Art Gallery Theorems and Algorithms

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Definitions

- $P$ is a simple polygon (i.e., does not cross over itself)
- Point $x \in P$ "covers" a point $y \in P$ if $xy \subseteq P$
- Let $G(P)$ be the minimum number $k$ of points of $P$, such that for any $y \in P$, some $x=x_1...x_k$ covers $y$
- Let $g(n)$ be the max($G(P)$) over all polygons of $n$ vertices
  - Thus, $g(n)$ guards are occasionally necessary and always sufficient

Art Gallery

- **Problem**: determine the minimum number of guards sufficient to cover the interior of an $n$-wall art gallery
  - Victor Klee, 1973
  - Vasek Chvatal, 1975

Main reference for this material:
Art Gallery Theorems and Algorithms, Joseph O'Rourke, Oxford University Press, 1987

Guard Placement

1. Can we just place one guard on every 3rd vertex?

[Diagram of a simple polygon with points labeled 0, 1, 2, 3, 4, and 5]
Guard Placement

1. Can we just place one guard on every 3rd vertex? – No!

2. If guards placed so they can see all the walls, does that imply they can see all the interior?
   - No!

3. If we restrict guards to vertices, is \( g_v(n) = g(n) \)?
   - In general, yes, equal for \( g(n) = \max(G(P)) \)

Art Gallery

- **Theorem**: \( \text{floor}(n/3) \) guards are occasionally necessary and always sufficient to cover a polygon of \( n \) vertices
  - “Chvatal’s Art Gallery Theorem”
  - “Watchman Theorem”

Fisk’s Proof

- \( g(n) = \text{floor}(n/3) \)
  - Published in 1978 (three years are Chvatal’s original proof, but it is much more compact)

- **Necessity**
  - \( g(n) \leq \text{floor}(n/3) \) are sometimes necessary

- **Sufficiency**
  - \( g(n) \leq \text{floor}(n/3) \) are always sufficient
Necessity: Base Cases

- $n=3$
- $n=5$
- $n=4$
- $n=6$

$$g(n) \geq \text{floor}(n/3)$$

Sufficiency: Fisk’s Proof

Step 1 of 3
- Triangulate the polygon $P$ by adding only internal diagonals

Step 2 of 3
- Perform a 3-coloring of the triangulation graph
  - Using three colors, no two adjacent nodes have the same color

Triangulation Theorem

- A polygon of $n$-vertices may be partitioned into $n-2$ triangles by the addition of $n-3$ internal diagonals

Four Color Theorem

- Problem stated in 1852 by Francis Guthrie and Augustus De Morgan
  - "Given a map on a flat plane, what is the minimum number of colors needed to color the different regions of the map in such a way that no two adjacent regions have the same color."
Four Color Theorem

- Several attempted proofs and algorithms
  - Kempe (1879), Tait (1880), Birkhoff (1922), …
  - Appel and Haken - first complete proof (1976)
  - Robertson, Sanders, Seymour, and Thomas - second more compact proof (1994)

- The proof creates a large number of cases (~1700 for Appel-Haken and ~600 for Robertson et al.)
- A computer is used to rigorously check the cases
- Solution is (still) controversial because of the use of a computer

Sufficiency: Fisk’s Proof

- Step 3 of 3
  - Note that one of three colors must be used no more than floor(1/3) of the time
    - Let \(a, b, c\) be \# of nodes of each color
    - \(a \leq b \leq c\) and \(n = a + b + c\)
    - If \(a > n/3\), then \((a+b+c) \geq n\)
    - Thus \(a \leq \text{floor}(n/3)\)
    - Since each triangle is a complete graph, each triangle has a node of color ‘\(a\)’
    - Since each triangle is convex and the triangles partition all of \(P\), at most ‘\(a\)’ guards are necessary!

Fisk’s Proof

- Necessity
  - \(g(n) \geq \text{floor}(n/3)\) are sometimes necessary

- Sufficiency
  - \(g(n) \leq \text{floor}(n/3)\) are always sufficient

- Thus, \(g(n) = \text{floor}(n/3)\)

- \(O(n\log n)\) overall algorithm

Reflex Vertices

- We wish to investigate the art gallery question as a function of \(r\) (the number of reflex vertices of a polygon)

  \[ r \leq (n-3) \]

  \[ \text{reflex vertex} \]

Reflex Vertices

- Necessity
  - How many reflex-vertex guards are necessary?

