In the last few lectures on centrality (Katz centrality, Hubs-and-Authorities, and PageRank), we came across examples where eigenvalues of matrices showed up. In this lecture (and the next) we take up spectral analysis as a topic. We will see different types of matrices that can be associated with a graph, and see how spectra can be used to analyze the graph.

Review of Basic Linear Algebra

We begin by reviewing basic concepts we need.

Let $M \in \mathbb{C}^{n \times n}$ be a square matrix whose entries are complex numbers. A nonzero vector $x \in \mathbb{C}^n$ is an eigenvector of $M$, and $\lambda \in \mathbb{C}$ is its corresponding eigenvalue, if

$$Mx = \lambda x.$$  \hfill (1)

The set of all the eigenvalues of a matrix $M$ is the spectrum of $M$, a subset of $\mathbb{C}$ denoted by $\Lambda(M)$.

Characteristic polynomial

The characteristic polynomial of a matrix $M \in \mathbb{C}^{n \times n}$, denoted by $p_M$, is the degree $n$ polynomial defined by

$$p_M(z) = \det(zI - M).$$  \hfill (2)

**Theorem 1** $\lambda$ is an eigenvalue of $M$ if and only if $p_M(\lambda) = 0$.

Theorem 1 implies that even if a matrix is real, some of its eigenvalues may be complex.

Algebraic multiplicity

Using the fundamental theorem of algebra, we can write the characteristic polynomial $p_M$ in the form

$$p_M(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$$

for some numbers $\lambda_j \in \mathbb{C}$. By Theorem 1, each $\lambda_j$ is an eigenvalue of $M$, and all eigenvalues of $M$ appear somewhere in the list. In general, an eigenvalue could appear more than once. The algebraic multiplicity of an eigenvalue $\lambda$ of $M$ is its multiplicity as a root of $p_M$.

The characteristic polynomial gives an easy way to count the number of eigenvalues of a matrix:

**Theorem 2** If $M \in \mathbb{C}^{n \times n}$, then $M$ has $n$ eigenvalues, counted with algebraic multiplicity.
Eigenvalue Decomposition

An eigenvalue decomposition of a square matrix $M$, when it exists, is a factorization

\[ M = X\Lambda X^{-1}. \] (3)

This can equivalently be written as

\[ MX = X\Lambda, \]

which makes it clear that if $x_j$ is the $j$th column of $X$ and $\lambda_j$ is the $j$th diagonal entry of $\Lambda$, then $Mx_j = \lambda_jx_j$. Thus the $j$th column of $X$ is an eigenvector of $M$ and the $j$th entry of $\Lambda$ is the corresponding eigenvalue.

Geometric multiplicity

The maximum number of linearly independent eigenvectors that can be found, each with the same eigenvalue $\lambda$, is called the geometric multiplicity of $\lambda$.

The geometric multiplicity and the algebraic multiplicity of an eigenvalue are related as follows: The algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity. An eigenvalue whose algebraic multiplicity exceeds its geometric multiplicity is a defective eigenvalue. A matrix that has one or more defective eigenvalues is a defective matrix.

Diagonalizability

Any diagonal matrix is nondefective. For a diagonal matrix, both the algebraic and the geometric multiplicities of an eigenvalue $\lambda$ are equal to the number of its occurrences along the diagonal. The class of nondefective matrices is precisely the class of matrices that have an eigenvalue decomposition (another term for nondefective is diagonalizable):

**Theorem 3** An $n \times n$ matrix $M$ is nondefective if and only if it has an eigenvalue decomposition $M = X\Lambda X^{-1}$.

Similarity

Two matrices $M$ and $M'$ are said to be similar if there exists a nonsingular matrix $X \in \mathbb{C}^{n \times n}$ such that $M' = X^{-1}MX$. Similar matrices share many properties.

**Theorem 4** Two similar matrices have the same characteristic polynomial, eigenvalues, and algebraic and geometric multiplicities.

Determinant and Trace

The trace of $M \in \mathbb{C}^{n \times n}$ is the sum of its diagonal elements: $tr(M) = \sum_{j=1}^{n} m_{jj}$. Both the trace and the determinant are related simply to the eigenvalues.

**Theorem 5** The determinant $\det(M)$ and trace $tr(M)$ are equal to the product and the sum of the eigenvalues of $M$, respectively, counted with algebraic multiplicity:

\[ \det(M) = \prod_{j=1}^{n} \lambda_j, \quad tr(M) = \sum_{j=1}^{n} \lambda_j. \] (4)
Unitary diagonalization

In some cases, an \( n \times n \) matrix \( M \) may possess both of the following two properties: (i) it has \( n \) linearly independent eigenvectors, and (ii) the eigenvectors can be chosen to be orthogonal. In such a case, \( M \) is said to be unitarily diagonalizable, that is, there exists a unitary matrix \( Q \) such that

\[
M = Q\Lambda Q^*.
\]  

This factorization is both an eigenvalue decomposition and a singular value decomposition.

Hermitian matrices (a matrix \( M \) is Hermitian if \( M = M^* \)) are unitarily diagonalizable. Moreover, the eigenvalues of a hermitian matrix are real. But, hermitians are not the only ones that are unitarily diagonalizable. In fact, the class of unitarily diagonalizable matrices have the following nice characterization.

