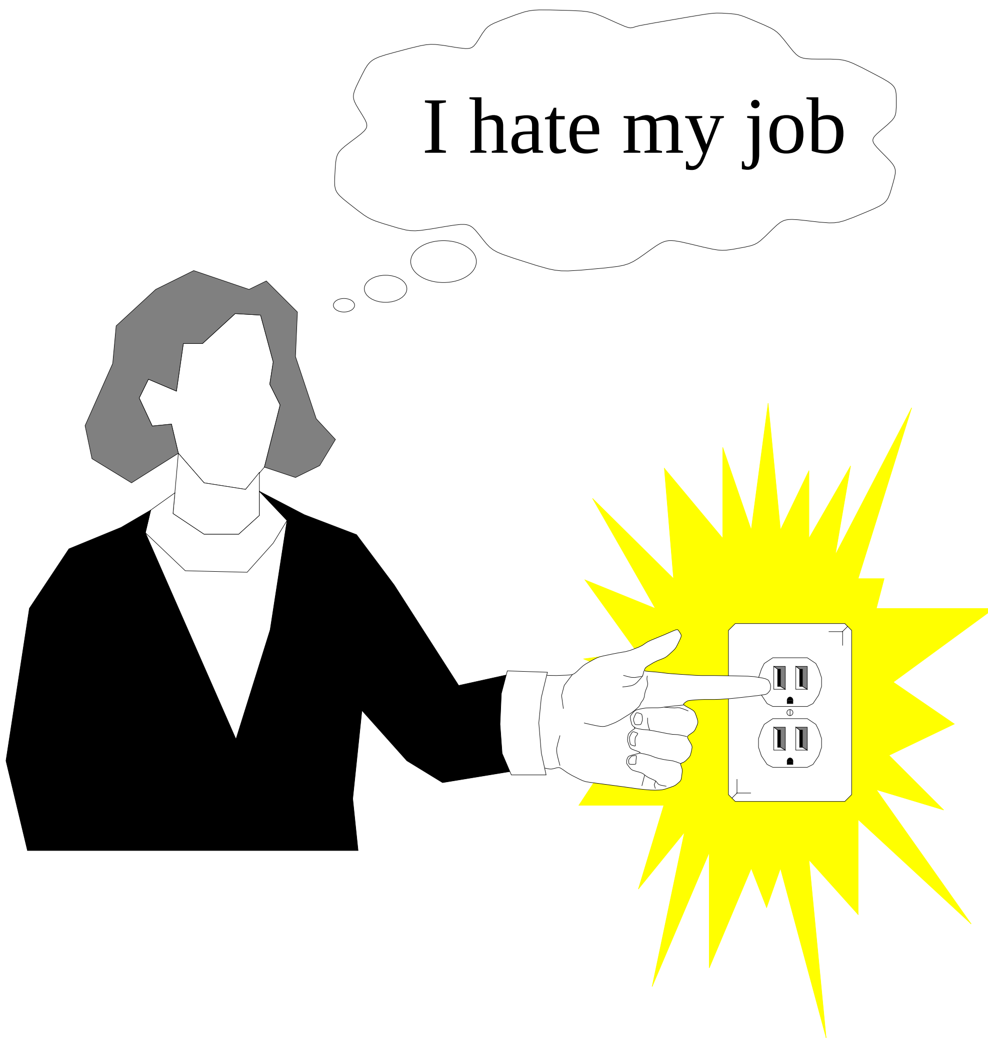
GEOMETRIC ALGORITHMS

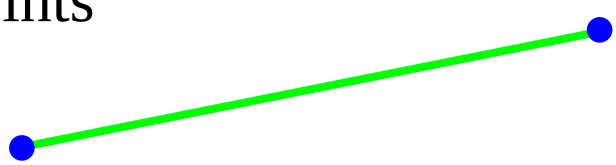
Yowzer! They're electrifying!



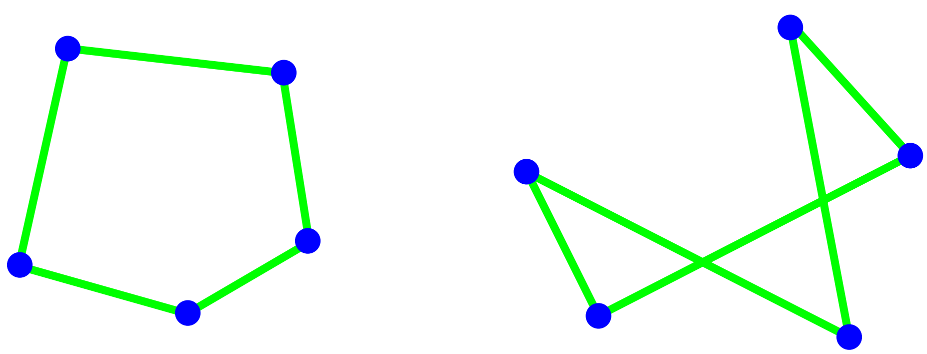
Basic Geometric Objects in the Plane

point: defined by a pair of coordinates (x,y)

