CALCULUS OF VARIATIONS TUTORIAL: (adapted from "Mathematics and Technology" by Rousseau and Saint-Aubin & John Strain's notes)

- **Introduction:**
  - branch of applied mathematics dealing with optimization over function spaces
  - many applications to physics & engineering
  - Used in Hamiltonian mechanics – bridge between Newtonian and quantum mechanics

- Recall Lagrange multiplier method:
  \[
  f: \mathbb{R}^n \to \mathbb{R}, \quad g: \mathbb{R}^n \to \mathbb{R}
  \]
  \[
  \min_{x} \quad f(x) \quad \text{s.t. } g(x) = c \quad \implies \quad \mathcal{L}(x, \lambda) = f(x) + \lambda (g(x) - c), \quad \nabla \mathcal{L} = 0
  \]

In variational calculus, we optimize over function spaces rather than \( \mathbb{R}^n \).

- **Example 1:** (Shortest Path)
  - Find shortest path between A & B.
  - Ans: Straight line \( \Delta \) inequality.

    ![Diagram of A and B points with a straight line connecting them](image)

    Formalism: Let \( y = y(x) \) \( \Rightarrow \) Path parametrized by \( (x, y(x)), \ x \in [a, b] \).

    Let \( I[y] = \) length of path between A & B

    \[
    I[y] \equiv \int_{A}^{B} ds = \int_{A}^{B} \sqrt{dx^2 + dy^2} = \int_{a}^{b} \sqrt{1 + (y'(x))^2} \ dx
    \]

    \( \therefore \) Our problem is: \( \min_{y(x): y(a) = y_1, \ y(b) = y_2} I[y] \) \( \Rightarrow \) can try to solve this \( \rightarrow \) DIRECT METHOD

- **Example 2:** (Brachistochrone "shortest time") \( \Rightarrow \) posed by Johann Bernoulli as contest & solved by Newton, Leibniz, L'Hôpital, Johann & Jacob Bernoulli

  What is the best "shape" of a skateboard ramp?

  - Want: Minimum time from A to B, powered only by gravity.
  - Let the path between A & B be \((x, y(x))\). Let \( I[y] \) = total time.

    ![Diagram of skateboard ramp](image)

    Formalism: (Conservation of energy)

    Let energy at A be \( E = 0 \) (stationary)

    \[
    \frac{1}{2}mv^2 = mg\frac{h}{y} \quad (\text{for some pt between A & B}) \quad \Rightarrow \quad v = \sqrt{2gh}
    \]

    \[
    I[y] \equiv \int_{A}^{B} dt = \int_{A}^{B} \frac{dx}{v} = \int_{A}^{B} \frac{dx}{\sqrt{g/h \cdot \sqrt{1 + (y'(x))^2}}} = \frac{1}{\sqrt{2gh}} \int_{a}^{b} \sqrt{1 + (y'(x))^2} \ dx
    \]

    \( \therefore \) Our problem is: \( \min_{y(x): y(a) = 0, \ y(b) = h} I[y] \)
• Fundamental Problem of Calculus of Variations:

Given a Lagrangian: \( L: [a,b] \times \mathbb{R}^3 \times \mathbb{R}^3 \), \( L(x,y,z) \)

admissible functions: \( C = \left\{ y: [a,b] \rightarrow \mathbb{R}^3 \mid y(a) = y_1, y(b) = y_2, y \text{ is twice differentiable} \right\} \)

boundary conditions

regularity conditions: Hölder derivatives

cost function: \( I[y] = \int_a^b L(x, y(x), y'(x)) \, dx \leftarrow \text{called action of physical system} \)

the problem is: \( \min_{y \in C} I[y] \), find extremal values


• Euler-Lagrange Equations: (Systematic / indirect method of solution)

- Thm: If \( y \in C \) minimizes \( I[y] \) over \( C \), then:

\[
\frac{\partial L}{\partial y} (x, y(x), y'(x)) - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} (x, y(x), y'(x)) \right) = 0.
\]

only necessary conditions

solution may be local optimum, or saddle point, etc.

Fundamental Lemma of Calculus of Variations: (FCLV)

\[
\int_a^b u(x) w(x) \, dx = 0 \quad \text{for all } w \in C
\]

if and only if \( u(x) = 0 \). compare with finite case (vectors)

Pf: \( \Rightarrow \) Let \( u = w = 0 \Rightarrow u(x) = 0 \). can make this measure theoretic

\( \Leftarrow \) Obvious.