  \[ 1 \text{ needed} \]
  \[ r \text{ needed} \]
Reflex Vertices

- **Necessity**
  - \( r \) guards are sometimes necessary
- **Sufficiency**
  - Place 1 guard at each reflex vertex
  - Proved via a convex partitioning of the polygon \( P \)
  - Any polygon \( P \) can be partitioned into at most \( r+1 \) convex pieces

Proved via a convex partitioning of the polygon \( P \)

Any polygon \( P \) can be partitioned into at most \( r+1 \) convex pieces

Convex Partitioning

- **Naive Algorithm (Chazelle 1980)**

  - Because at most two reflex vertices can be resolved by a single cut, the minimum number of pieces is \( m=\text{ceil}(r/2)+1 \)
  - This approach achieves no more than \( r+1 \leq 2m \) in \( O(rn)=O(n^2) \) time

Convex Partitioning

- **A fast algorithm: \( O(n \log \log n) \)**
  - Any triangulation can be divided into \( 2r+1 \) convex pieces by removing diagonals

Orthogonal Polygons

- **Kahn, Klawe, Kleitman 1980**
  - \( \text{Floor}(n/4) \) guards are occasionally necessary and always sufficient
  - Based on convex quadrilateralization
    - Any orthogonal polygon \( P \) is convexly quadrilateralizable (theorem)
Orthogonal Polygons

- **Necessity**
  \[ g(n) \geq \text{floor}(n/4) \]

- **Sufficiency**
  Four-colorable, and thus:
  \[ g(n) \leq \text{floor}(n/4) \]

- **Theorem:** \( g(n) = \text{floor}(n/4) \)

Quadrilateralization

- **Sacks's Algorithm**
  \[ O(n \log n) \]

- **Lubiw's Algorithm**
  \[ O(n \log n) \]

Mobile Guards

- **Theorem**
  \[
  \begin{array}{|c|c|c|}
    \hline
    \text{Shape} & \text{Stationary} & \text{Mobile} \\
    \hline
    \text{General} & \text{floor}(n/3) & \text{floor}(n/4) \\
    \text{Orthogonal} & \text{floor}(n/4) & \text{floor}((3n+4)/16) \\
    \hline
  \end{array}
  \]

  - In general, only \( \frac{1}{3} \) as many mobile guards are needed as stationary guards.

- **Goal of the proof**
  - Given a triangulation graph \( T \)
    - Vertex guard = node
    - Edge guard = adjacent arc
    - Diagonal guard = any arc
  - The analog of covering is domination
  - A collection of guards \( C = \{g_1, \ldots, g_k\} \) dominates triangulation graph \( T \) if every face has at least one of its three nodes in some \( g_i \in C. \)
### Mobile Guards

- **Necessity**
  
  Polygon that requires floor(n/4) edge, diagonal (or line) guards

- **Sufficiency**: a little more complicated...

### Miscellaneous Shapes

- (General polygon, convex, orthogonal)
- Star, spiral, monotone

### Star Shape

- A star polygon $P$ is a polygon that may be covered by a single point guard

- **Toussaint's Theorem**
  - A star polygon $P$ requires floor(n/3) vertex guards

- **Toussaint's Theorem**
  - A star polygon $P$ requires at least floor(n/5) edge guards
Star Shape

- Toussaint's Theorem
  - For a star polygon $P$
    - Unrestricted patrol, one line guard is needed
    - Restricted to diagonal lines, two are needed

Spiral Polygon

- A spiral polygon is a polygon with at most one chain of reflex vertices

Monotone Polygon

- A polygon with no "doubling back" with respect to a line

Spiral and Monotone Polygons

- Aggarwal's Theorem
  - $\lceil n/3 \rceil$ vertex guards are needed
  - $\lceil r/2 \rceil + 1$ reflex-vertex guards are needed
  - $\lceil (n+2)/5 \rceil$ diagonals guards are needed

Exterior Visibility

- "Fortress Problem"
  - "Prison Yard Problem"

(Independently stated by Derick Wood and Joseph Malkelvitch, early 1980s)
Fortress Problem

- How many vertex guards are needed to see the exterior of a polygon $P$?