segment: portion of a straight line between two points

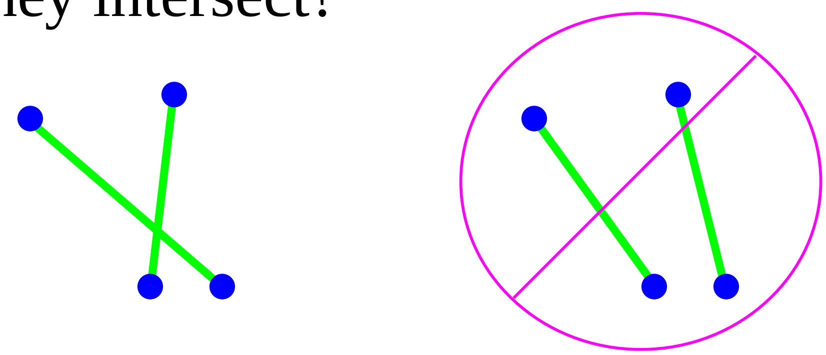


polygon: a circular sequence of points (vertices) and segments (edges) between them

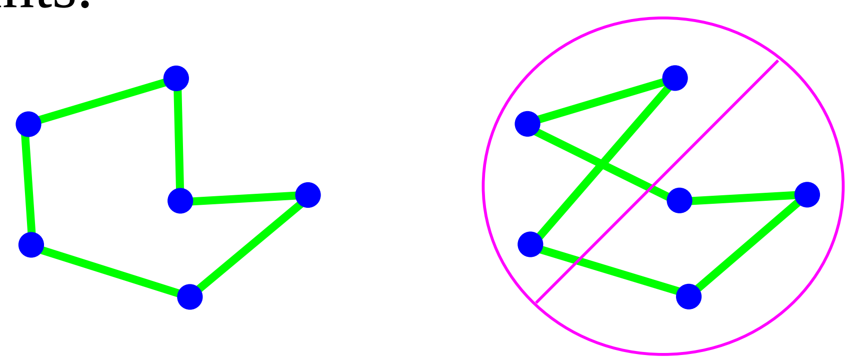


Some Geometric Problems

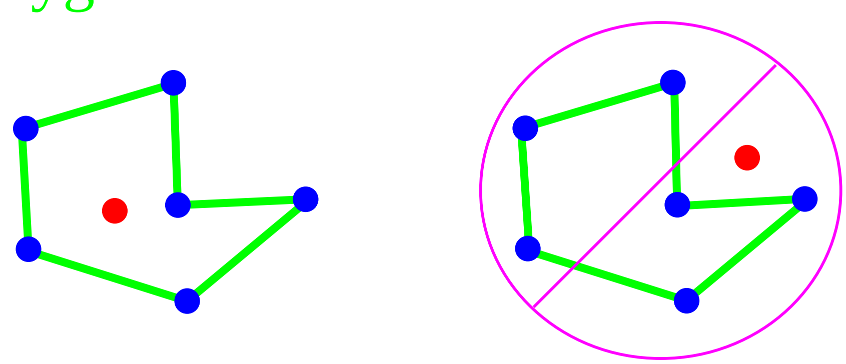
Segment intersection: Given two segments, do they intersect?



Simple closed path: Given a set of points, find a nonintersecting polygon with vertices on the points.

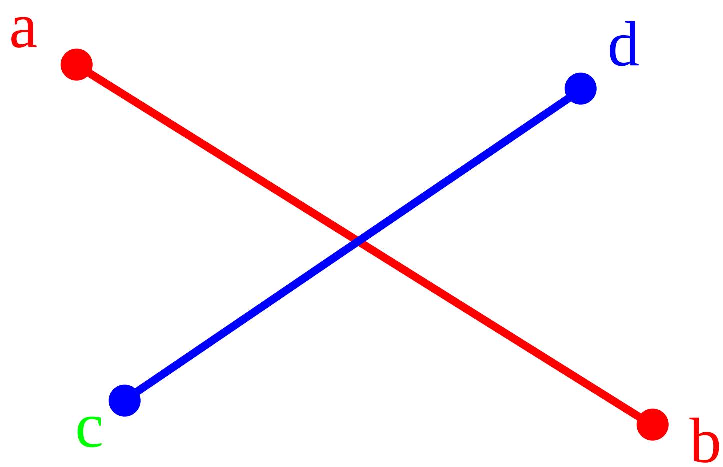


Inclusion in polygon: Is a point inside or outside a polygon?



An Apparently Simple Problem: Segment Intersection

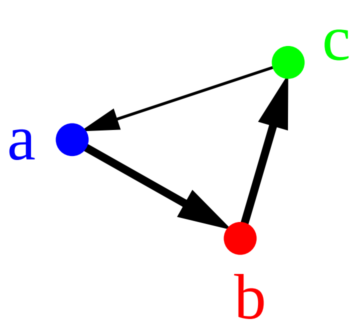
- Test whether segments (a,b) and (c,d) intersect.
  How do we do it?



