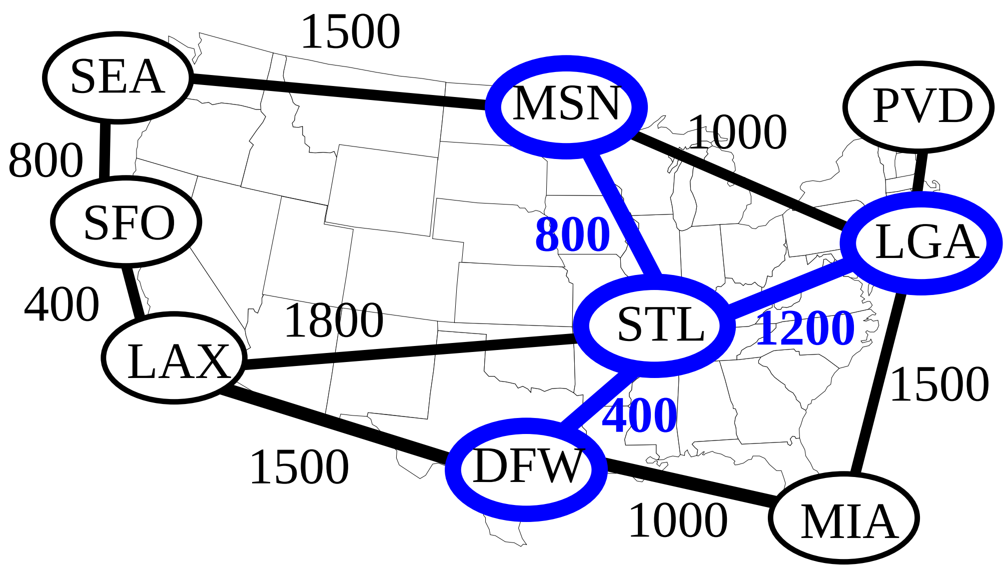
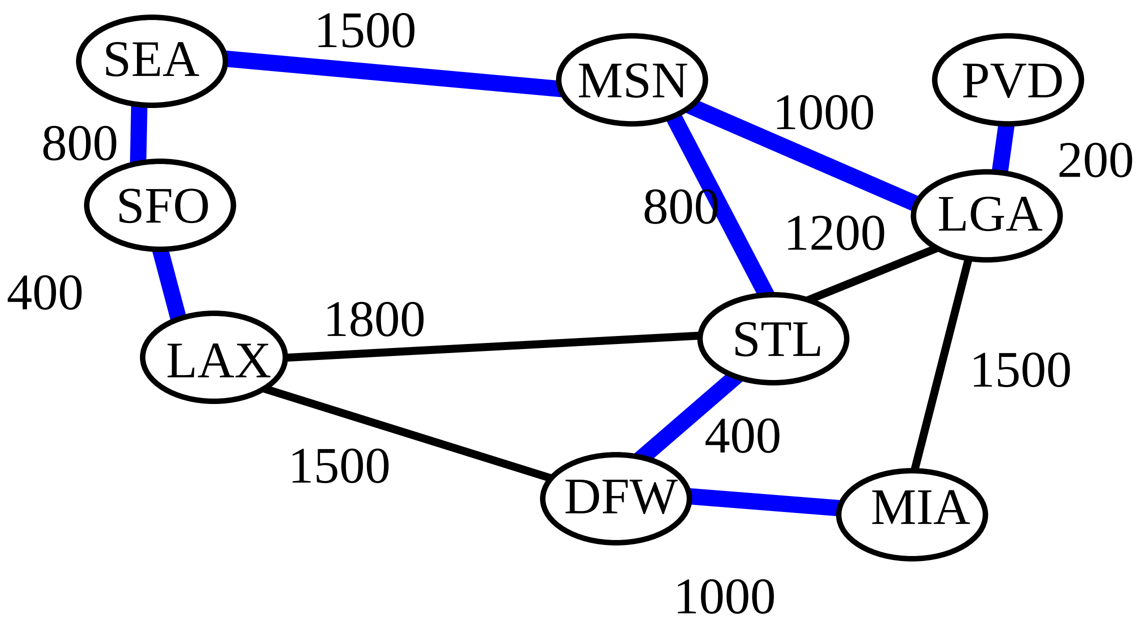
MINIMUM SPANNING TREE

- Prim-Jarnik algorithm
- Kruskal algorithm

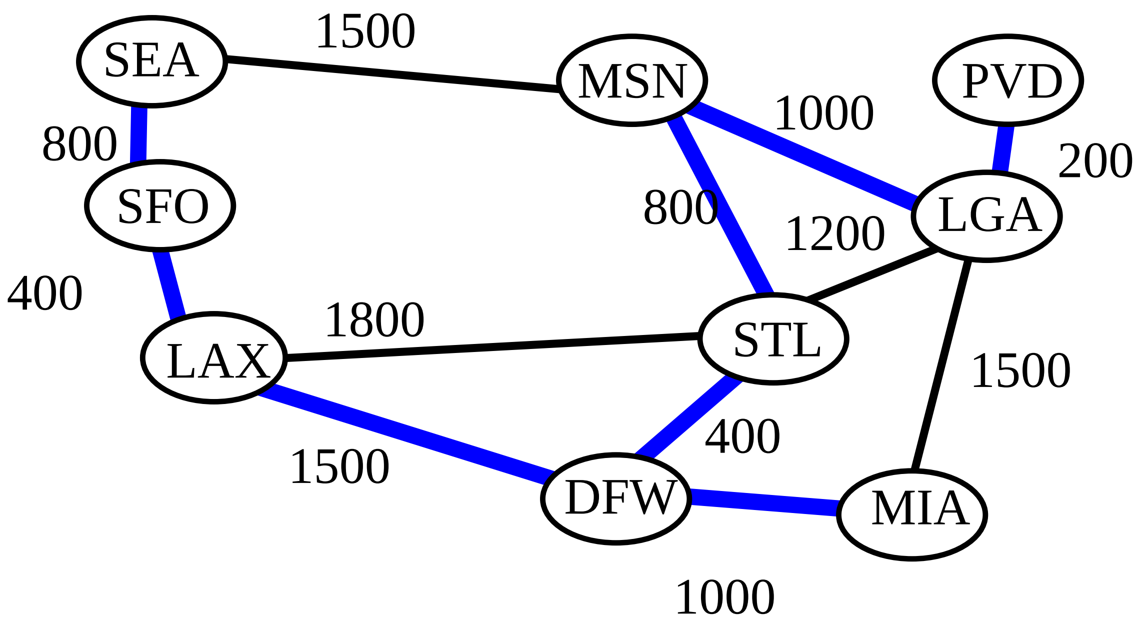


Minimum Spanning Tree

- spanning tree of minimum total weight
- e.g., connect all the computers in a building with the least amount of cable
- example