Pf: Suppose \( y \) minimizes \( I[y] \) over \( C \). Let \( w: [a,b] \rightarrow \mathbb{R} \) be any function with \( w(a) = w(b) = 0 \) and appropriate regularity conditions.

\[
\frac{d}{dt} I[y_{0}+tw] \bigg|_{t=0} = 0 \quad \text{as } y_0 \text{ is minimizer}
\]

DCT/Leibniz rule

\[
0 = \frac{d}{dt} \int_a^b L(x, y_0 + tw, y_0' + tw') \, dx \bigg|_{t=0} = \int_a^b \frac{\partial L}{\partial y} (x, y_0 + tw, y_0' + tw') w(x) \, dx + \int_a^b \frac{\partial L}{\partial y'} (x, y_0 + tw, y_0' + tw') w'(x) \, dx
\]

[Chain rule]

\[
= \int_a^b \left( \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right) w(x) \, dx + \int_a^b \frac{\partial L}{\partial y'} w(x) \, dx
\]

\[
= 0, \text{ as } w(a) = w(b) = 0
\]

\[
\Rightarrow \frac{dL}{dy} - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0
\]

- Fermat’s Principle of Optics: Light follows the trajectory that takes the shortest time to travel.

\( \Rightarrow \) Can use variational calculus & E-L eqns to derive laws of reflection & refraction.

Snell’s Law
Example 1 Solution: (Shortest Path)

\[ L(x, y, z) = \sqrt{1 + z^2}, \quad \min_{y(x)}: I[y] = \int_a^b L(x, y, y') \, dx \]
\[ y(a) = y_1, y(b) = y_2 \]

\[ \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial z} = \frac{z}{\sqrt{1 + z^2}} \]

Euler-Lagrange equations:

\[ \frac{\partial L}{\partial y} - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0 \quad \Rightarrow \quad \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = 0 \]

\[ \Rightarrow \quad \frac{y''}{(1 + (y')^2)^{3/2}} = 0 \quad \Rightarrow \quad y'' = 0, \quad y(a) = y_1, \quad y(b) = y_2 \]

So, \( y(x) \) is a straight line connecting \((a, y_1)\) and \((b, y_2)\).

Remark:

\[ \frac{y''}{(1 + (y')^2)^{3/2}} \]

is the signed curvature

- \( 0 \) curvature corresponds to lines.
- Clairaut’s Thm [cont. second deriv] - constant curvature is a circle.

\[ \frac{d^2 (y')}{dx^2} = \frac{d}{dx} \left( \frac{d}{dt} \left( \frac{dy}{dx} \right) \right) = \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = \frac{y''}{(1 + (y')^2)^{3/2}} \]

rate of change of gradient with \( ds \)

Noether’s Thm: Any differentiable symmetry of the action integral (or Lagrangian) has a corresponding conservation law.

1. \( L = L(x, z) \) independent of \( y \).

\[ \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \text{By Euler-Lagrange eqns,} \quad \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial z} = \text{constant} \]

\[ \text{differential symmetry} \]

eg: (Shortest Path)

\[ L(x, y, z) = \sqrt{1 + z^2} \quad \Rightarrow \quad \frac{\partial L}{\partial z} = \frac{y'}{\sqrt{1 + (y')^2}} = \text{constant} \quad \Rightarrow \quad y' \text{ constant} \quad \Rightarrow \quad y \text{ linear.} \]

eg: (Conservation of Linear Momentum)

\[ L(t, x, x') = \frac{1}{2} m(x')^2 \]

\[ \frac{\partial L}{\partial x} = m \quad \text{constant} \]

Linear momentum due to symmetry of physical laws in position.

2. Thm: \( L = L(y, z) \) independent of \( x \) \( \Rightarrow \) BELTRAMI Identity:

\[ y' \frac{\partial L}{\partial z} - L(y, y') = \text{constant} \]

Pf: \( \frac{d}{dx} \left( y' \frac{\partial L}{\partial z} - L \right) = \frac{\partial L}{\partial y} y'' + \frac{\partial L}{\partial z} z' - y' \frac{\partial}{\partial z} \left( \frac{\partial L}{\partial z} \right) = y' \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right) = 0 \]

by Euler-Lagrange equation

[continued.]
Hamilton's Principle: (The last example motivates Lagrangian mechanics.)

A system in motion follows a trajectory that minimizes
\[ \int_{t_1}^{t_2} L(t, x, x') \, dt, \]
where the Lagrangian \( L = T - V \). (Hamiltonian is \( T + V \))

- also called principle of least action (as we minimize action integral)

- solve \( E = L \) eqns: \( \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial x'} \right) = 0 \).

