COUNTING: BASIC RULES

Lecture 14

Problems we want to solve

How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

How many cards must Bob select from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Alice likes 4 types of bagels. How many ways are there to select a dozen bagels when she chooses only among the 4 types she likes?

BASICS OF COUNTING (6.1)

- Basic Counting Principles and Counting Rules

Counting Rules

- Product Rule
- Sum Rule
- Subtraction Rule
- Division Rule

The Product Rule

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.'

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are

- n_1 ways to do the first task and
- n_2 ways to do the second task.

Then, there are $n_1 \cdot n_2$ ways to do the procedure.

Example: License Plates

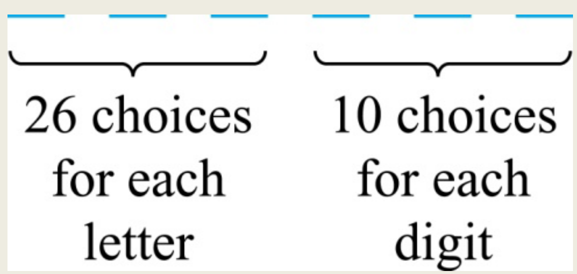
How many different license plates can be made if each plate is a sequence of three uppercase English letters followed by three digits?

e.g., AGF349

Solution: By the product rule, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$$

different possible license plates.



Example: Telephone Numbering Plan

The *North American Numbering Plan* (*NANP*) specifies that a telephone number consists of 10 digits:
3-digit area code, 3-digit office code, a 4-digit station code.

There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- Format in old plan (used in 1960s): NYX-NNX-$XXXX$.
- Format in new plan: NXX-NXX-$XXXX$.

How many different telephone numbers are possible under the old plan and the new plan?

$X = \{0, \ldots, 9\}$, $N = \{2, \ldots, 9\}$, $Y = \{0, 1\}$

old plan format: *NYX-NNX-XXXX*

new plan format: *NXX-NXX-XXXX*

How many different telephone numbers are possible?

$X = \{0, \ldots, 9\}$, $N = \{2, \ldots, 9\}$, $Y = \{0, 1\}$

old plan format: *NYX-NNX-XXXX*

new plan format: *NXX-NXX-XXXX*

How many different telephone numbers are possible?

Solution: Use the Product Rule.

- $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX.*
- $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX.*
- $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX.*
- $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX.*

Number of old plan telephone numbers:

$$160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000.$$

Number of new plan telephone numbers:

$$800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000.$$

Example: Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set *S* is $2^{|S|}$.

Solution: List the elements of S, |S|=k, in an arbitrary order.

There is a one-to-one correspondence between subsets of *S* and bit strings of length k.

- When the *i*-th element is in the subset, the bit string has a 1 in the *i*-th position and a 0 otherwise.

By the product rule, there are 2^k such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, …, and an element in A_m.
- By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

The Sum Rule

Example:

Friday night you can see one of five movies, go to one of two concerts, or stay home. How many choices do you have for spending Friday night?

There are 5 + 2 + 1 = 8 choices

Sum Rule:

If there are n_1 ways for one task and n_2 ways for another task and the two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to select one of these tasks.

Sum Rule Example

The CS department chooses either a student or a faculty as a representative for a committee.

How many choices are there for this representative if there are 47 CS faculty and 558 CS majors and no one is both a faculty member and a student.

There are 47 + 558 = 605 possible ways

Sum Rule in terms of sets

The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \emptyset \text{ for all } i, j.$$

Combining Sum and Product Rule

Example:

Suppose an ID can be either a two letters or a letter followed by a digit.

Find the number of possible IDs.

Solution: $(26 \cdot 26) + (26 \cdot 10) = 936$

Example: Counting Passwords

A password must be 6-8 characters long; each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?

Solution:

Let P be the total number of passwords.

Let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

By the sum rule, $P = P_6 + P_7 + P_8$.

Example: Counting Passwords

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Let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
By the sum rule, $P = P_6 + P_7 + P_8$.

To find each of P_6, P_7, and P_8, find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters.

$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$

$P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176$
$= 70,332,353,920.$

$P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576$
$= 2,612,282,842,880.$

Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

The Subtraction Rule

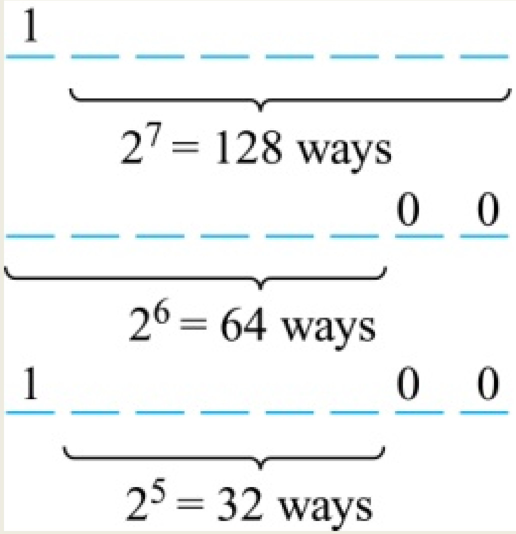
Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Counting Bit Strings

How many bit strings of length eight either start with a 1 or end with 00?



Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1: $2^7 = 128$
- Number of bit strings of length eight that end with 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 <u>and</u> end with 00: $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

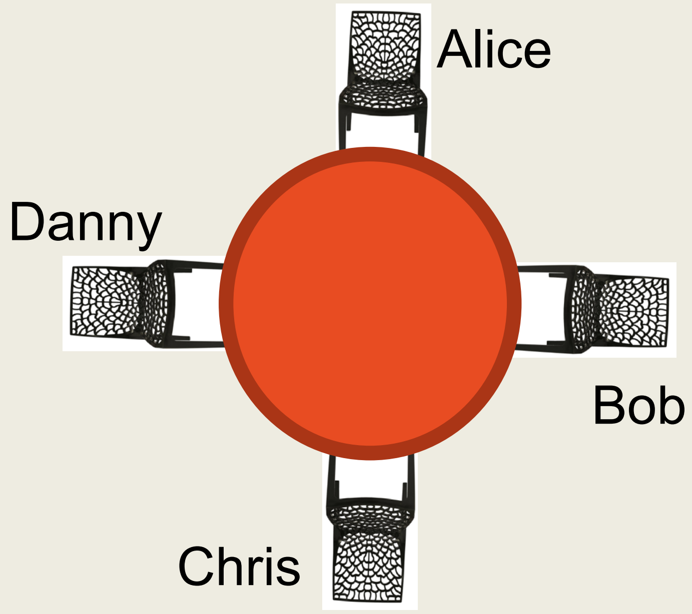
The Division Rule

Bob counts the number of people in the room by courting the number of ears (assume every person has two ears). To get the number of people, he needs to divide the number of ears by 2.

Division Rule: If a task can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w, then there n/d ways to do the task.

The Division Rule

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?



Solution:

Number the seats around the table from 1 to 4 proceeding clockwise.

There are four ways to select the person for seat 1, 3 ways for seat 2, 2 ways for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people.

But, two seatings are the same when each person has the same left and right neighbor. Each of the four choices for seat 1 leads to the same seating arrangement.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.