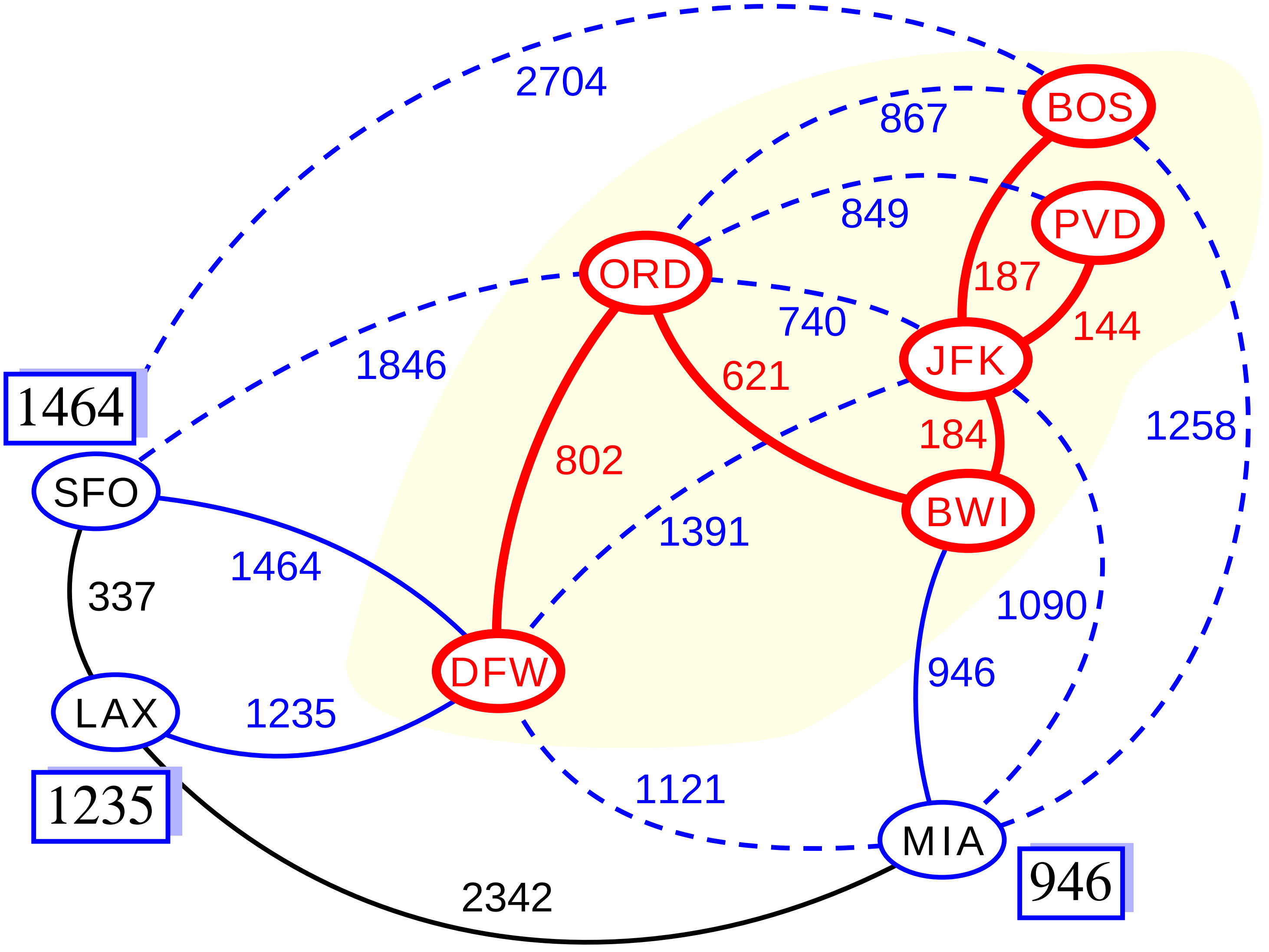


- not unique in general

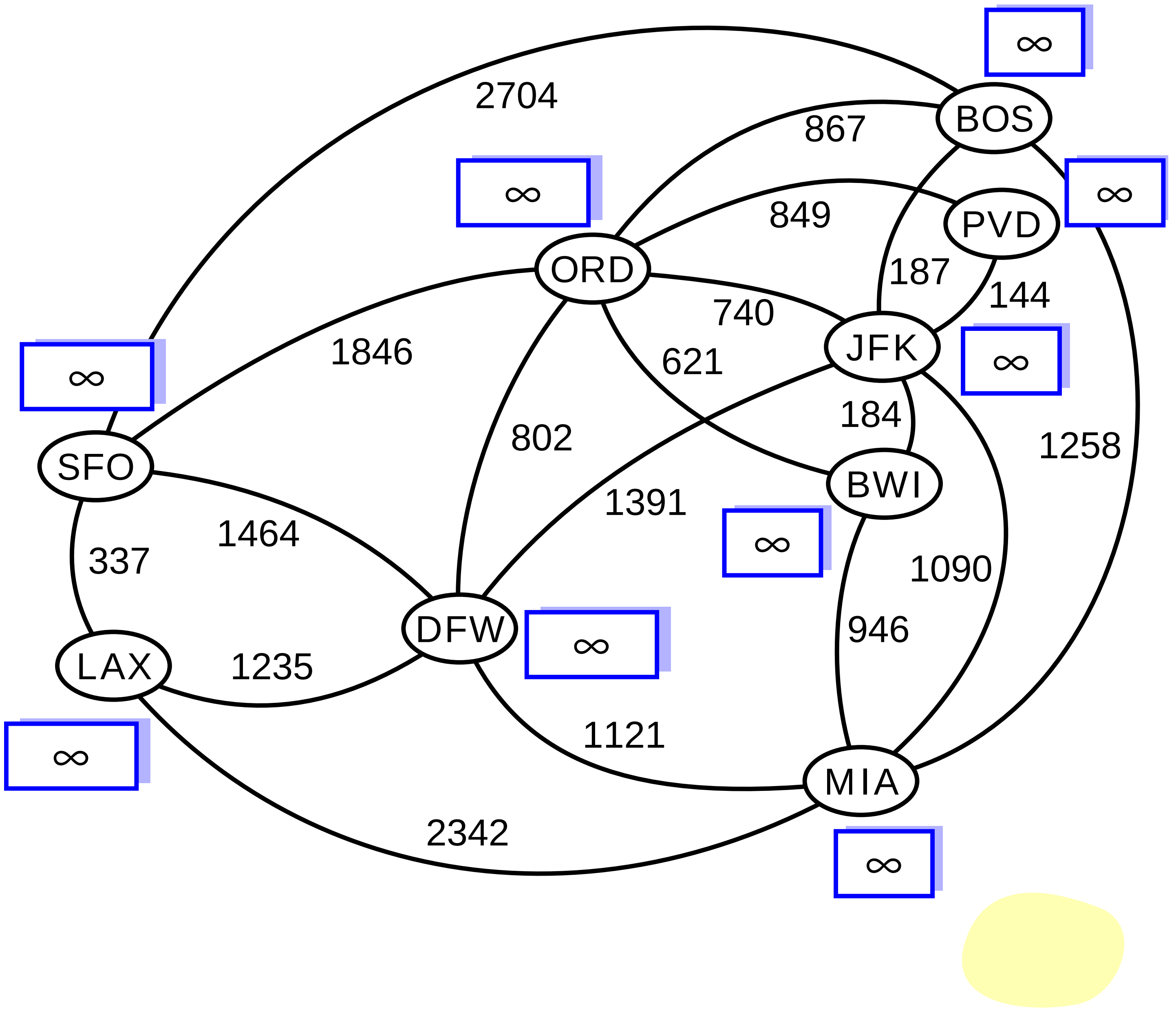


Prim-Jarnik Algorithm

- similar to Dijkstra's algorithm
- grows the tree T one vertex at a time
- cloud covering the portion of T already computed
- labels D[v] associated with vertex v
- if v is not in the cloud, then D[v] is the minimum weight of an edge connecting v to the tree



Example



BOS
PVD
ORD
JFK
SFO
BWI
LAX
DFW
MIA
2704
867
849
187
144
740
621
1846
802
184
1258
1391
1464
337
1090
946
1235
1121
2342
∞
∞
∞
∞
∞
∞
∞
∞
∞
∞

