COUNTING: PIGEON HOLE PRINCIPLE, PERMUTATIONS, COMBINATIONS

Lecture 15

Tree Diagrams

We can solve many counting problems through the use of ***tree diagrams***

- a branch represents a possible choice
- the leaves of the tree represent possible outcomes.

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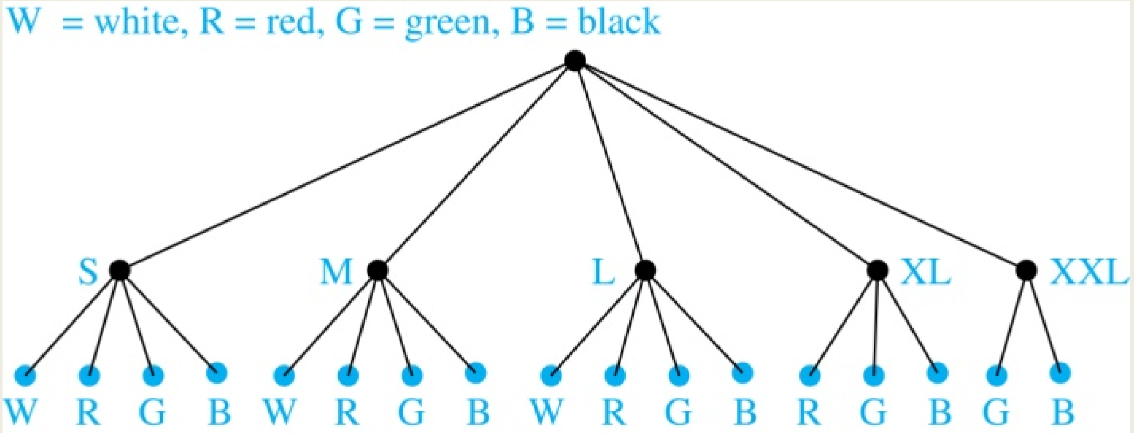


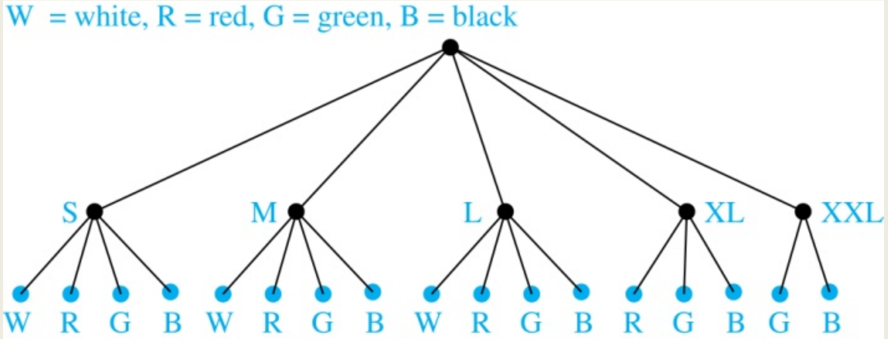
Example: Tree Diagrams

A T-shirt comes in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors: white, red, green, and black, except

- XL comes only in red, green, and black
- XXL comes only in green and black.

What is the minimum number of T-shirts that a store needs to stock to have one of each size and color available?

Solution:
Draw the tree diagram.
17 T-shirts must be stocked.



PIGEONHOLE PRINCIPLE (6.2)

- Basic principle
- Applications

The Pigeonhole Principle

If a flock of 26 pigeons roosts in a set of 25 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

https://en.wikipedia.org/wiki/Pigeonhole_principle

The Pigeonhole Principle

Pigeonhole Principle:

If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contradiction.

Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. ◀

Example: Among any group of 367 people, there must be at least two with the same birthday because there are only 366 possible birthdays.

Example: Pigeonhole Principle

Every positive integer *n* has a multiple that has only 0’s and 1’s in its decimal expansion.
For example, for n=6, 1110 = 185×6.

Solution: Let *n* be a positive integer.

Every positive integer *n* has a multiple that has only 0’s and 1’s in its decimal expansion; e.g., for n=6, 1110 = 185×6.