Example 2 Solution: (Brachistochrone Problem)

\[ L(x, y, z) = \sqrt{1 + z^2}, \quad \text{min.} \quad I[y] = \int \frac{1 + (y')^2}{\sqrt{y}} \, dx \]

remove constant (0)

\[ \frac{d^2}{dz^2}(y) - \frac{1}{\sqrt{1 + (y')^2}} = C \leftarrow \text{constant} \]

\[ \frac{dy}{dx} = \sqrt{k \frac{y - 1}{y}}, \quad k = \frac{1}{c^2} (\text{constant}) \]

Let \( \tan(\phi) = \sqrt{k} \frac{y - 1}{y} \Rightarrow d\phi \frac{dx}{dx} = \frac{1}{2k \sin(\phi)} \left( \phi \text{ func of } x \right) \]

as \( y = k \sin^2(\phi) \) & \( \frac{d\phi}{dx} = \frac{dy}{dx} \) constant of integration

\[ x = 2k \int \sin^2(\phi) \, d\phi = 2k \left( \frac{\phi}{2} - \frac{\sin(2\phi)}{4} \right) + c_1 \]

\[ y = k \sin(\phi) \, d\phi = -\frac{k \cos(2\phi)}{2} + c_2 \]

Boundary condition: \( y(0) = 0 \Rightarrow \phi = 0 \text{ & } x = 0 \Rightarrow c_1 = 0 \text{ & } c_2 = \frac{a}{2} \)

Let \( k = 2a \text{ & } 2\phi = \theta \). Then,
\[ x = a(\theta - \sin(\theta)) \]
\[ y = a(1 - \cos(\theta)) \]

- Cycloid:
  - path traced by point on rolling circle of radius a
  - solution to brachistochrone problem
  - solution to tautochrone problem (same period of oscillation of ball regardless of starting amplitude)

- Hamiltonian = total energy

- Symmetry of physical laws with time

- Beltrami: \[ x' \frac{\partial L}{\partial x'} - L(x, x') = m(x'^2) - (\frac{1}{2}m(x^2) - V(x)) = \text{constant} \]

- energy conservation

- best brachistochrone shape
Constrained Optimization:
- What if we have constraints? \( \max_{y \in C} \int_a^b L(x, y, y') \, dx \) subj to: \( \int_a^b K(x, y, y') \, dx = K_0 \)

Lagrangean \( C = \{ y : [a, b] \rightarrow \mathbb{R} | y(a) = y_1, y(b) = y_2, y \text{ twice differentiable} \} \)

Augmented Lagrangian: \( L(x, y, y') + \lambda K(x, y, y') \)

\[ I[y] = \int_a^b L(x, y, y') + \lambda K(x, y, y') \, dx \]

\[ \max_{y \in C} I[y] \Rightarrow \text{Euler-Lagrange Equations: } (\frac{\partial L}{\partial y} + \lambda \frac{\partial K}{\partial y}) - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} + \lambda \frac{\partial K}{\partial y'} \right) = 0, \]

& explicitly impose \( \int_a^b K(x, y, y') \, dx = K_0 \)

Example 3: (Dido's Isoperimetric Problem)
Legend is that around 850 B.C., Dido (Queen of Carthage) purchased land from a local king in the North African coastline that could be enclosed by the hide of an ox. The area she enclosed became the city of Carthage.

Want: Max area given arc length of A to B fixed.

\[ L(x, y, z) = y \]

\[ K(x, y, z) = \sqrt{1 + y'^2} \]

\[ \max_{y \in C} \int_a^b L(x, y, y') \, dx = \int_a^b y(x) \, dx \] \{ area \}

\[ \int_a^b K(x, y, y') \, dx = \int_a^b \sqrt{1 + (y')^2} \, dx = K_0 \] \{ fixed length \}

equivalent to minimizing arc length for fixed area

Augmented Lagrangian: \( L(x, y, y') + \lambda K(x, y, y') \)

Euler-Lagrange eqns: \( (\frac{\partial L}{\partial y} + \lambda \frac{\partial K}{\partial y}) - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} + \lambda \frac{\partial K}{\partial y'} \right) = 0 \)

\[ 1 - \frac{d}{dx} \left( \lambda \frac{y'}{\sqrt{1 + y'^2}} \right) = 0 \]

\[ \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) = \frac{1}{\lambda} \]

\[ \frac{y''}{(1 + (y')^2)^{3/2}} = \text{constant} \]

Constant curvature \( \Rightarrow \) Solution is circle:

Choose circle so that this length is \( K_0 \) & area is maximized

Solution to isoperimetric problem