Surface Triangulation
and Voronoi Regions

CS334
Fall 2017

Daniel G. Aliaga
Department of Computer Science
Purdue University

[Slides with help from Michael Kazhdan @ JHU,
Ioannis Stamos @ CUNY, and
Profs. Shmuel Wimer and Andy Mirzaian]
Motivation

• Time of flight
• Structured light
• Stereo images
• Shape from shading
• Etc.

http://graphics.stanford.edu/projects/mich/
Motivation

Surface reconstruction

Geometry processing

Parameterization

Decimation

Filtering

etc.
One Option: Marching Cubes

If the function is sampled on a regular voxel grid, we can independently triangulate each voxel.
Ball-pivoting

Bernardini et al., IBM

Fixed-radius ball “rolling” over points selects subset of alpha-shape.
Pivoting in 2D

(a) Circle of radius $\rho$ pivots from point to point, connecting them with edges.

(b) When sampling density is low, some of the edges will not be created, leaving holes.

(c) When the curvature of the manifold is larger than $1/\rho$, some of the points will not be reached by the pivoting ball, and features will be missed.
Ball Pivoting Algorithm
Ball Pivoting Algorithm
Implicit Representation

Another option is representing a 3D model by an implicit function for:

- Reconstruction
- Fluid Dynamics
- 3D Texturing

Kazhdan 2005
Losasso et al. 2004
Implicit Function Fitting

Given point samples:
– Define a function with value zero at the points.
– Extract the zero isosurface.
Triangulation Complexity (in general)

• Theorem: (Gary et. al. 1978) A simple n-vertex polygon can be triangulated in $O(n \log n)$ time and $O(n)$ storage

• The problem has been studied extensively between 1978 and 1991, when in 1991 Chazelle presented an $O(n)$ time complexity algorithm.
Delaunay Triangulation

• Another very popular algorithm...
• But first, Voronoi Diagrams...
Voronoi Diagram

\[ P = \{ p_1, p_2, \ldots, p_n \} \text{ a set of } n \text{ points in the plane.} \]
Voronoi Diagram:

Voronoi(P): # regions = n, # edges ≤ 3n-6, # vertices ≤ 2n-5.
Delaunay Triangulation = Dual of the Voronoi Diagram

DT(P): # vertices = n, # edges ≤ 3n-6, # triangles ≤ 2n-5.
Delaunay triangulation

Delaunay triangles have the “empty circle” property.
Voronoi Diagram and Delaunay Triangulation
Computing Delaunay Triangulation

• Many algorithms: $O(n \log n)$

• Lets use flipping:
  – Recall: A Delaunay Triangulation is a set of triangles $T$ in which each edge of $T$ possesses at least one empty circumcircle.
  – Empty: A circumcircle is said to be empty if it contains no nodes of the set $V$
What is a flip?

A non-Delaunay edge flipped
Flip Algorithm

• ??
Flip Algorithm

1. Let \( V \) be the set of input vertices.
2. \( T = \) Any Triangulation of \( V \).
3. Repeat until all edges of \( T \) are Delaunay edges.
   a. Find a non-Delaunay edge that is flippable
   b. Flip

Naïve Complexity: \( O(n^2) \)
Locally Delaunay $\Rightarrow$ Globally Delaunay

• If $T$ is a triangulation with all its edges locally Delaunay, then $T$ is the Delaunay triangulation.

• Proof by contradiction:
  – Let all edges of $T$ be locally Delaunay but an edge of $T$ is not Delaunay, so flip it...
Flipping

• Other flipping ideas?
Randomized Incremental Flipping

• Complexity can be $O(n \log n)$
Popular Method

• Fortune’s Algorithm
Simultaneously drop pebbles on calm lake at \( n \) sites.

Watch the intersection of expanding waves.
Let Time be the 3\textsuperscript{rd} Dimension

All sites have identical opaque cones.
Let Time be the 3rd Dimension

All sites have identical opaque cones. cone(p) ∩ cone(q) = vertical hyperbola h(p,q). vertical projection of h(p,q) on the xy base plane is PB(p,q).
Voronoi Diagrams

Examples Triangulations
Voronoi Diagram

- [Link](http://www.raymondhill.net/voronoi/rhill-voronoi-demo5.html)
And Beyond...

- Not “relaxation” but more general:
  - Reaction Diffusion
    - [https://pmneila.github.io/jsexp/grayscott/](https://pmneila.github.io/jsexp/grayscott/)
    - Textures: