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A framework for object modeling

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Abstract

Modeling of products (objects) form a critical task in design and manufacturing. CAD–CAM techniques based on solid/geometric modeling have been developed for this purpose. Primarily, these methods capture the shape of the object. However, recent developments in diverse fields demand modeling schemes, which extend beyond the shape to include other relevant attributes of the object. In this paper, this issue is addressed and a new modeling framework is proposed. This framework enables the modeling of geometry and several attributes simultaneously in an integrated fashion.

Keywords: Solid modeling; CAD–CAM techniques; Geometry and attribute representation

1. Introduction

Solid modeling has been an application driven field with new developments stemming from the needs dictated by evolving applications [1–3]. Solid modeling has focused mainly on modeling objects to capture their shape or geometry (inclusive of related topological aspects) [4,5]. Solid modeling has been extensively used in mechanical design, analysis and manufacturing where the geometric information present in the solid models is utilized and processed for various purposes. That is, the geometric model of the net shape is used to compute material characteristics and performance parameters, possibly after converting to a mesh representation. This computation assumes homogeneous material distribution throughout the interior of the solid. Other uses of the geometry include annotations with attributes of selected areas of the net shape surface, for example with surface finish, etc. Thus, in traditional CAD models of products, surface characteristics may vary over a part's surface, but the volumetric characteristics are considered constant throughout the interior.

With the advent of new manufacturing technologies and applications, there is a need to represent parts with inhomogeneous interior material distributions. Material distribution becomes one of a number of attributes, associated with the volumetric domain defined by the net shape [1]. One such recent advent is the design and fabrication of heterogeneous objects, for example with solid free form fabrication machinery.

Heterogeneous objects are objects composed of different constituent materials and could exhibit continuously varying composition and/or microstructure thus producing gradation in their properties [6–11]. There exist structural and material design methods which are capable of deriving optimal designs (geometry, topology and material) for mechanical/structural components that are made of heterogeneous materials [12–16]. The host of manufacturing processes that can fabricate such 3D heterogeneous objects is termed Solid Freeform Fabrication (SFF) or Layered Manufacturing (LM). This fabrication technique involves deposition of material to create an object unlike conventional methods where material is removed to obtain the final object [17–20]. The deposition of material can be explicitly controlled thereby providing unique opportunities to selectively deposit material. In other words, the material deposited can be varied continuously to yield a heterogeneous object with varying material distribution [21–25,44].

For the purpose of design, analysis and manufacture of heterogeneous objects, a CAD model of the object is required which has not only the geometry information but also the information on material, property, etc. at each point of the object [8,26,27]. Below, we present an example to illustrate the need imposed by heterogeneous objects.

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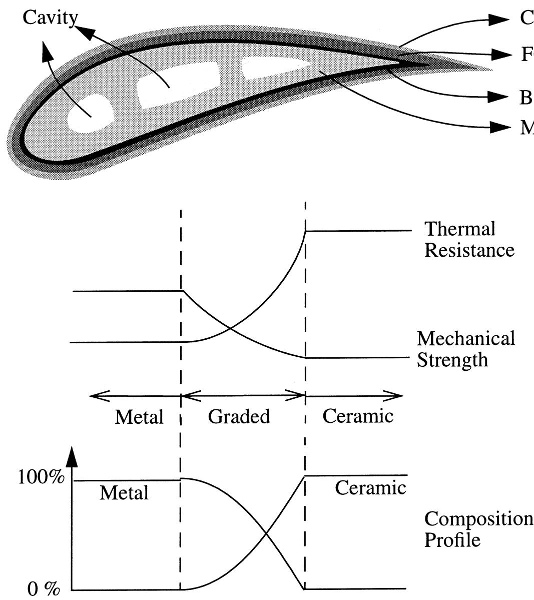


Fig. 1. Use of graded materials in turbine blades.

1.1. Motivating example from aerospace design

Many structural elements used in aerospace applications (such as turbine blades, vanes, outer plane body, etc.) are subject to severe thermomechanical loading giving rise to intense thermal stresses. Ideally the materials used must possess the following properties—heat resistance and anti-oxidation properties on the high temperature side, mechanical toughness and strength on the low temperature side, and effective thermal stress relaxation throughout the material [6]. Initial designs of these elements used metals on the low temperature side and ceramics on the high temperature side. However, the property difference between the two materials generated high stress concentrations at the interfaces resulting in cracks, plastic deformations and interfacial decohesion. Recently, this problem was solved by using a mixture of metal and ceramic with varying proportions (heterogeneous or graded materials). An example of a turbine blade design using such a mixture is shown in Fig. 1. The sharp interface between the metal and ceramic is eliminated by using a graded zone of metal/ceramic, denoted as FGM (functionally graded material) in Fig. 1. In such a structure, the properties can be adjusted by controlling the composition, microstructure and porosity ratios from metal to ceramic. The graphs in the figure show typical variation in properties due to the variation in material composition at the FGM region.

A complete representation of the above turbine blade (and similar such designs) must include not only the geometry of various regions (metal, ceramic, bond coat and graded zones) but also the material information for each region. Specifically, the material variation for the graded zones must be captured which is essential for the fabrication of these designs. Also, it is necessary to represent the variation in properties (strength, conductivity, etc.) for analysis purposes.

1.2. Extending beyond geometry representation

Recent developments in diverse fields such as mechanical design, aerospace, etc. (as the above example illustrates) generate objects that have attributes which vary throughout the interior of the geometric shape—attributes such as material density, varying proportions of mixed materials, etc. Current CAD and solid models cannot represent such attribute information. Hence, new representation schemata should be explored in which the volume of a shape becomes the domain of attribute functions, and for which, at every interior point of a solid, we can determine in principle the value of each attribute under consideration. The availability of such modeling techniques is bound to assume an increasingly prominent role in the design, analysis and fabrication of technologically advanced artifacts.

In our earlier work [8,28], the rigorous inclusion of material composition along with geometry was considered. The term “heterogeneous solid model” was used to refer to models possessing both geometry and material composition information. Heterogeneous models are sufficient for the purpose of design and manufacture of heterogeneous objects. However, to explicitly capture property variation, the heterogeneous solid models must be generalized to store several attributes simultaneously, material variation being one such attribute. Such a generic model is termed as an “object model”.

Table 1
Evolution of modeling strategies for objects

Year	Motivating applications	Object attributes	Computer representation
1950s	Computer graphics and NC machining	Geometry	Electronic drafting and wireframe
1960s			Polygonal and surface models
1970s and 1980s	CAD/CAM	Geometry and topology	Solid Models
1990s	Heterogeneous objects	Geometry, topology and material	Heterogeneous solid models
Future	Heterogeneous objects and physical modeling	Geometry, topology, material, physical, attributes etc.	Object models

The need for an object model also arises from another application—physical modeling. Physical modeling aims to establish a formal link between several activities related to engineering design and analysis. The goal is to provide uniform and rigorous computational models, which explicitly link form to function. One of the key ingredients to achieve this goal is an object model which captures all information related to an object in a mathematically rigorous fashion. Earlier work in this regard is presented in Ref. [29,30].

In this paper, the aim is to develop the object models. Table 1 shows the historical perspective and evolution of various models of object over time.

1.3. Prior models in the literature

Solid modeling focuses on creating a valid representation of the geometry of an object. Several mathematical models have been proposed in the solid modeling literature for this purpose. These models are bounded manifold solids, *r*-sets, *s*-sets, non-manifold solids (non-homogeneous pointsets), CNRG and SGC. A comprehensive survey is presented in Ref. [31].

Manifold solid [2,4]: A manifold solid is a finite collection of disjoint compact connected 3-manifolds embedded in $\mathbf{E}^3$ such that the boundary of each 3-manifold is a compact, oriented 2-manifold without boundary, embedded in $\mathbf{E}^3$ with bounded variation. Also, for every 3-manifold, the connected components of its boundary are pair-wise consistently oriented (i.e. pointing to the interior of the 3-manifold consistently). Regularized operations can be used to manipulate manifold solids. However, they are not closed under these operations.