Pseudo Code

```
Algorithm PrimJarnik($G$):
   Input: A weighted graph $G$.
   Output: A minimum spanning tree $T$ for $G$.
pick any vertex $v$ of G
{grow the tree starting with vertex $v$}
T $\leftarrow$ {$v$}
   $D[u] \leftarrow 0$
   $E[u] \leftarrow \varnothing$
for each vertex $u \neq v$ do
   $D[u] \leftarrow +\infty$
let $Q$ be a priority queue that contains all the
      vertices using the $D$ labels as keys
while $Q \neq \varnothing$ do
      {pull $u$ into the cloud C}
      $u \leftarrow Q$.removeMinElement()
      add vertex $u$ and edge $(u,E[u])$ to $T$
      for each vertex $z$ adjacent to $u$ do
         if $z$ is in $Q$
            {perform the relaxation operation on edge $(u, z)$ }
            if weight$(u, z) < D[z]$ then
               $D[z] \leftarrow$ weight$(u, z)$
               $E[z] \leftarrow (u, z)$
               change the key of $z$ in $Q$ to $D[z]$
   return tree $T$
```

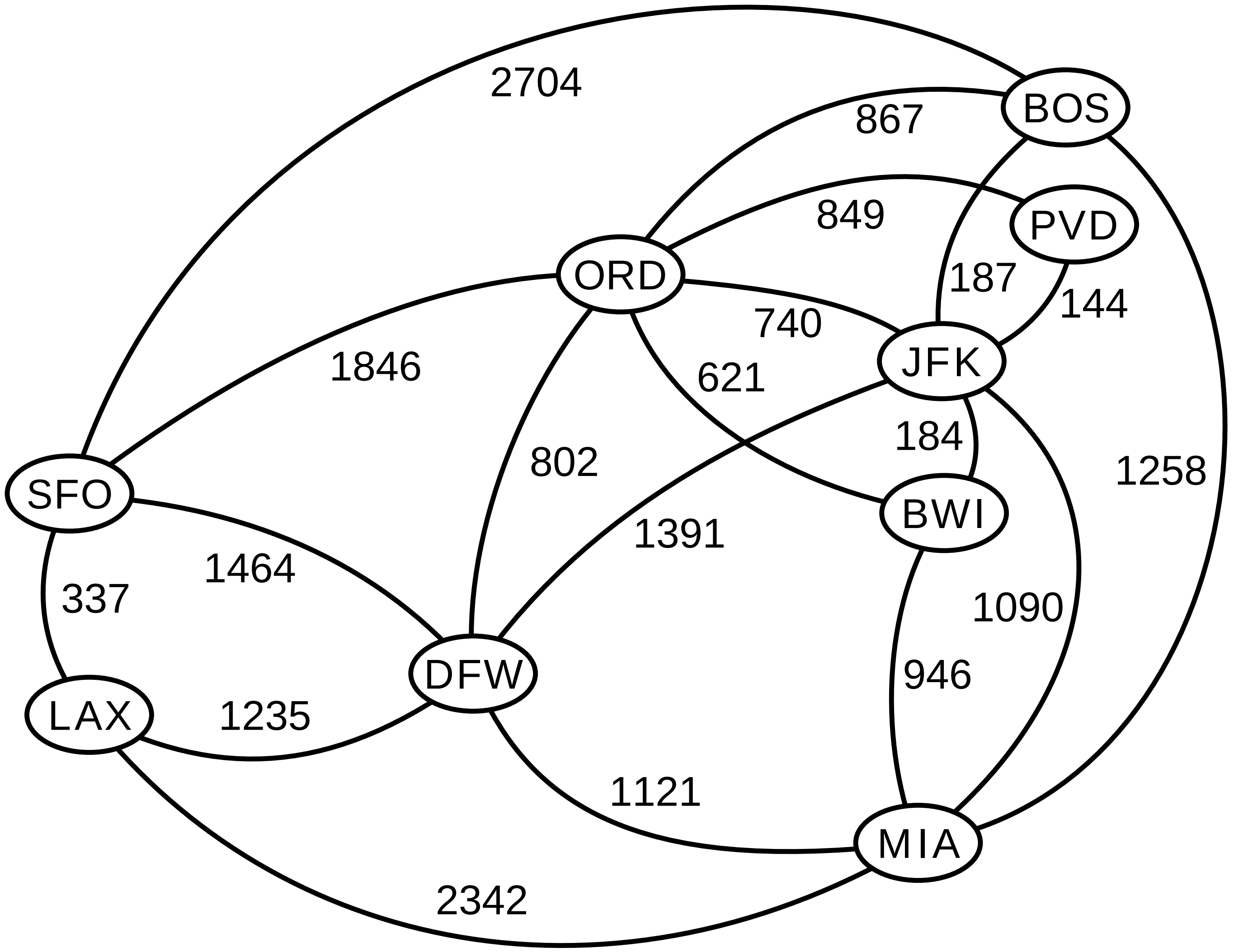
Running Time

```
T ← {$v$}
   $D[u] \leftarrow 0$
   $E[u] \leftarrow \emptyset$
for each vertex $u \neq v$ do
   $D[u] \leftarrow +\infty$
let $Q$ be a priority queue that contains all the
      vertices using the $D$ labels as keys
while $Q \neq \emptyset$ do
      $u \leftarrow Q$.removeMinElement()
      add vertex $u$ and edge $(u, E[u])$ to $T$
      for each vertex $z$ adjacent to $u$ do
          if $z$ is in $Q$
            if weight$(u, z) < D[z]$ then
             $D[z] \leftarrow$ weight$(u, z)$
             $E[z] \leftarrow (u, z)$
             change the key of $z$ in $Q$ to $D[z]$
return tree $T$
```

O((n+m) log n)

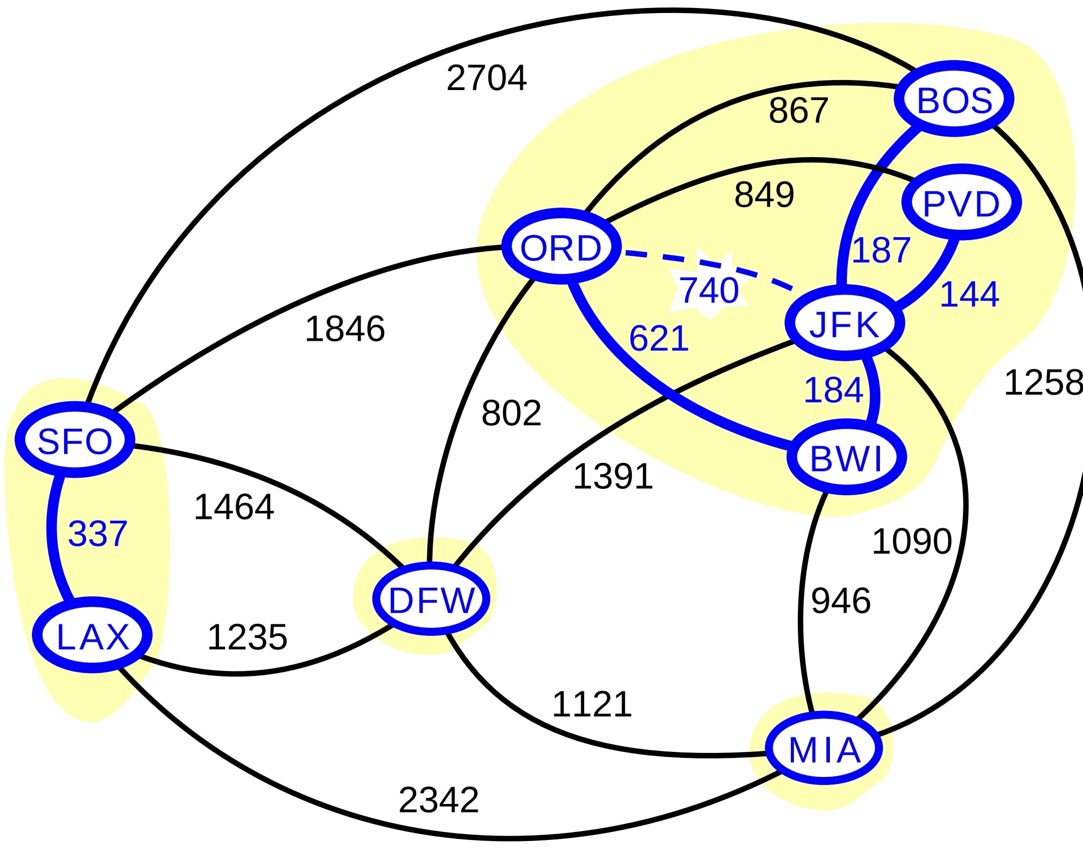
Kruskal Algorithm

- add the edges one at a time, by increasing weight
- accept an edge if it does not create a cycle



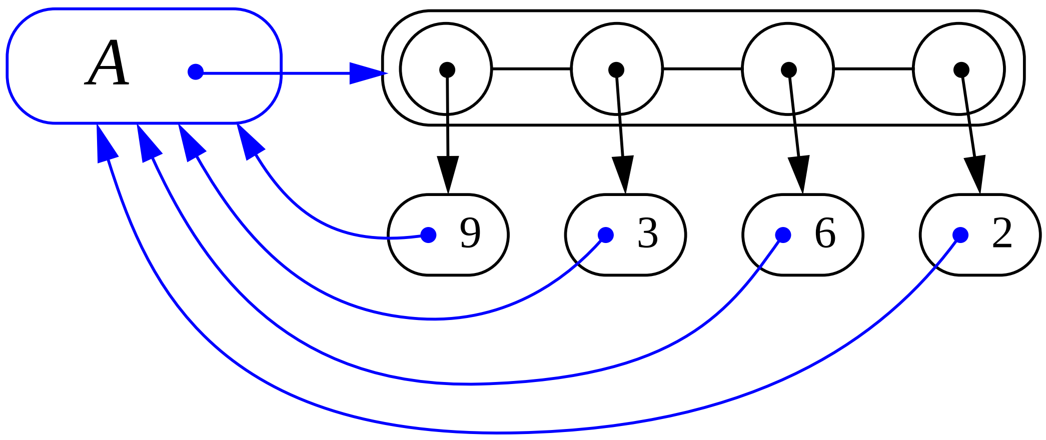
Data Structure for Kruskal Algortihm

- the algorithm maintains a forest of trees
- an edge is accepted it if connects vertices of distinct trees
- we need a data structure that maintains a partition, i.e.,a collection of disjoint sets, with the following operations
  - **find**(u): return the set storing u
  - **union**(u,v): replace the sets storing u and v with their union



Representation of a Partition

- each set is stored in a sequence
- each element has a reference back to the set



- operation **find**(u) takes O(1) time
- in operation **union**(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
- the time for operation **union**(u,v) is min(n_u,n_v), where n_u and n_v are the sizes of the sets storing u and v
- whenever an element is processed, it goes into a set of size at least double
- hence, each element is processed at most log n times

Pseudo Code

```
**Algorithm** Kruskal($G$):
  Input: A weighted graph $G$.
  Output: A minimum spanning tree $T$ for $G$.

let $P$ be a partition of the vertices of $G$, where each
vertex forms a separate set

let $Q$ be a priority queue storing the edges of $G$ and
their weights

$T \leftarrow \emptyset$
while $Q \neq \emptyset$ do
  (u,v) $\leftarrow$ $Q$.removeMinElement()
  if $P$.find(u) $\neq$ $P$.find(u) then
    add edge (u,v) to $T$
    $P$.union(u,v)

return $T$
```

Running time: O((n+m) log n)