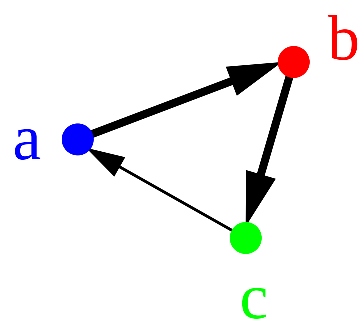
- We could start by writing down the equations of the lines through the segments, then test whether the lines intersect, then ...
- An alternative (and simpler) approach is based in the notion of **orientation** of an ordered triplet of points in the plane

Orientation in the Plane

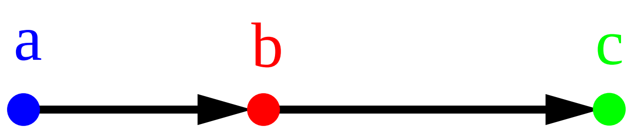
- The orientation of an ordered triplet of points in the plane can be
  - **counterclockwise** (**left turn**)
  - **clockwise** (**right turn**)
  - **collinear** (**no turn**)
- Examples:



counterclockwise (**left turn**)



clockwise (**right turn**)

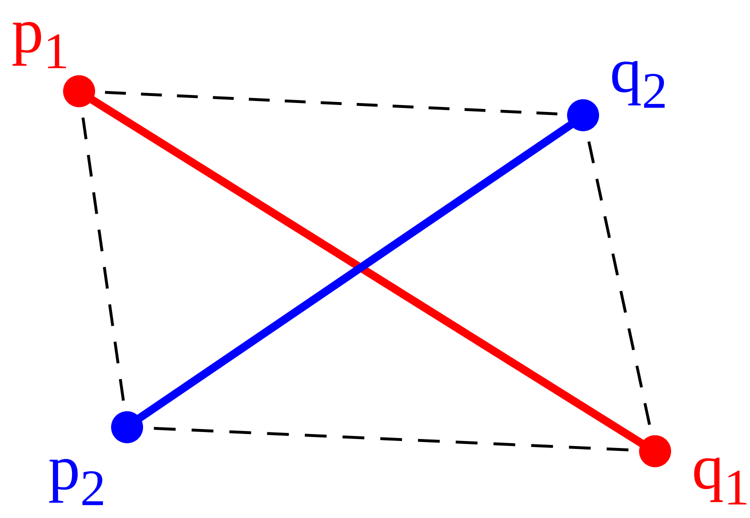


collinear (**no turn**)

Intersection and Orientation

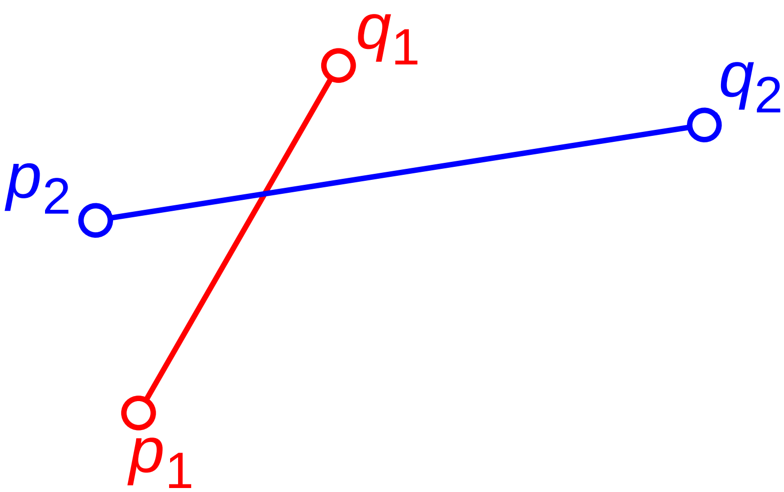
Two segments (p_1,q_1) and (p_2,q_2) intersect if and only if one of the following two conditions is verified

- general case:
  - (p_1,q_1,p_2) and (p_1,q_1,q_2) have different orientations **and**
  - (p_2,q_2,p_1) and (p_2,q_2,q_1) have different orientations
- special case
  - (p_1,q_1,p_2), (p_1,q_1,q_2), (p_2,q_2,p_1), and (p_2,q_2,q_1) are all collinear **and**
  - the *x*-projections of (p_1,q_1) and (p_2,q_2) intersect
  - the *y*-projections of (p_1,q_1) and (p_2,q_2) intersect



Examples (General Case)

- general case:
  - (p_1,q_1,p_2) and (p_1,q_1,q_2) have different orientations **and**
  - (p_2,q_2,p_1) and (p_2,q_2,q_1) have different orientations