R-set [5]: An *r*-set is defined as a compact, regular, semi-analytic subset of $\mathbf{E}^3$. An *r*-set can be disconnected and hence, an object with many components (components are connected sets) can be modeled with a single *r*-set. Regularized boolean operations are used which regularize the resulting pointset to obtain valid solids. An *r*-set can also be obtained from manifold solids by relaxing the condition of 3-manifolds being disjoint and allowing their boundaries to intersect with dimension 0 or 1 [2]. It can be seen that manifold solids form a subset of *r*-sets.

S-set [32]: An *s*-set is defined as a bounded open subset of $\mathbf{E}^3$ whose boundary is a patchwise smooth surface (smoothness being defined by homeomorphisms, a weaker condition than analyticity used for *r*-sets). The modeling scheme based on *s*-sets was called Realizable Shape Calculus and operations were defined to manipulate these *s*-sets. *S*-sets are more suitable for modeling assemblies when compared to *r*-sets and can potentially model more variety of objects than *r*-sets.

Selective geometric complex (SGC) [33]: SGC is a non-regularized non-homogeneous pointset represented through enumeration as union of mutually disjoint connected open cells. The cells are dimensionally homogeneous and singularity-free manifolds of algebraic varieties. SGC includes pointsets with internal geometric structures and incomplete boundaries to extend traditional coverage of solid modeling. Appropriate operations (sub-division, selection and simplification) are used to manipulate the cells.

Non-manifold solid [34]: A non-manifold solid is an extension of manifold solids to allow non-manifold entities that includes dimensionally non-homogeneous pointsets. The non-manifold boundaries defined are quasi-disjoint enumerations of compact pointsets of dimension one and two. The radial data structure was developed to allow the representation of these solids.

Constructive non-regularized geometry (CNRG) [35]: A collection of non-regularized regions that need not be connected, bounded, nor dimensionally homogeneous are constructed through set-theoretic and topological operators from primitive regions. A generative approach is followed similar to CSG (Constructive Solid Geometry), but can represent assemblies and solids with internal structures.

Manifold solids and *r*-sets do not capture internal (geometrical) structures such as cracks or separations while the other models do. *S*-sets can capture these structures, however, these information are lost when boolean or set-theoretic operations are performed on them. Non-manifold solids, SGC and CNRG can capture these internal structures and operations defined on them can be used to create and manipulate them. Manifolds solids, *r*-sets and *s*-sets enforce dimensional homogeneity whereas the more general models of non-manifold solids, SGC and CNRG allow lower dimensional entities as well as non-manifold neighborhoods. This enables them to capture internal structures as well as assemblies.

It must be noted that all these models, in varying degree of generality, represent only geometric structures, i.e. geometric entities of various topological nature and complexity are modeled. Thus, any sort of purely geometric information corresponding to physical situations can be represented by these models. The examples include cracks, internal boundaries, separations, etc. that exist in a solid. However, the domain of representation still remains geometric. Our emphasis in this paper is to go beyond geometry to include other information which are not geometric but can be represented with geometry as the basis. Such information include attributes like material, microstructure, properties, etc. These models are not capable of modeling these attributes of the object, as discussed in the previous example. However, these models do provide the basis for such a generalization into representation of multiple attributes of solids. Previous work in this direction are described below:

Heterogeneous solid model [8,28]: In our earlier work, heterogeneous solid modeling system was developed to incorporate material composition along with the geometry. A heterogeneous solid model (r_m-object) is developed as a finite collection of material domains with each domain being discrete or mixed/graded. A discrete material domain

Table 2
Classification of various attributes of an object

Attribute	Specifications	
Geometry	Geometry (and related topology)	
	Composition	
Material	Microstructure	Inclusions: Shape parameters
		Voids
		Orientation/Spin
		Phase transition
		Crystal structure
		Dislocations
		Density
	Property	Moduli (elastic, bulk, shear, etc.)
		Thermal expansion coefficient
		Thermal conductivity
Physical parameters		Temperature
		Velocity
		Stress

is made of only one material, which could also be an embedded component. In a mixed material domain, each point in the object is made up of many materials and could possibly contain voids. Each material domain (r_m-set) is modeled as an r-set and a material distribution function is attached to it that specifies how the material composition varies in the particular r-set. Mathematically stating:

$$S = \{(P_j, F_j)\}, \qquad j = 1\ldots k \ \text{(finite)} \tag{1}$$

where

$$F_j = \{\underline{v}^j(\underline{x} \in P_j) \equiv (v_1^j(\underline{x}), \ldots, v_n^j(\underline{x}))\}$$

Here, S denotes the heterogeneous model, P_j denotes an r-set which models the geometry of the j-th material domain and F_j denotes the material distribution function in that domain. F_j contains component functions $v_i^j(\underline{x})$ which denote the variation of volume fraction of a particular material (tagged with index i) in that material domain P_j. Hence, the term "material domain" refers to a region of the object whose material could be discrete (constant), mixed (graded or varying) or an embedded component. This model contains both external and internal surfaces, external surfaces bounding the entire object and internal surfaces separating different material domains.

Chain model [29,36]: Chain model was proposed for physical modeling and is composed of cell complexes and algebraic chains defined over them. An object is defined as a set of physical quantities that is distributed over space and time. The cell complex models the geometry of the object as collection of cells (spatial decomposition) which are finite and oriented. The chains are used to associate physical quantities to each cell in the complex through functions. Chains are essentially mappings from the cells of the complex to a vector space, elements of which are used to represent physical quantities.

Hermite hyperpatch [37]: The use of hyperpatches based on multivariate Hermite interpolation for shape and physical properties is explored. In this approach, the shape and the physical data is modeled simultaneously with the same parametric variables.

FR-set [30]: FR-sets were proposed to provide a mechanism to attach physical attributes to r-sets. The FR-sets aim to provide the required physical information during analysis. FR-sets are based on the concept of fiber bundles. The shape of the object is defined by r-sets and physical attributes are attached as fibers. Refer texts on differential geometry for details on fiber bundles [38,39]. Our work follows a similar approach in exploiting the fiber bundle framework.

All these models aim to include additional information along with the geometry. Heterogeneous solid model provides the facility to have continuous material functions attached to geometry. Chain models use a combinatorial approach and provide a more general capability than the heterogeneous solid models by allowing attachment of multiple attributes. The use of hermite hyperpatches restricts the type of geometry that can be represented. Also, simultaneous representation of geometry and attributes using the same parametric variables is not always feasible. FR sets use a differential geometric approach to model several attributes along with geometry. It must be mentioned that SGC also has the provision of attaching discrete values to each cell, but is not fully explored in the context of physical modeling.

The aim of this work is to model geometry along with the continuous variations of physical quantities. The object model proposed shares the same aim as chain models and FR sets (for object representation) and uses the well-established concepts of differential geometry.

1.4. Outline

In this paper, a new modeling strategy (object model) is based on the concepts of product manifolds and trivial fiber bundles. This model is generic enough to represent varying attributes of an object in a rigorous and integrated way. As mentioned earlier, the use of fiber bundles was also explored in Ref. [30].

2. Attributes of an object

As mentioned, a complete description of an object must include all the characteristics and attributes of an object. Below, a list of attributes is presented (Table 2):

- *Geometry (Shape):* Capturing shape information of an object is addressed in solid/geometric modeling and involves modeling and representing the geometry along with the required topological information of an object.
- *Material:* The most general type of objects are heterogeneous objects which possess varying material composition and microstructure.
- *Properties:* Though the material properties of an object can be estimated from the material composition

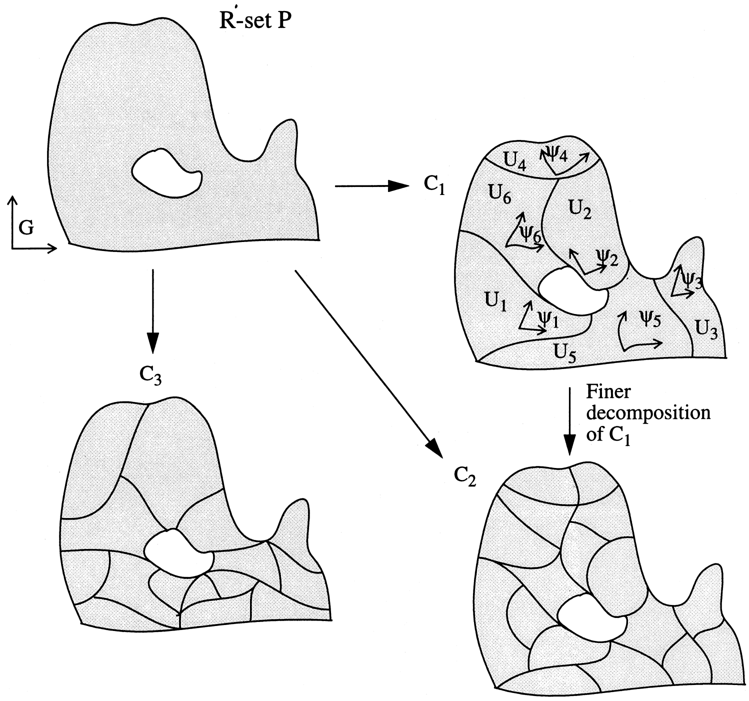


Fig. 2. Geometry model—*r*-set and its decompositions (atlases).

and microstructure, it might be useful to explicitly evaluate and store them. This is required for performing any analysis of the object such as in FEM, etc.

- *Physical parameters:* These parameters associated with an object are specified based on the physical process which the object undergoes. Examples of such parameters are temperature, velocity, stress, strain, etc.

2.1. Geometry as a base attribute

Geometry of an object is the most fundamental attribute used in any description of object. Each point in the object is represented by a unique geometrical point in the euclidean space $\mathbf{E}^3$ (one-to-one mapping). All other attributes are typically described as a function of geometry, expressed in terms of spatial variables. Also, any physical process or phenomena modeled through partial differential equations uses geometry (spatial variables) for their description. Hence, "geometry" of the object is termed as the "base attribute" and this distinction proves to be useful in defining the mathematical model of an object.

3. Mathematical model for an object

In this section, a new mathematical model is introduced to model multiple attributes of an object. The inclusion of other attributes can be achieved by constructing a separate model for each attribute. An object model is defined as a composite of its attribute models:

$$\mathscr{M} = \mathscr{M}_{\mathrm{G}} \otimes \mathscr{M}_{A_1} \otimes \mathscr{M}_{A_2} \otimes \ldots \otimes \mathscr{M}_{A_n} \tag{2}$$

Here, $\mathscr{M}$ denotes a model with the subscripts denoting the attributes. G represents the geometry (inclusive of topology) and $A_1, \ldots A_n$ represent the other attributes. The symbol $\otimes$ indicates that the object model is a composite (collection) of individual attribute models, the precise form of this operator is discussed later. Geometry being the base attribute is used as a model parameter for all other attribute models. Hence, the object model can be stated as:

$$\mathscr{M} = \mathscr{M}_{\mathrm{G}} \otimes \mathscr{M}_{A_1}(\mathscr{M}_{\mathrm{G}}) \otimes \mathscr{M}_{A_2}(\mathscr{M}_{\mathrm{G}}) \otimes \ldots \otimes \mathscr{M}_{A_n}(\mathscr{M}_{\mathrm{G}}) \tag{3}$$

The above equation gives a general form of an object model. It is obvious that the geometry model $\mathscr{M}_{\mathrm{G}}$ has to be generated first. First, we describe the mathematical model used to describe the geometry.

3.1. Geometry model

The geometry model of an object is defined as $\mathscr{M}_G = (P, \{C_i\})$ where:

- $P \subset \mathbf{E}^3$ is an *r*-set.
- $\{C_i\}$ is a finite set of disjoint decompositions of P i.e. each C_i partitions P into a finite set of closed 3-cells, which are mutually interior disjoint. Thus, each C_i can be considered as a geometric cell complex (as in Ref. [38]) where each n-cell ($n \leq 3$) in C_i is a compact

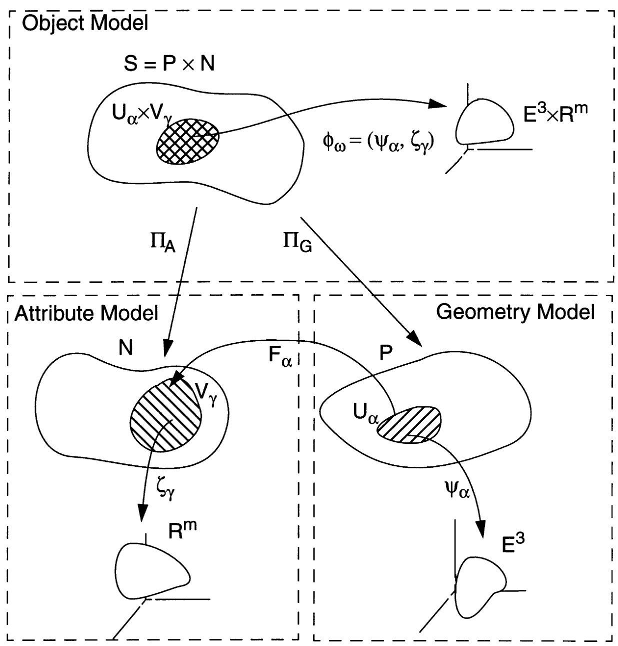


Fig. 3. Object model as a topological product, a trivial fiber bundle.

connected n-manifold. The boundary of each n-cell is a finite union of $(n-1)$ cells. Each 3-cell U_α in C_i possesses a local coordinate system, which is related to the global coordinate system through the C^∞ compatible coordinate map ψ_α. All the coordinate systems are assumed to be C^∞ compatible (non-vanishing Jacobian) and consistently oriented. For each local coordinate system ψ_α in each C_i, the appropriate metric tensor $[g]$ is generated. Each 3-cell and its coordinate map (U_α,ψ_α) is called a chart and the collection of these charts (i.e. the entire decomposition) is called an atlas.

An example is shown in Fig. 2. The overall geometry is defined by the r-set P. This model has three decompositions, C_1, C_2, and C_3, with C_2, being a finer decomposition of C_1. The local charts $((U_1,\psi_1)$ to $(U_6,\psi_6))$ are illustrated for C_1 in the figure. Similar charts (not shown) exist for the atlases C_2 and C_3. Several atlases (decompositions) $\{C_i\}$ are defined in the geometry model to help in modeling the attributes. However, these decompositions are specified only when required and is not essential for the initial definition of the geometry (r-set) P. The geometry model could possess only the r-set P with no decompositions. Depending on the attributes to be modeled, new decomposition(s) can be generated and added to the geometry model. Each attribute is defined on one of the decompositions as described later. The identification of appropriate decompositions of P for the purpose of defining and evaluating attributes efficiently is a continuing issue for research.

For each chart in a decomposition the local coordinate system can be defined in three possible ways. First, the local system is same as the global coordinate system. Second, the local system is intrinsically defined from the geometry of the region (like using spherical coordinates for the local spherical region). Thus, any CSG primitive can be given a local chart when the geometry is created. Also, any primitive generated as a parametric solid has a chart defined in the parametric space. Third, the local coordinate system is imposed on the geometry derived from physical constraints depending on the applications, and is not intrinsic to the geometry.

As all charts are required to be C^∞-related to the global system, all the atlases are C^∞-compatible with each other. In fact, all the atlases $\{C_i\}$ can be combined to a single atlas and would belong to the complete atlas for the r-set. However, the explicit separation facilitates the definition of models for different attributes. Each attribute has to be defined only on a finite set of charts which cover the r-set P. These charts are explicitly put together as a separate atlas. The condition that the 3-cells in an atlas must be mutually interior disjoint is imposed intentionally and can be relaxed if required.

3.2. Attribute model

The generic model for the attribute A is a manifold N, which could be a vector or a tensor space. Additional properties on the model have to be imposed depending on the attribute that is being modeled.

Each point in the object, modeled geometrically as a point $\underline{x}$ in the r-set P, is mapped to its corresponding attribute value in N through the attribute function $F = \{F_\alpha\}$. This attribute function F is defined on a particular atlas C_j in the geometry model:

$$F_\alpha : (U_\alpha \in C_j) \rightarrow (V_\gamma \subseteq N) \tag{4}$$

One particular atlas C_j in the geometry model is associated with the attribute A in order to define the correspondence between the geometry model and the attribute model. In general, F is a collection of functions F_α which are required to be C^k, $k > 0$. The exact continuity class as well as other constraints on these functions can be dictated by the attribute that is being modeled.