- Simplex convex polygon

- Arbitrary polygon

- Three-color the resulting triangulation graph $T$ (of $n+2$ nodes)
Fortress Problem

- **Arbitrary polygon**
  - If least frequently used color is red and \( v_\infty \) is not red then,
    - \( \lceil (n+2)/3 \rceil \) vertex guards are needed

Fortress Problem

- **Arbitrary polygon**
  - If least frequently used color is red and \( v_\infty \) is red then,
    - No guard can be placed at \( v_\infty \) because it's not part of original polygon
    - Thus, place guards at second least frequently used color
    - \( a \leq b \leq c \) and \( a + b + c = n + 2 \)
    - \( a \geq 1 \) and \( b + c \leq n + 1 \)
    - \( b \leq \lceil (n+1)/2 \rceil = \lceil n/2 \rceil \) vertex guards are needed

Fortress Problem

- **Arbitrary polygon (Summary)**
  - 1. Triangulate the convex hull of the polygon \( P \)
  - 2. Add edges from all exterior vertices to new vertex \( v_\infty \)
  - 3. Split a vertex \( x \) into \( x' \) and \( x'' \)
  - 4. Open up the convex hull, straighten the lines to \( v_\infty \), and form a triangulation graph \( T \) of \( (n+2) \) nodes
  - 5. Three-color graph \( T \)
  - 6. Use least or second least frequently used color
    - At most \( \lceil n/2 \rceil \) vertex guards are needed

Fortress Problem

- **Orthogonal polygon**
  - \( \end{equation} 

Fortress Problem

- **Orthogonal polygon**
  - \( \end{equation} 

Fortress Problem

- **Orthogonal polygon**
  - \( \end{equation} 

Fortress Problem

- **Orthogonal polygon**
  - \( \end{equation} 

**Fortress Problem**

- Orthogonal polygon
  - Solution A
  - Solution B
  ceil(n/4)+1 vertex guards necessary

**Fortress Problem**

- Orthogonal polygon

**Fortress Problem**

- Orthogonal polygon
  - Interiors of new polygon P coincide with the immediate exterior of P, except for Q which is exterior to both

**Fortress Problem**

- Orthogonal polygon
  - For P of m+4 vertices, floor(r/2)+1 or floor((m+4)/4) vertex guards suffice to cover the interior
  - None of the new vertices of P are reflex vertices
  - Need an additional one for Q
  - Thus, floor(n/4)+2 vertex guards are sufficient
  - For n mod 4=0, ceil(n/4)+1

**Guards in the plane**

- Necessity
  - ceil(n/3) point guards needed
    (n=3k+4 and k=2 guards)
Fortress Problem

- Guards in the plane
  - Sufficiency
    - New triangulated polygon $P'$ of $n+3$ vertices
      - $\text{floor}(n+3)/3 = \text{ceil}(n+1)/3$ point guards

- More lengthy proof to remove "1/3 of a guard"
  - Add only 2 guards and 3-color triangulation
  - If even hull vertices, trivial
  - If odd hull vertices, need some extra work
  - Result: $\text{ceil}(n/3)$ point guards are necessary to cover the exterior of a polygon $P$ of $n$ vertices
    - Nice duality with $\text{floor}(n/3)$ for the interior

Prison Yard Problem

- How many vertex guards are needed to simultaneously see the exterior and interior of polygon $P$?

General Polygons

- Worst-case is a convex polygon
  - $\text{ceil}(n/2)$ vertex guards needed

- Multiply-connected polygons
  - $\text{min}(\text{ceil}(n/2), \text{floor}(n/2) + \text{ceil}(n/4), 2\text{ceil}(n/3))$

Orthogonal Polygons

- $\text{floor}((7n/16)+5)$ vertex guards are needed
Fortress/Prison Yard Problem

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<th>Techniques</th>
<th>Guards</th>
<th>Time</th>
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<td><strong>General</strong></td>
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<tr>
<td></td>
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<td>$\text{floor}((n+\text{ceil}(h/2))/2)$</td>
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<td>Exterior, triangulation, 4-coloring</td>
<td>$\text{floor}(2n/3)$</td>
<td>$O(n^2)$</td>
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<tr>
<td></td>
<td>Exterior, triangulation, 3-coloring</td>
<td>$\text{floor}(2n/3+1)$</td>
<td>$O(T)$</td>
</tr>
<tr>
<td></td>
<td>Exterior, quadrantal, 4-coloring</td>
<td>$\text{floor}(7n/16)+5$</td>
<td>$O(T)$</td>
</tr>
</tbody>
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Minimal Guard Coverage

- Seek the placement of a minimal number of guards that cover a polygon $P$
  - In general, a NP-complete problem

Minimal Guard Coverage

<table>
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<tr>
<th>Polygon</th>
<th>Cover</th>
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