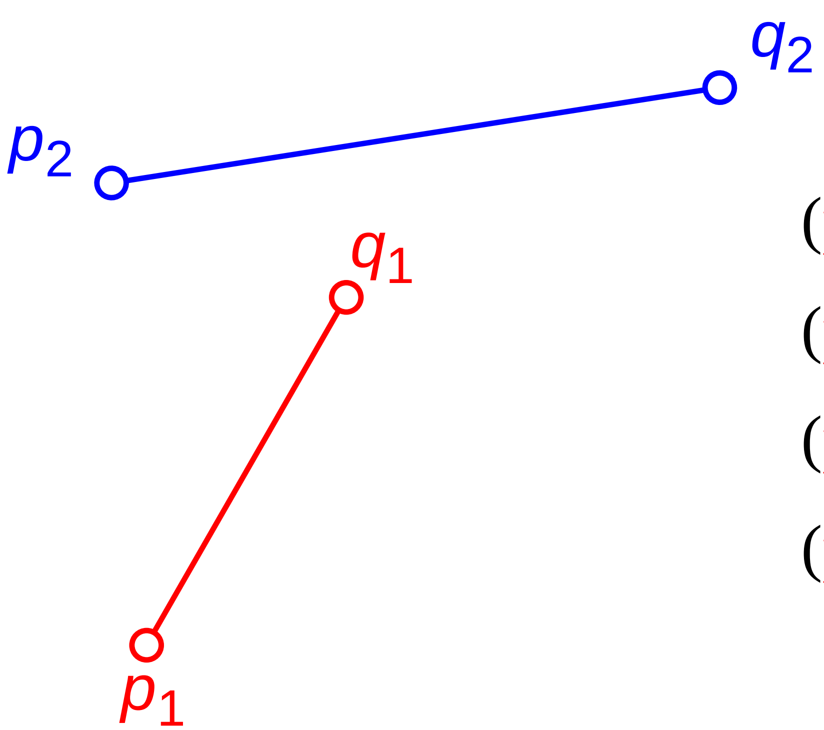


(p_1,q_1,p_2)

(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)



(p_1,q_1,p_2)

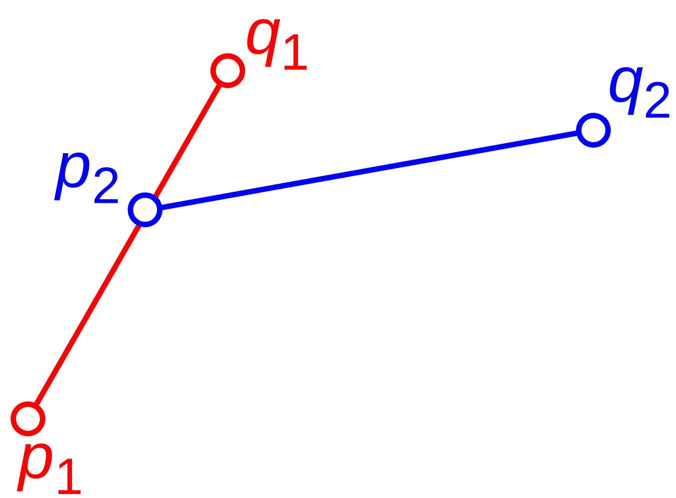
(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)

Examples (General Case)

- general case:
  - (p_1,q_1,p_2) and (p_1,q_1,q_2) have different orientations **and**
  - (p_2,q_2,p_1) and (p_2,q_2,q_1) have different orientations

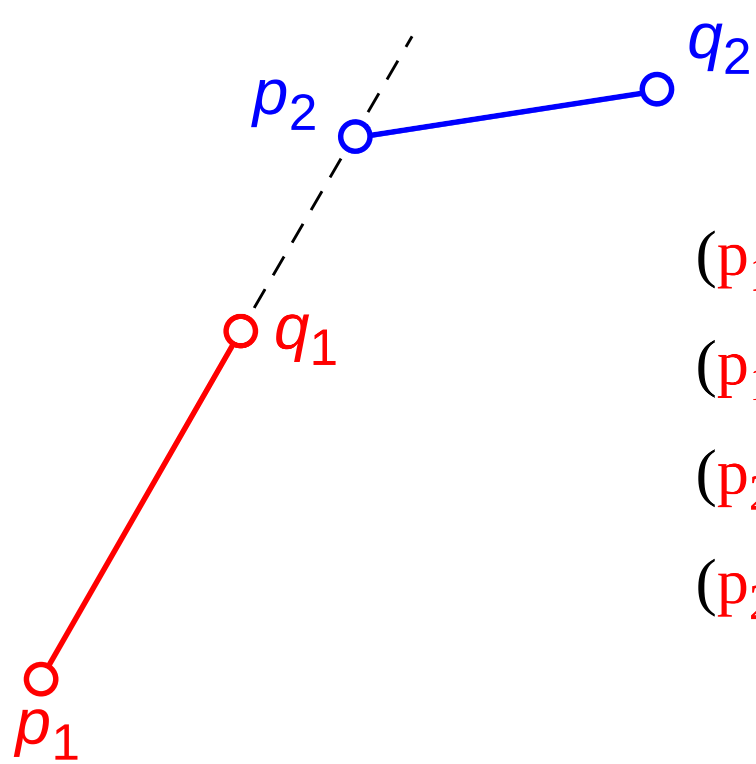


(p_1,q_1,p_2)