**Theorem 6** A matrix \( M \) is unitarily diagonalizable if and only if it is normal, that is \( M^*M = MM^* \).

Symmetric matrices

In these lectures, we will mostly be concerned with symmetric matrices. We can restate the implication of the above results for a real, symmetric matrix \( M \in \mathbb{R}^{n \times n} \) in the following manner (this is also known as the Spectral Theorem):

**Theorem 7** If \( M \) is an \( n \times n \) symmetric matrix, then \( M \) has real eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) and \( n \) orthonormal eigenvectors forming a basis of \( \mathbb{R}^n \).

Eigenvalues and eigenvectors of symmetric matrices have many characterizations. Some are connected to optimizing the Rayleigh quotient.

**Definition 1** The Rayleigh quotient of a vector \( y \) with respect to a matrix \( M \) is the ratio

\[
\frac{y^TM y}{y^Ty}.
\]

Observe that if \( x \) is an eigenvector of \( M \) of eigenvalue \( \lambda \), then

\[
\frac{x^T M x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \frac{\lambda x^T x}{x^T x} = \lambda.
\]

The following is an important result in spectral theory.

**Theorem 8** Let \( M \) be a symmetric matrix and let \( y \) be the non-zero vector that maximizes the Rayleigh quotient with respect to \( M \). Then, \( y \) is an eigenvector of \( M \) with eigenvalue equal to the Rayleigh quotient. Moreover, this eigenvalue is the largest eigenvalue of \( M \).
Matrices for Graphs

We will look at three matrices associated with graphs: the adjacency matrix, the Laplacian, and the normalized Laplacian. Let $G = (V, E)$ be the graph, and let $V = \{1, 2, \ldots, n\}$.

Spectrum of the adjacency matrix

The adjacency matrix $A_G$ is defined such that entry $A_G(u, v) = 1$ if $(u, v) \in E$ and 0 otherwise.

Let $w \in \mathbb{C}^n$ be an arbitrary vector and let $\omega : V \to \mathbb{C}$ map each $i \in V$ on $w_i$. Then, the $i$th component of $A_G w$, $\sum_{j=1}^n a_{ij} w_j$, can be written as $\sum_{j \in N(i)} \omega(j)$. Now the equation $A_G x = \lambda x$ has the following useful interpretation.

Remark 1 Two points:

- $A_G$ has eigenvalue $\lambda$ if and only if there exists a nonzero weight function $\omega : V \to \mathbb{C}$ such that for all $i \in V$, $\omega(i) = \sum_{j \in N(i)} \omega(j)$.
- The Spectral Theorem ensures that we can restrict ourselves to considering real-valued weight functions. Moreover, we can assume the maximum weight to be non-negative (if $\max\{\omega(i); i \in V\} < 0$ then $\omega(i) < 0$ for all $i \in V$ and we can consider $-\omega$ instead of $\omega$).

Remark 1 enables us prove the following claims:

Lemma 1 Let $G = (V, E)$ be a graph on $n$ vertices with adjacency matrix $A_G$ and eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Let $\Delta$ be the maximum vertex degree in $G$.

1. $\lambda_n \leq \Delta$.
2. If $G = G_1 \cup G_2$ is the union of two disjoint graphs $G_1$ and $G_2$ then $\text{spectral}(G) = \text{spectral}(G_1) \cup \text{spectral}(G_2)$.
3. If $G$ is bipartite then $\lambda \in \text{spectral}(G) \iff -\lambda \in \text{spectral}(G)$.
4. If $G$ is a simple cycle then $\text{spectral}(G) = \{2 \cos(\frac{2\pi k}{n}); k \in \{1, \ldots, n\}\}$.

Let us now see the spectra of the complete bipartite graph $K_{n_1, n_2}$ and the complete graph $K_n$.

Lemma 2 Let $n_1$, $n_2$ and $n$ be positive integers.

1. For $G = K_{n_1, n_2}$, $\lambda_1 = -\sqrt{n_1 n_2}, \lambda_2 = \ldots = \lambda_{n-1} = 0$, and $\lambda_n = \sqrt{n_1 n_2}$.
2. For $G = K_n$, $\lambda_1 = \ldots = \lambda_{n-1} = -1, \lambda_n = n - 1$.

The adjacency matrix is also useful for counting paths of length $k$ in a graph.

Lemma 3 Let $G$ be a multi-digraph possibly with loops. The $(i, j)$th entry of $A^k$ counts the $i \to j$-paths of length $k$. The eigenvalues of $A^k$ are $\lambda_i^k$.

Corollary 1 The following hold true:

- $\sum_{i=1}^n \lambda_i = \text{number of loops in } G$.
- $\sum_{i=1}^n \lambda_i^2 = 2 \cdot |E|$.
- $\sum_{i=1}^n \lambda_i^3 = 6 \times \text{the number of triangles in } G$.

Using these we can prove the third claim in Lemma 1.

Lemma 4 $G$ is bipartite if and only if the eigenvalues of $A_G$ occur in pairs $\lambda, \lambda'$ such that $\lambda = -\lambda'$. 


Things the spectrum of the adjacency matrix does not help with  Consider two graphs: a star graph on 5 vertices, and a five-vertices graph made up of a four-cycle and an isolated vertex. The two graphs have eigenvalues $\lambda_1 = -2, \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_5 = 2$, but they are not isomorphic. Such graphs are called cospectral. Also, we cannot determine from the spectrum of $A_G$ whether or not $G$ is connected. Fortunately, this can be addressed using the spectrum of the Laplacian, which we will see in the next lecture.