A few points that must be noted are:

- The function F need to be surjective, i.e. $F(P) = \cup\{F_\alpha(U_\alpha)\} = \cup\{V_\gamma\} \subseteq N$.
- Each attribute model is associated with only one atlas in the geometry model. This atlas is used to define the attribute corresponding to the geometry.
- More than one attribute can use the same atlas in the geometry model.
- The atlases defined in the geometry model must be suitable to define the attribute functions $F(P)$. If no such atlas exist, a new atlas has to be created.

Thus, the attribute model is defined as $\mathscr{M}_A = (N, F)$ where $F = \{F_\alpha\}$ relates the geometry P to the attribute manifold N through the atlas C_j. The image of F in N (i.e. $\cup\{V_\gamma\}$) is implicitly defined.

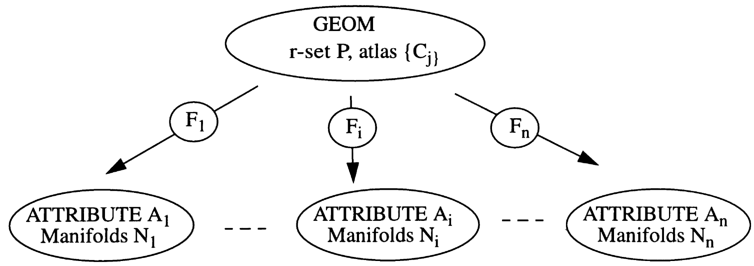


Fig. 4. Schematic of the object model.

3.3. Object model

An object with a single attribute is modeled as a product set $S = P \times N$ where P is the r-set describing the geometry and N is the manifold describing that particular attribute. The operator $\otimes$ (Eq. 3) is precisely defined as a topological product. The function $F = \{F_\alpha\}$ is specified which maps the geometry P to the attribute N. Refer Fig. 3.

The above model can be treated as a trivial fiber bundle where S is the total space (product set), P is the base space (r-set) and N is the fiber space (attribute manifold). Refer Appendix A for a detailed mathematical definition of fiber bundles. If the fiber space N is a vector (tensor) space, the bundle becomes a vector (tensor) bundle. The bundle is trivial if it can be expressed as a global product. Then, both the projections Π_{G} and Π_A are defined everywhere. However, for true fiber bundles (non-trivial), only Π_{G} is defined everywhere, and Π_A is defined only locally with respect to geometric base variables. The charts on the base space P and the attribute functions define the charts on the total space S. The use of fiber bundle concept with r-sets, termed FR-sets, was also proposed in Ref. [30].

For an object having n attributes, the product set S (equivalent to (Eq. 3)) would be:

$$S = P \times (\prod_{i=1}^{n} N_i) \tag{5}$$

where P is the r-set model describing geometry and each N_i is a manifold describing the attribute A_i. It is possible that the attribute model N_i for each attribute A_i could be a finite collection of disjoint manifolds. The mathematical space in which the object model S is represented can be given as:

$$\mathbf{T} = \mathbf{E}^3 \times (\prod_{i=1}^{n} \mathbf{R}^{m_i}) \tag{6}$$

The set of mappings $F_i = \{F_\alpha\}$ is defined for each attribute A_i which maps the r-set P to the attribute manifold N_i using one particular atlas C_j:

$$F_i : P \longrightarrow N_i, \text{ using some atlas } C_j \tag{7}$$

The projection functions for S are defined as:

$$\Pi_{\mathrm{G}} : S \longrightarrow P \tag{8}$$

$$\Pi_i : S \longrightarrow N_i$$

The geometry projection Π_{G} is a continuous surjective while the other projections Π_i need not be. Thus, in the object model, the geometry model has a special significance as the base model for other attribute models. This can mathematically specified by letting the product S (Eq. 5) to be a trivial fiber bundle where the geometry P is the base and all other attribute manifolds are attached to the base as fiber spaces.

To summarize, the object model (Refer Fig. 4) is defined as a trivial fiber bundle $\mathscr{M} = \{S, P, \{C_j\}, \{(N_i, F_i)\}\}$ where:

- Object S is a product space (called the total space) as defined in (Eq. (5)).
- Geometry is defined by r-set P and its atlases $\{C_j\}$ (base model).
- Each N_i describes the attribute A_i (fiber manifolds attached to base P). The attributes are rigorously attached to geometry (section of the bundle) using the functions F_i using one particular atlas C_j defined on P.
- The projections are defined in (Eq. (8)).

Note that the explicit modeling of the attribute manifolds N_i is not necessary if the attribute functions F_i are defined properly and yield valid attribute values.

3.4. Object model as a generalization of heterogeneous solid model

Heterogeneous solid models (r_{m}-objects) were proposed in Ref. [8] for representing material variation. These models can be viewed as a special case of an object model (fiber bundle model) with single attribute. In r_{m}-objects, the geometry model consists of a collection of r-sets which is equivalent to one particular atlas in the object model. Hence, the space of volume fractions V (defined in Table 4) which represent the material composition is the attribute manifold N and the material functions correspond to the attribute functions which map the geometry to the attribute manifold V.

4. Example

In Section 1.1, it was mentioned that the variation in

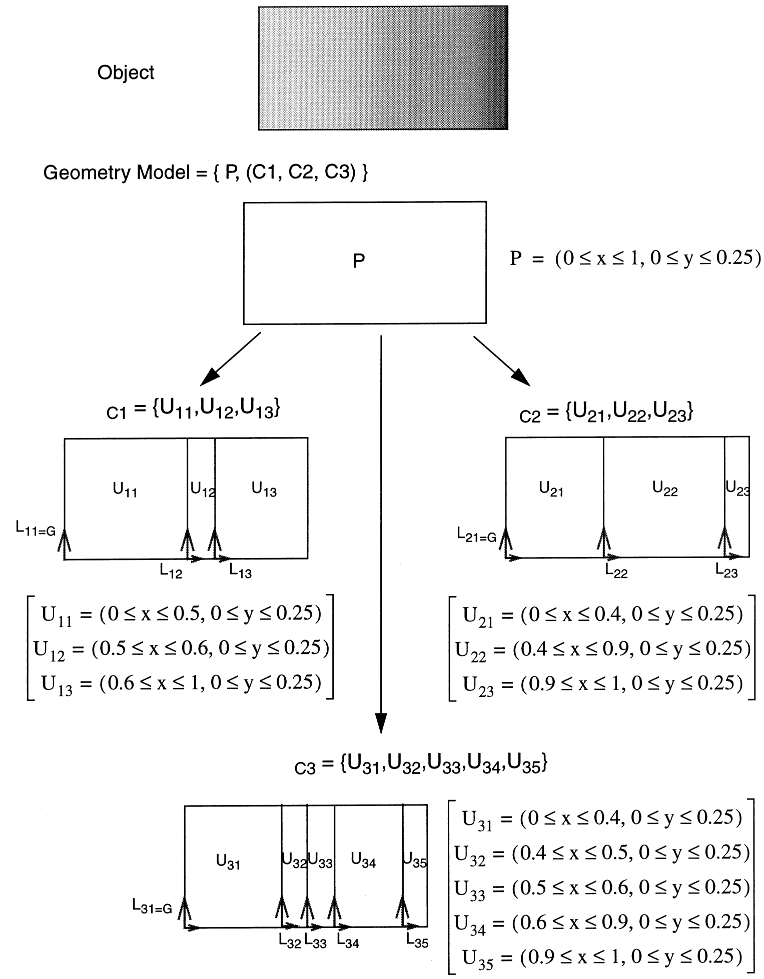


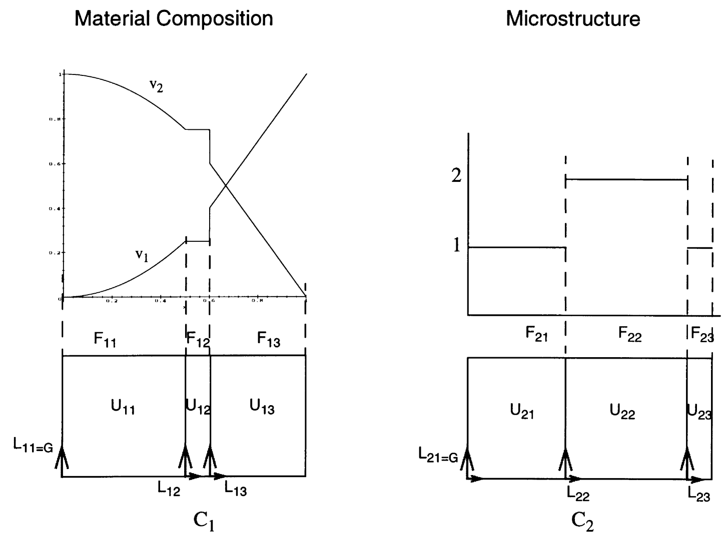
Fig. 5. Object and its geometry model.