(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)



(p_1,q_1,p_2)

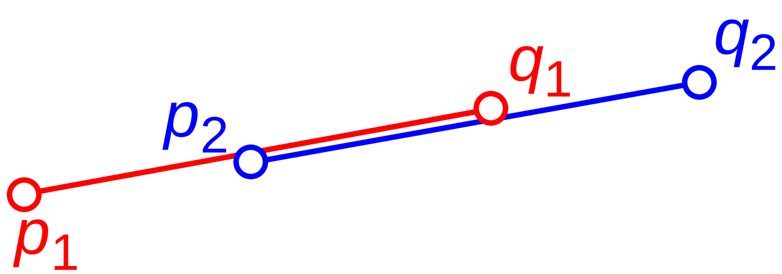
(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)

Examples (Special Case)

- special case
  - (p_1,q_1,p_2), (p_1,q_1,q_2), (p_2,q_2,p_1), and (p_2,q_2,q_1) are all collinear **and**
  - the *x*-projections of (p_1,q_1) and (p_2,q_2) intersect
  - the *y*-projections of (p_1,q_1) and (p_2,q_2) intersect

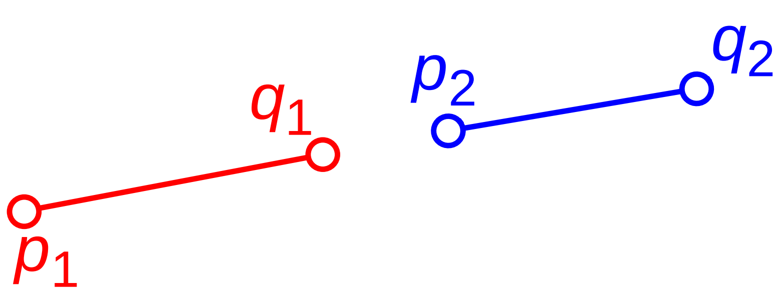


(p_1,q_1,p_2)

(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)



(p_1,q_1,p_2)

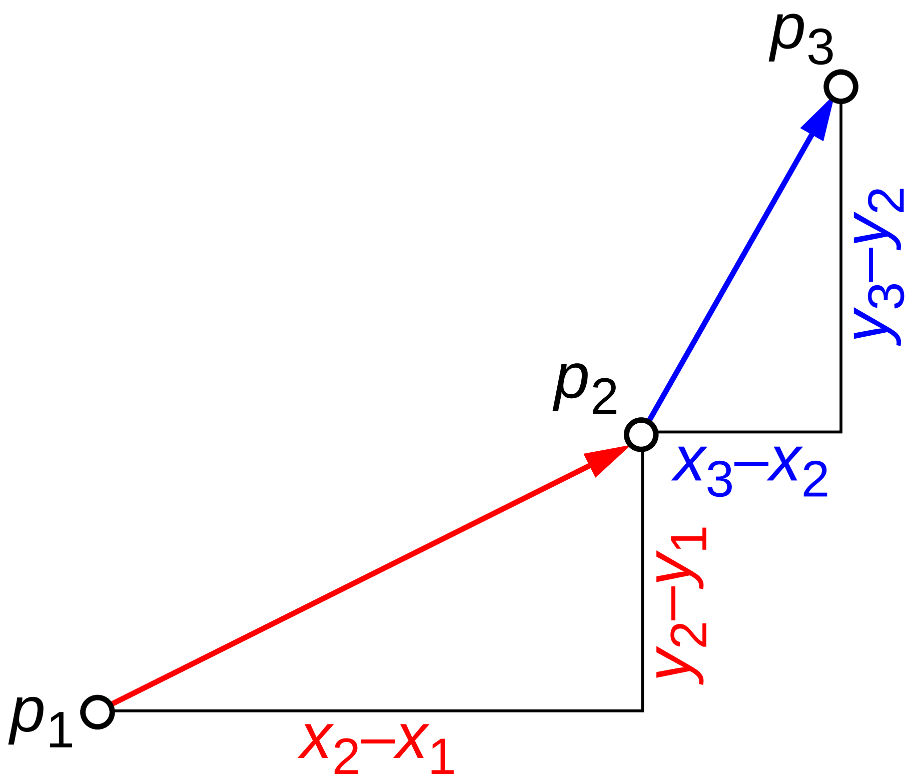
(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)