Solution:

Let *n* be a positive integer.
Consider the *n* + 1 integers 1, 11, 111, …., 11…1 (where the last integer has (*n* + 1) 1’s).
There are *n* possible remainders when an integer is divided by *n*.

Divide each of the *n* + 1 integers by *n*. By the pigeonhole principle, at least two integers must have the same remainder (i.e., s = kn+r, t = jn+r)

Subtract the smaller from the larger.
The result is a multiple of *n* that has only 0’s and 1’s in its decimal expansion.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Prove by contradiction: If all boxes contain at most $\lceil N/k \rceil$ -1 objects, the total number of objects cannot be N.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example: Generalized Pigeonhole Principle

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

Example: Generalized Pigeonhole Principle

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

Assume there are four boxes, one for each suit.
We place cards in the box reserved for its suit.
After N cards have been placed into boxes, at least one box contains at least $\lceil N/4 \rceil$ cards.

At least three cards of one suit have been selected if $\lceil N/4 \rceil \geq 3$.
The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is
$N = 2 \cdot 4 + 1 = 9$.
Hence, **select 9 cards**.

Pigeonhole Principle Example (sort of…)

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

Solution:

A deck contains 52 cards and 13 hearts.
Hence, 39 cards are not hearts.

If we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.
However, when we select **42 cards**, we must have at least three hearts.

PERMUTATIONS AND COMBINATIONS (6.3)

- Permutations and r-permutations
- Combinations and r-combinations
- Binomial coefficients

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects.
An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

The ordered arrangement 3,1, 2 is a **permutation** of *S*.

The ordered arrangement 3, 2 is a **2-permutation** of *S*.

Permutations

The number of ***r*-permutations** of a set with *n* elements is denoted by $P(n,r)$.

The 2-permutations of $S = \{1,2,3\}$ are

1,2; 1,3; 2,1; 2,3; 3,1; 3,2

Hence, $P(3,2) = 6$.

$P(n,r) = n(n-1)(n-2) \ldots (n-r+1)$ with $1 \leq r \leq n$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations

Example: Suppose a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order.

How many possible orders exist?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If you need to find the tour with the shortest path that visits all the cities, do you need to consider all 5040 paths?

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$\boldsymbol{P(n,r)} = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule.

- The first element can be chosen in n ways.
- The second element can be chosen in $n-1$ ways,

  .

  .
- until there are $(n - (r - 1))$ ways to choose the last element.

Note: $P(n,0) = 1$. There is only one way to order zero elements.

Solving Counting Problems by Counting Permutations

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set.

An *r*-combination is a subset with *r* elements.

The number of *r*-combinations of a set with n distinct elements is denoted by *C*(*n*, *r*).

Notation: $C(n, r) = \binom{n}{r}$ is called a *binomial coefficient*.

Example: Combinations

$S = \{a, b, c, d\}$

$\{a, c, d\}$ is a 3-combination from S.
It is the same as $\{d, c, a\}$ since the order does not matter.

$C(4,2) = 6$

The 2-combinations of set $\{a, b, c, d\}$ are six subsets:
$\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem: The number of *r*-combinations of a set with *n* elements, $n \geq r \geq 0$, is $C(n,r) = \frac{n!}{(n-r)!r!}$.

Proof:

The P(n,r) r-permutations of the set can be obtained by

- forming the $C(n,r)$ r-combinations and then
- ordering the elements in each which can be done in r! ways

By the product rule $P(n, r) = C(n,r) \cdot r!$ The result follows.

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Useful identities

$$C(n,r) = \frac{P(n,r)}{r!}$$

$$P(n,r) = C(n,r) \cdot r!$$

$$C(n,r) = C(n,n-r)$$

Example: Combinations

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

Example: Combinations

How many ways are there to select 47 cards from a deck of 52 cards?

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations

Corollary: Let n and r be nonnegative integers, $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: Since $C(n,r) = \frac{n!}{(n-r)!r!}$

and $C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$.

$C(n, r) = C(n, n - r)$ follows. ◀