material composition of heterogeneous objects results in variation of material properties, illustrated with an example of turbine blade design. Here, a sample heterogeneous object with a rectangular geometry is used to explain the concept of object model for modeling property variation. Refer Fig. 5. The geometry of the object is modeled by an *r*-set *P*. Three decompositions are used for this model. As mentioned earlier, identification of appropriate decompositions of an irregular/complex geometry for defining and evaluating attribute functions is a topic of ongoing research. In the example considered in Fig. 5, the decomposition C_1 is used for describing the material composition. The decomposition C_2 describes the microstructural regions in the object. Decomposition C_3 is an intersection of C_1 and C_2 and is used for the purpose of defining the material properties. Each decomposition has a set of 2-cells U_{ij} and the corresponding local coordinate systems L_{ij} (G represents the global coordinate system). For the sake of simplicity, all U_{ij} shown in Fig. 5 are defined in the global system G.

For the purpose of evaluating the properties, it is assumed that the heterogeneous object is made of carbon and silicon carbide—a C/SiC system. The material composition of these materials is defined on the decomposition C_1 where v_1 and v_2 are the volume fractions of C and SiC (see Fig. 6). The volume fractions are defined for each of the three regions (U_{11}, U_{12}, U_{13}) by the functions F_{11}, F_{12}, F_{13}.

Table 3
Various models used for evaluation of bulk and shear moduli

Volume fraction range	v_1:0.0–0.16 v_2:1.0–0.84	v_1:0.16–0.85 v_2:0.84–0.15			v_1:0.85–1.0 v_2:0.15–0.0
Geometry range (range of x)	0.0–0.4	0.4–0.5	0.5–0.6	0.6–0.9	0.9–1.0
Model 1	Dilute estimates	Wakashima–Tsukamoto method (or, Mori–Tanaka method) with averaging			Dilute estimates
Model 2	Wakashima–Tsukamoto Method (or, Mori–Tanaka Method) combined with fuzzy techniques				
Model 3	Self consistent method				



$$\left[\begin{array}{ll} \multicolumn{2}{c}{F_1 = (F_{11}, F_{12}, F_{13})} \\ F_{11} = (x^2, 1 - x^2) & \text{in } U_{11} \\ F_{12} = (0.25, 0.75) & \text{in } U_{12} \\ F_{13} = (1.5x - 0.5, -1.5x + 1.5) & \text{in } U_{13} \end{array}\right] \qquad \left[\begin{array}{ll} \multicolumn{2}{l}{F_2 = (F_{21}, F_{22}, F_{23})} \\ F_{11} = 1 & \text{in } U_{11} \\ F_{12} = 2 & \text{in } U_{12} \\ F_{13} = 1 & \text{in } U_{13} \end{array}\right]$$

Fig. 6. Models for material composition and microstructural type.

In addition to material composition, we also require information about microstructure in order to evaluate accurately material properties. Modeling of microstructure is a complicated problem and is not fully solved. Methods based on stereology, fractals, percolation theory, topology etc. are being explored to characterize the microstructure [9,24]. For our example, a simple microstructure is assumed where one material is embedded in the other as spherical particulates. Three microstructure domains are defined for the sample using the decomposition C_2–U_{21} is a dispersed microstructure where the spherical C particles are included in continuous SiC matrix, U_{22} is a network structure with both C and SiC exist as an interconnected phase and, U_{23} is a dispersed microstructure with SiC particles included in C matrix. In a dispersive structure, the volume fraction of the dispersed material in the continuous matrix is typically less than 0.2. A limit of 0.16 is used for this example, i.e. in U_{21} and U_{23} the volume fraction of the dispersed material (C in SiC and SiC in C, respectively) is less than 0.16. A simplified model is used for modeling the microstructure by representing its type (dispersed as type 1 and network as type 2) using the functions F_{21}, F_{22}, F_{23}. Additional information such as the size distribution of the spherical inclusions can be added to this model, if needed.

Several models exist to estimate the bulk modulus and shear modulus based on material composition [9,27,40]. Here, three models are considered as shown in Table 3. The appropriate methods used for various volume fraction ranges and their corresponding geometry range is shown.

Model 1 uses the method of dilute estimates for the dispersed microstructure i.e. in the regions U_{21} and U_{23} where the volume fraction of one material (either C or SiC) is less than 0.16. In the middle zone U_{22}, Model 1 uses Wakashima–Tsukamoto Method with averaging which yields identical results as Mori–Tanaka Method in this case. Model 2 uses Wakashima–Tsukamoto Method entirely with a combination of fuzzy techniques. Model 3 uses the Self Consistent Method for the entire region.

Table 4
Various attributes of an object and their nodes

Attribute	Manifold
Material composition	$V \subset R^N, V = \{v \in R^n/\|v\|_1 = 1, v_i \geq 0\}$
Inclusions: shape parameters	M (could be $\mathbf{R}^{\mathbf{m}}$)
Voids	$(0, v) \subset R, v < 1$
Planar spin	S^1
Orientation/spin	S^2 (unit vectors)
Liquid crystals	P^2, projective plane
Cosserat medium	$\mathrm{Orth}^{+(3)}$
Vector microstructure	T, translational space of $\mathbf{E^3}$
Crystal orientation	$\mathrm{Orth}^{+(3)}/G$ (crystal symmetry)
Density	$\mathbf{R}$
Moduli	C_{ijkl}, 4th order tensor in $\mathbf{E^3}$
Thermal expansion coefficient (similar coefficients)	$\mathbf{R}$
Thermal conductivity	$\mathbf{R}$
Temperature	$\mathbf{R}$
Velocity	$\mathbf{R^3}$
Stress	$Sym \subset GL(3, \mathbf{R})$