How to Compute the Orientation

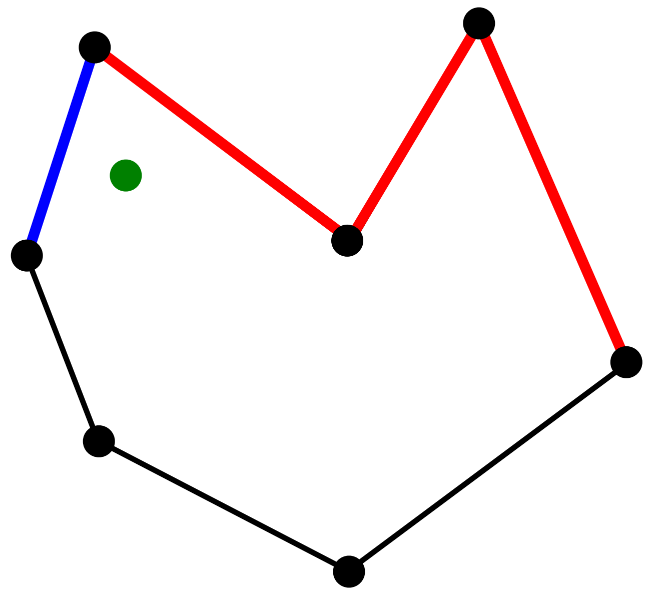
- slope of segment (p_1,p_2): $\sigma = (y_2 - y_1) / (x_2 - x_1)$
- slope of segment (p_2,p_3): $\tau = (y_3 - y_2) / (x_3 - x_2)$

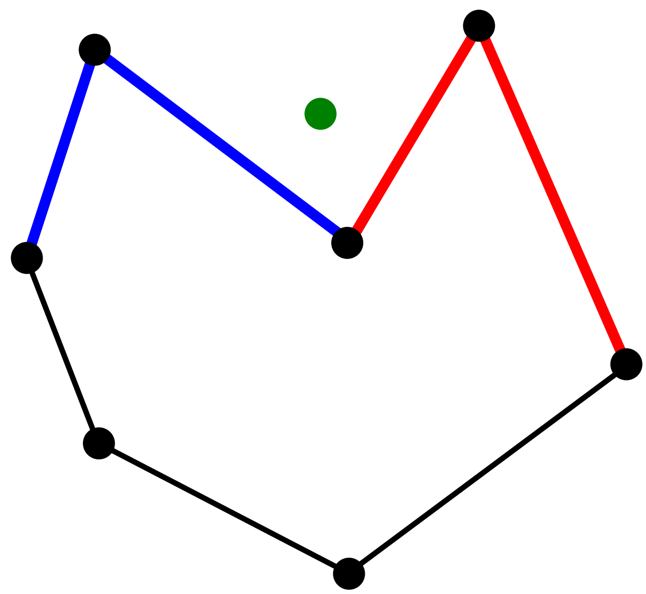


- Orientation test
  - counterclockwise (left turn): $\sigma < \tau$
  - clockwise (right turn): $\sigma > \tau$
  - collinear (left turn): $\sigma = \tau$
- The orientation depends on whether the expression
  $(y_2 - y_1)(x_3 - x_2) - (y_3 - y_2)(x_2 - x_1)$
  is positive, negative, or null.

Point Inclusion

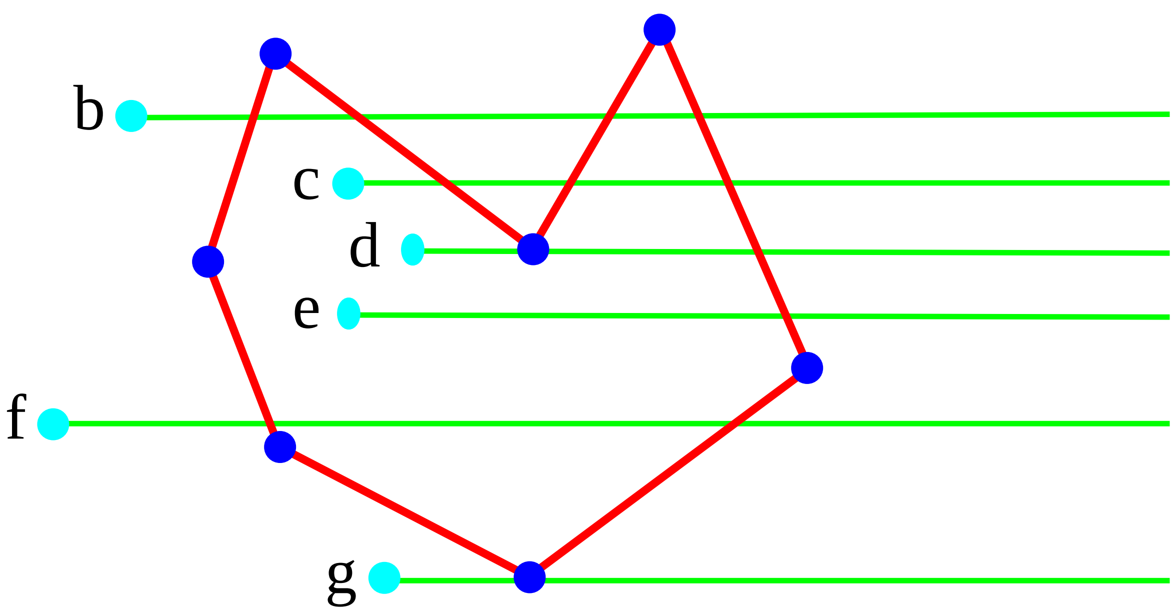
- given a polygon and a point, is the point inside or outside the polygon?
- orientation helps solving this problem in linear time





