  rebalance(zPos);
  }

public Object remove(Object key)
      throws InvalidKeyException {
   Object toReturn = super.remove(key); // may throw
                            // an InvalidKeyException
    if (toReturn != NO_SUCH_KEY) {
      Position zPos = actionPos; // start at the
                                 // removal position
      rebalance(zPos);
    }
    return toReturn;
  }
}
```