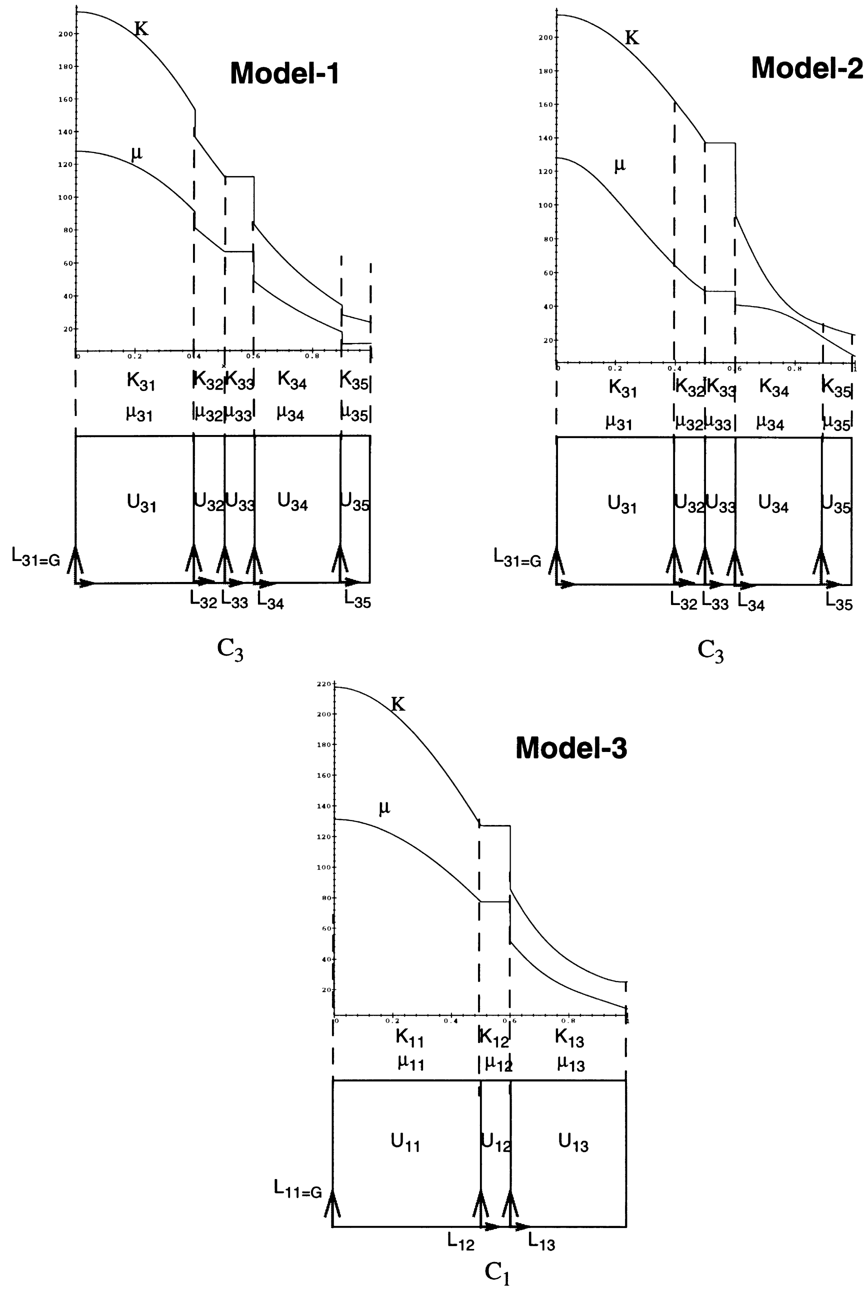


Fig. 7. Models for bulk modulus (K) and shear modulus (μ).

These three models were computed precisely. The actual equations are described in Ref. [27] and are not shown here. The results of these computations for the three models are plotted in Fig. 7. The variations of bulk and shear moduli as derived from the three models are shown. The decomposition C_3 is used to represent the Models 1 and 2, and decomposition C_1 is used to represent Model 3. Usually, it is preferable to have continuous property variation if the material composition variation is continuous. Model 1 yields discontinuous results. Model 2 is continuous but is not C^1 continuous. Hence C_3 is used to represent these two models. Model 3 is generated as an interpolated polynomial function, which is continuous over the entire region and is represented using C_1.

Finally, the object representation for this heterogeneous example is a trivial fiber bundle, which is generated as a product of the geometry and attribute models. As shown in Fig. 5, three decompositions, C_1, C_2 and C_3 were defined for the geometry P. The first attribute $A1$ is the material composition, whose attribute manifold is defined as the space of volume fractions $V \subset R^2$. The volume fractions are defined by function F_1 on the decomposition C_1 as shown in Fig. 6. The second attribute $A2$ is the microstructure type which is described as an integer and hence, the attribute manifold is the set of integers $\mathbf{Z}$. The type is defined using F_2 on the decomposition C_2 (Fig. 6). The third and fourth attributes $A3$ and $A4$ are the bulk and shear moduli. The attribute manifold for both is the real space $\mathbf{R}$. These attributes are defined by the functions K and μ and the decomposition used depends on the property model used. Three possibilities were illustrated in Fig. 7, one of which can be selected. The entire object is now modeled as a product set S, a product of geometry and attribute sets. The object model $\mathcal{M}$ for this example is summarized as:

$$\mathcal{M}_{\mathrm{G}} = (P, \{C_1, C_2, C_3\})$$

$$\mathcal{M}_{A1} = (V, F_1(C_1)), \qquad V = \{(v_1, v_2) \in \mathbf{R}^2 | v_1 + v_2 = 1\}$$

$$\mathcal{M}_{A2} = (Z, F_2(C_2))$$

$$\mathcal{M}_{A3} = (\mathbf{R}, K(C_i)), \qquad i = 1 \text{ or } 3$$

$$\mathcal{M}_{A4} = (\mathbf{R}, \mu(C_i)), \qquad i = 1 \text{ or } 3$$

$$S = P \times V \times \mathbf{Z} \times \mathbf{R} \times \mathbf{R}$$

$$\mathcal{M} = \{S, \mathcal{M}_{\mathrm{G}}, \mathcal{M}_{A1}, \mathcal{M}_{A2}, \mathcal{M}_{A3}, \mathcal{M}_{A4}\} \tag{9}$$

The above model describes the geometry of the heterogeneous object as well as its attributes (material composition, microstructural type, bulk and shear moduli) as a function of geometry.

In this example, though the decomposition C_1 was sufficient for specifying the material composition, another decomposition C_2 was required due to the presence of different microstructure regimes (based on volume fraction ranges). Note that C_1 and C_2 are two different subdivisions of geometry. Property variations in the geometry depend on both the material composition and the microstructural type. Hence, a new decomposition C_3 (combination of C_1 and C_2) was required to specify the physical properties (using Models 1 and 2).

As mentioned in the introduction, layered manufacturing (LM) processes have shown potential to fabricate heterogeneous objects. The LM processes demand a decomposition of the geometry dictated by manufacturability constraints. Specifically, the process planning of heterogeneous objects for LM require the geometry to be broken into “compacts” [41,42]. SFF-Compacts are proposed in Ref. [41] which are compacts with necessary process planning information attached to it. The collection of SFF-Compacts is termed as SFF-Object. The SFF-Object is equivalent to one particular decomposition of the geometry and each SFF-Compact is equivalent to a 3-cell in that decomposition. The attributes that are attached to the geometry are process planning information such as build direction, build sequence, etc. The object modeling framework presented in this paper can be used to represent the SFF-Object. Thus, the object modeling framework can provide a unified representation of heterogeneous objects that can be used for their design, analysis and manufacture.

4.1. Models of attributes

Depending on the object and its applications, several attributes may be added to the model. In Table 4, a list of some attributes is presented along with their manifold model (some of these are cited in Ref. [43]). Depending on the object and its applications, these attributes can be added to the model by means of appropriate atlases and attribute functions.

5. Modeling operations for object model

The modeling operations on object models depend on the operations that are defined for manipulating the geometry models as well as the attribute models.

5.1. Operations on geometry model

Geometric affine transformations can be applied to the geometry model (r-set and its atlases) to transform them in $\mathbf{E^3}$. The general form of this transformation (combination of rotation $[T]$ and translation $\underline{c}$) can be given as:

$$\Gamma : \mathbf{R}^n \rightarrow \mathbf{R}^n \tag{10}$$

$$\Gamma(\underline{x}) = [T]\underline{x} + \underline{c}$$

Any two r-sets can be combined using a set of modeling operations called regularized operations (reg-union $\cup^*$, reg-intersect $\cap^*$ and reg-difference $/^*$). Given two r-sets P and Q, the regularized booleans (denoted collectively as $\square^*$) are defined as:

$$\square^* : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A} \tag{11}$$

where

$$\square^*(P \in \mathbf{A}, Q \in \mathbf{A}) \equiv P \square^* Q = clo(int(P \square Q))$$

where $\mathbf{A}$ denotes the set of r-sets, $\square$ represents one of the set theoretic operations (union $\cup$, intersection $\cap$, and difference /) and, $col(int(\,))$ represents regularization (closure of interior) of the pointset. These regularized operations are algebraically closed in the class of r-sets and preserve the dimensionality of the r-sets. The r-sets with the regularized boolean operations from a boolean ring.