Point Inclusion — Part II

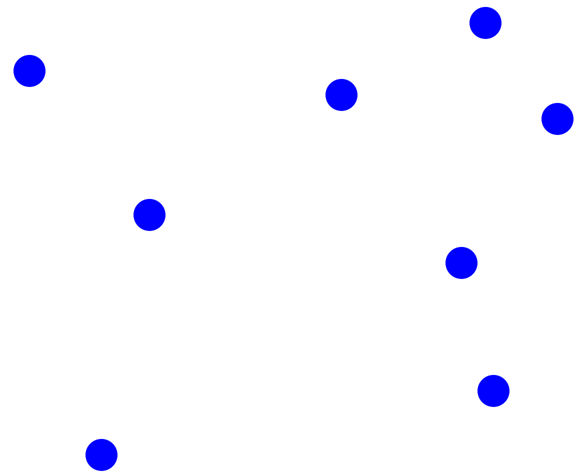
- Draw a horizontal line to the right of each point and extend it to infinity
- Count the number of times a line intersects the polygon. We have:
  - even number $\Rightarrow$ point is outside
  - odd number $\Rightarrow$ point is inside
- Why?



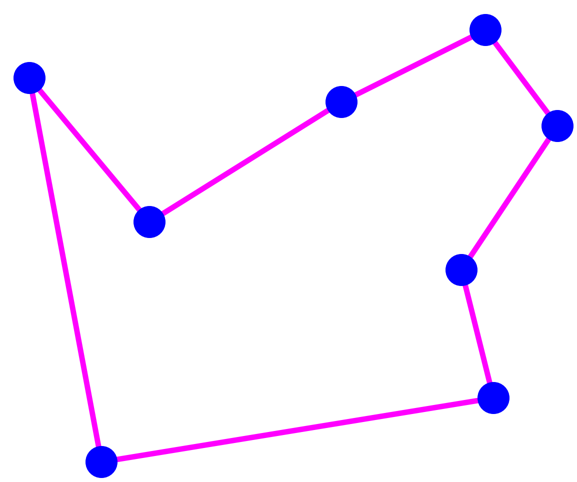
- What about points d and g ?? Degeneracy!

Simple Closed Path — Part I

- Problem: Given a set of points ...

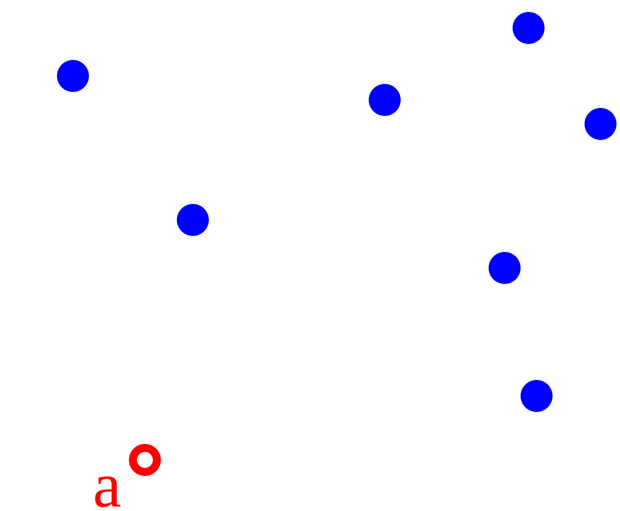


- “Connect the dots” without crossings

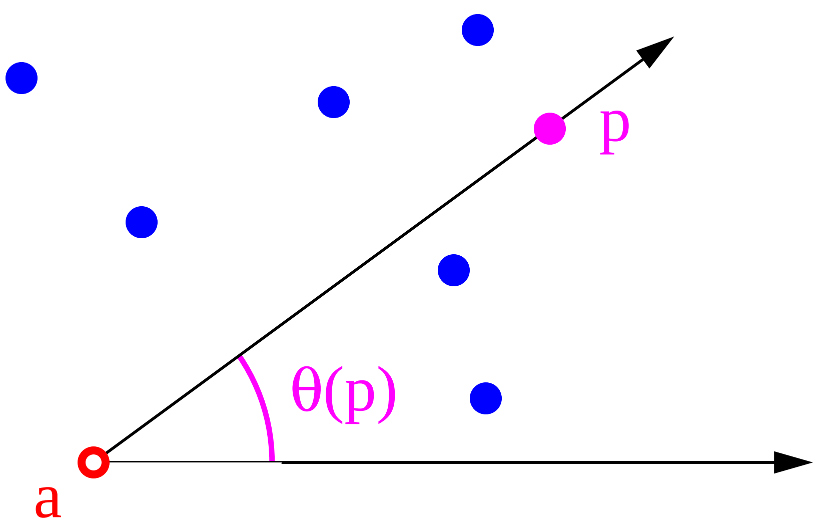


Simple Closed Path — Part II

- Pick the bottommost point a as the anchor point

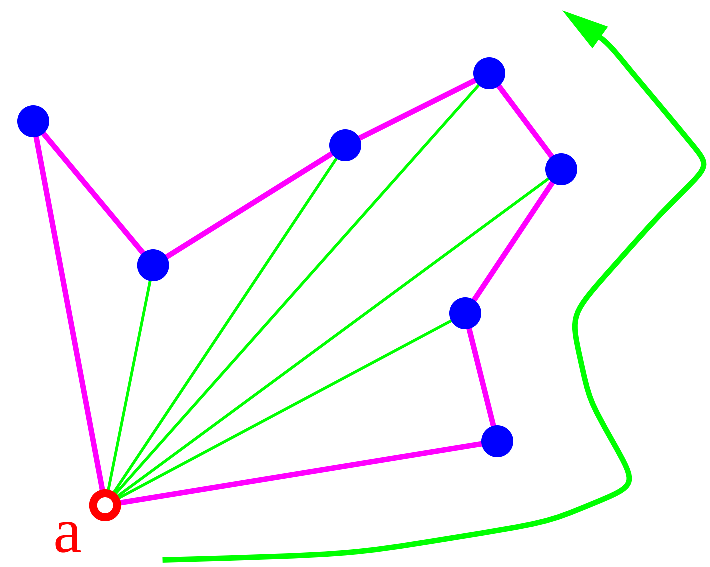


- For each point p, compute the angle q(p) of the segment (a,p) with respect to the x-axis:



Simple Closed Path — Part III

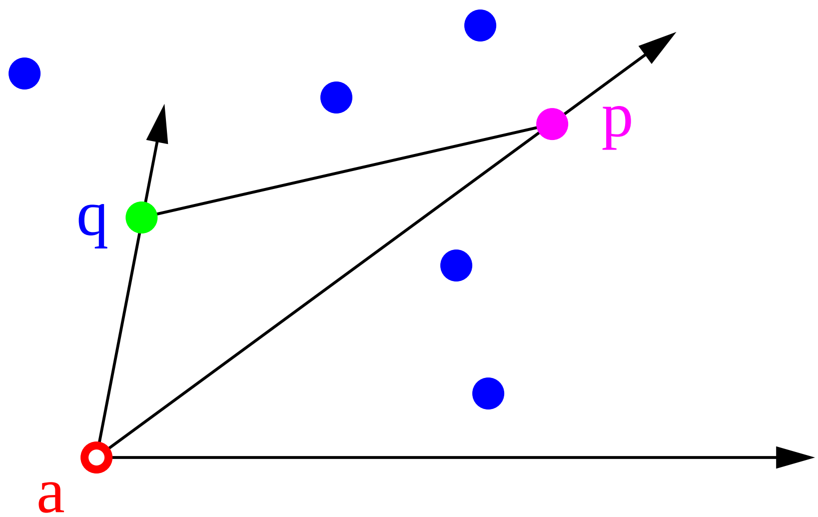
- Traversing the points by increasing angle yields a simple closed path:



- The question is: how do we compute angles?
  - We could use trigonometry (e.g., arctan).
  - However, the computation would be inefficient since trigonometric functions are not in the normal instruction set of a computer and need a call to a math-library routine.
  - Observation:, we don't care about the actual values of the angles. We just want to sort by angle.
  - Idea: use the orientation to compare angles without actually computing them!!

Simple Closed Path — Part IV

- the orientationcan be used to compare angles without actually computing them ... Cool!



$\theta(p) < \theta(q) \;\Leftrightarrow\;$ orientation(a,p,q) = CCW

- We can sort the points by angle by using any “sorting-by-comparison” algorithm (e.g., heapsort or merge-sort) and replacing angle comparisons with orientation tests
- We obtain an O(N log N)-time algorithm for the simple closed path problem on N points

Graham Scan Algorithm

Algorithm Scan(S, a):
Input: A sequence S of points in the plane beginning with point a such that:
1) a is a vertex of the convex hull of the points of S
2) the remaining points of S are counterclockwise around a.

Output: Sequence S from which the points that are not vertices of the convex hull have been removed.

```
S.insertLast(a)              {add a copy of a at the end of S}
prev ← S.first()             {so that prev = a initially}
curr ← S.after(prev)         {the next point is on the}
                             {current convex chain}
repeat
   next ← S.after(curr)      {advance}
   if points (point(prev), point(curr), point(next))
make a left turn then
      prev ← curr
   else
      S.remove(curr)         {point curr is on the convex hull}
      prev ← S.before(prev)
   curr ← S.after(prev)
until curr = S.last()
S.remove(S.last())           {remove the copy of a}
```