If two geometry models are combined, the regularized operations defined above in (Eq. (11)) yields the resulting geometry. However, the set of atlases must be modified to reflect this change. For the difference operation $P/^*Q$, only the atlases of P have to be modified:

$$\forall i, \qquad C_i/^*Q = \{U_\alpha, \psi_\alpha\}/^*Q = \{U_\alpha/^*Q, \psi_\alpha\} \tag{12}$$

For the intersection $P \cap^* Q$, if the same attribute exists in both P and Q, only one set of the charts corresponding to that attribute is retained. If the attribute exists only in one of the object models, then the atlas corresponding to the attribute is modified by restricting it to the intersecting region.

$$\begin{aligned} &C_i \cap^* C_j = \{(U_\alpha \cap^* V_\beta, \psi_\alpha) \text{ or } (U_\alpha \cap^* V_\beta, \varphi_\beta)\} \quad \text{common attribute} \\ &C_i \cap^* Q = \{(U_\alpha \cap^* Q, \psi_\alpha)\} \text{attribute not on } Q \\ &C_j \cap^* P = \{(V_\beta \cap^* P, \varphi_\beta)\} \text{attribute not on } P \end{aligned} \tag{13}$$

If more than one attribute is defined on an atlas, (for example, on atlas C_i in P), then the above operation yields two atlases (i.e. $C_i \cap^* C_j$ for the common attribute and $C_i \cap^* Q$ for the attribute not in Q).

Finally, for the union $P \cup^* Q$, a new atlas is created as a collection of all charts from P and Q, if the attribute is the same. If the attribute is defined only for one geometry, then the remaining region in the union where the attribute is undefined is added as a single chart.

$$\begin{aligned} &C_i \cup^* C_j = \{C_i/^*Q, C_i \cap^* C_j, C_j/^*P\} \text{ if attribute is common} \\ &C_i \cup^* Q = \{C_i, (Q/^*P, \psi_G)\} \text{ if the attribute does not exist in } Q \\ &C_j \cup^* P = \{C_j, (P/^*Q, \psi_G)\} \text{ if the attribute does not exist in } P \end{aligned} \tag{14}$$

Using the above equations, the *r*-sets and its atlases are updated when Boolean operations are performed. An example illustrating these ideas is presented later in this section. Henceforth, it is assumed that the operations on the geometry models implies the boolean operations on the *r*-sets as well as on their atlases.

5.2. Operations on attribute model

Operations on attribute model depend on the attribute being modeled. Here, a generic set of operations can be defined based on the following operations:

- The vector space operations $+$ (sum) and $*$ (product with scalar).
- The operations $\cup$ (union), $\cap$ (intersection) and $\sim$ (complement) on the subsets of attribute manifold.

These operations have to be modified to satisfy certain conditions such as algebraic closure, dimensional homogeneity etc. In the rest of the paper, we denote these operators generically as $\diamond$. A typical example of such operators is the "combine" operator defined for manipulating material composition values (points of volume fraction space V) for heterogeneous objects [8].

5.3. Operations on object model

The operations on the object model are driven by the operations on the geometry model. The attribute operations are used during the geometry operations in order to retain or derive the appropriate attribute values for the new geometry.

Operations for creating object models are

- *Construction of base space (r-set):* The construction of geometry (*r*-set P) has traditionally been done using primitives (like in CSG) or using sculptured objects. A vast amount of literature exists in CAGD and solid modeling on geometry creation.
- *Construction of atlases:* The atlases ($\{C_j\}$) can be specified as a part of geometry creation (local coordinate systems) or can be separately defined for a particular attribute.
- *Construction of fiber space (attribute manifold):* The fiber space (N_i), in general, is a collection of manifolds or a vector/tensor space. Though, it is not necessary to explicitly model the fiber space, it is useful to identify the constraints imposed by this space to ensure the validity of the attribute model generated. It must be noted that the attribute space may have higher dimensions (>3). Most often, these manifolds are simple geometrically and hence possess simple representations.
- *Construction of section (attribute functions):* The attribute model is specified using the attribute functions (F_i) and attribute manifolds N_i (or its constraints). The subsets of the attribute (fiber) space which models the attribute is implicitly defined through F_i.
- *Construction of object (product) model:* This construction essentially ties the attribute model to the geometry model.

Operations for manipulating/modifying/querying object models are

Attribute modifications: The following operations are needed to modify the attribute model and are used in conjunction with the geometry model operations described next.

- Replacing the fiber space by its own subspace, which includes component-wise projection.
- Combining several different fibers to form a new fiber of which the original fibers form a subspace (component product, inverse of above).
- Combining two fibers (same attribute) into a single fiber. The operations used to combine them are based on those mentioned in Section 5.2 and generically denoted by $\diamond$.
- Comparing attribute functions between cells. This would involve performing coordinate transformations for the attribute functions using the coordinate maps defined for the cells.

Geometry modifications: These operations involve modifying the geometry model(s) and creating new geometry(s). The attributes (fibers) must be appropriately carried over to the resulting object. The common operations used to modify the geometry are:

- Generating a new geometry from the charts of a particular atlas. The attributes defined on these charts are carried over to the new object model.
- Combining two geometry models (Section 5.1).
- Slicing the geometry into a set of subspaces. The

Table 5
Attribute definitions for the two objects

Object	Attribute	Atlas used	Function
O_1	A_1	C_1	F_1
O_1	A_2	C_2	F_2
O_2	A_1	D_1	G_1

geometry model can be sliced by a plane or, split into two geometry models. The attributes (fibers) must be appropriately carried over to the generated geometries.

- Geometry reconstructed from a set of subspaces of the base (inverse of the above).

The regularized boolean operations are used for manipulating the geometry model (r-sets and their atlases, refer (Eqs. (11)–(14)). Thus, new object modeling operations are defined (similar to the regularized boolean operations on the r-sets) to help manipulate object models:

$$/^* \to (/^*, I, \ldots, I) \qquad \cap^* \to (\cap^*, \diamond_1, \ldots, \diamond_n)$$
$$\cup^* \to (\cup^*, \vee_1, \ldots, \vee_n) \tag{15}$$

The symbol I above denotes the identity operation indicating that the attributes need not be altered for the difference operation. The operation $\diamond_i$ on each attribute A_i can be different and depends on the individual attribute models. The operation $\vee_i$ is a combination of I and $\diamond_i$. The example below illustrates the operations with an example. Refer [27] for details of these modeling operations.

5.3.1. Example

A sample example is presented to show how two object models can be combined. Consider two objects O_1 and O_2. O_1 possesses two decompositions C_1 and C_2, and O_2 has one decomposition D_1.

Consider two attributes A_1 and A_2. Object O_1 possesses both the attributes whereas object O_2 possesses only one attribute A_1. The functions and atlases used to define these attributes for each object are shown in Table 5.

Thus, the two object models are defined as $O_1 = \{P, \{C_1, C_2\}, \{(A_1, F_1(C_1)), (A_2, F_2(C_2))\}\}$ and $O_2 = \{Q, \{D_1\}, \{(A_1, G_1(D_1))\}\}$. When these two objects are combined, their decompositions have to be updated and new ones created if necessary. Table 6 lists the decompositions generated as a result of combining these two objects for all attributes. New atlases are created depending on the modeling operation performed.

For the union operation, the operator $\vee$ is defined as:

$$(F_1 \vee G_1)(C_1 \cup^* D_1)$$
$$= \begin{cases} F_1(C_1/^*Q) & \text{for } P/^*Q \\ (F_1 \diamond G_1)(C_1 \cap^* D_1) & \text{for } P \cap^* Q\,. \\ G_1(D_1/^*P) & \text{for } Q/^*P \end{cases} \tag{16}$$

where

$$C_1 \cup^* D_1 = \{C_1/^*Q, C_1 \cap^* D_1, D_1/^*P\}$$

6. Computer representation

A representation scheme for the fiber bundle model is shown in Fig. 8. Some of the data elements are included to provide clarity. This representation has not yet been implemented completely.

Each object model (OBJECT) is comprised of three representation modules—the geometry (GEOM), the attribute section (ATTR-SECTION) and the attribute manifold (ATTRIBUTE). GEOM represents the r-set P and its atlases $\{C_i\}$. The attribute model is broken into two separate modules ATTR-SECTION to represent the attribute functions F_i and ATTRIBUTE to represent the attribute manifold (N_i). As mentioned earlier, ATTRIBUTE need not be explicitly created if the validity of the ATTR-SECTION is ensured. However, it might be good idea to partially represent the attribute manifold in terms of the constraints. This would also ease the creation of ATTR-SECTION especially if several objects have the same attribute through different mappings.

The geometry representation (GEOM) captures the geometrical and topological information of the shape. A B-Rep scheme (G-BREP) is used to model the entire r-set (R-SET). The r-set can possess several atlases denoted by ATLAS. Each cell (CELL) in the atlas is represented by a B-Rep scheme (L-BREP). The cell has the information about its coordinate map (COORD) from its local coordinate

Table 6
Combining object models

Object	Geometry	Atlases	Attributes	
			A_1	A_2
O_1	P	C_1,C_2	$F_1(C_1)$	$F_2(C_2)$
O_2	Q	D_1	$G_1(D_1)$	
$O_1 \cap^* O_2$	$P \cap^* Q$	$C_1 \cap^* D_1, C_2 \cap^* Q$	$(F_1 \diamond G_1)(C_1 \cap^* D_1)$	$F_2(C_2 \cap^* Q)$
$O_1 \cup^* O_2$	$P \cup^* Q$	$C_1 \cup^* D_1, C_2 \cup^* Q$	$(F_1 \vee G_1)(C_1 \cup^* D_1)$	$F_2(C_2 \cup^* Q)$

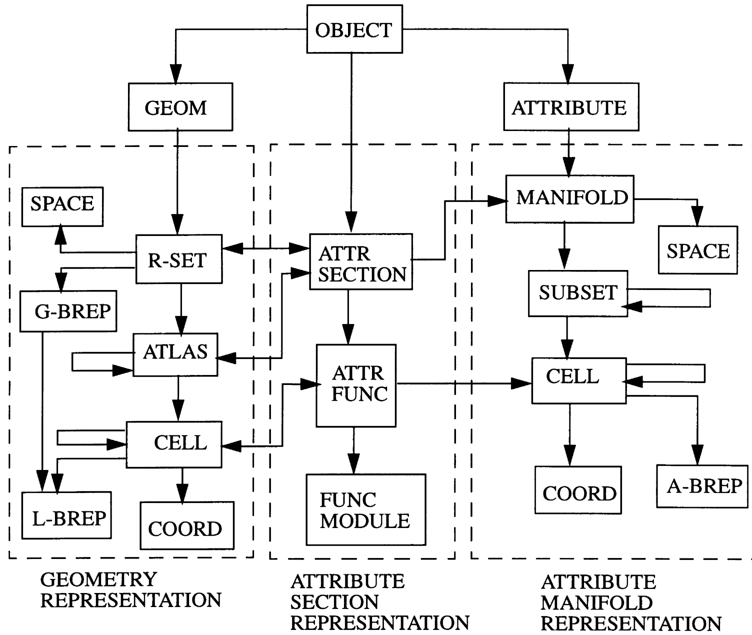


Fig. 8. Computer representation for the object model.

system to the global system. The information about the global system is attached to the *r*-set as SPACE.

The attribute manifold representation (ATTRIBUTE) implements the attribute model as a manifold (MANIFOLD), similar to that of *R*-SET. The hierarchy for manifold includes SUBSET, CELL, COORD and SPACE are identical to those implemented for *R*-SET. The CELL represents the subset of the attribute manifold which is mapped from a single cell in the *r*-set. The collection of these cells is SUBSET. Each CELL is implemented through a B-Rep scheme called A-BREP.

The attribute section representation (ATTR_SECTION) relates GEOM with MANIFOLD. This hierarchy implements the attribute functions ATTR-FUNC. The ATTR-FUNC contains the FUNC-MODULE to represent the individual functions.

7. Summary

Solid modeling schemes are used in various CADCAM applications to create, manipulate and query shapes of objects. Several models exist in the literature such as *r*-sets, manifold solids, SGC etc. to represent the shape of the objects. Recent applications such as design and manufacture of heterogeneous objects demand models of object, which represent not only the geometry but also include other attributes of an object.

In this paper, we have presented a trivial fiber bundle (global product set) model to represent several attributes of an object along with the geometry. This object model aims to provide a rigorous framework for mathematically integrating the various attributes of an object, as demanded by the applications. The geometry model represents the geometry of the object and forms the basis for modeling other attributes. *R*-set are used in this paper for representing geometry and, decompositions of the *r*-set are used to define the attributes. For each attribute, an attribute function is defined on a particular decomposition to map the geometry model to the attribute model. As mentioned, the attribute model need not be explicitly created if the validity of the attribute functions can be ensured. However, some applications might require the explicit presence of an attribute model. In order to create and manipulate these models, modeling operations are also defined.

The framework presented in this paper is a preliminary step towards the development of generic models of objects that are responsive to the needs of emerging manufacturing and engineering technologies. The models extend well beyond representing solid geometry alone, and can, in principle, represent all important engineering properties and attributes. At this time, it remains an open problem to ascertain that the proposed representation is a convenient basis from which to launch future methods and algorithms that help the designer explore and master new manufacturing and engineering technologies.

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Appendix

Fiber bundle: A fiber bundle ($\boldsymbol{E}$, $\boldsymbol{\Pi}$, $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{X}$) is [38,39]:

- A topological space $\boldsymbol{E}$ (total space)
- A topological space $\boldsymbol{X}$ (base space)
- A projection $\boldsymbol{\Pi}$ of $\boldsymbol{E}$ onto $\boldsymbol{X}$:

$$\boldsymbol{\Pi} : \boldsymbol{E} \rightarrow \boldsymbol{X}$$

- A topological space $\boldsymbol{F}$ called fiber
- A group of homeomorphisms of the fiber $\boldsymbol{F}$.
- A set of coordinate neighborhoods $\boldsymbol{U_\alpha}$ covering $\boldsymbol{X}$ which makes the bundle locally trivial as a product space

$$\boldsymbol{\phi_\alpha} : \boldsymbol{\Pi}^{-1}(\boldsymbol{U_\alpha}) \rightarrow \boldsymbol{U} \times \boldsymbol{F}$$

where $\boldsymbol{\Pi\phi_\alpha^{-1}(x, f) = x, x \in U_\alpha, f \in F}$

- A group $\boldsymbol{G}$ of homeomorphisms of the fiber $\boldsymbol{F}$ defined through the homeomorphisms between coordinate neighborhoods as:

$$\boldsymbol{\phi_\alpha \cdot \phi_\beta^{-1}} : (\boldsymbol{U_\alpha} \cap \boldsymbol{U_\beta}) \times \boldsymbol{F} \rightarrow (\boldsymbol{U_\alpha} \cap \boldsymbol{U_\beta}) \times \boldsymbol{F}$$

For every point $\boldsymbol{x} \in \boldsymbol{U_\alpha} \cap \boldsymbol{U_\beta}$, the map $\boldsymbol{\phi_\alpha \cdot \phi_\beta^{-1}}$ defines a map from $\boldsymbol{F}$ to $\boldsymbol{F}$. This map is termed as the transition function, $\boldsymbol{g_{\alpha\beta}(x)}$, which forms a homeomorphism on the fiber $\boldsymbol{F}$. The set of all these homeomorphisms for entire family $\mathscr{U} = \{\boldsymbol{U_\alpha}, \boldsymbol{\phi_\alpha}\}$ form a group called the structure group $\boldsymbol{G}$ of the fiber bundle $\boldsymbol{E}$.

There are similarities between the definition of a manifold and a fiber bundle. The manifold is locally $\mathbf{R^n}$ and a fiber bundle is locally a product space. The maps $\phi_\alpha \cdot \phi_\beta^{-1}$ are homeomorphisms between charts in a manifolds and the transition functions $g_{\alpha\beta}$ are homeomorphisms between fibers.

A fiber bundle whose fiber F is a vector space is called a vector bundle. A similar name extends to tensor bundle whose fiber is a space of tensors.

Principal bundle: If the fiber space F of a bundle E is replaced by G itself, then the bundle obtained is called the principal bundle $P(E)$. Section of a bundle E is a continuous map

$$\sigma : X \rightarrow E$$

satisfying $\Pi\sigma(x) = x, x \in X$. If the principal bundle $P(E)$ has a section then it is trivial. Then, $P(E)$ is globally a product of its fiber G and its base X, i.e. $P(E) = G \times X$. Hence, the bundle E is trivial and is a product space $E = F \times X$.

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