# How Solid Is Solid Modeling?

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## 1 On the Semantics of CSG and BRep

Constructive Solid Geometry (CSG) and Boundary Representations (Brep) are two major approaches to representing rigid solids dating back to the 1970s; see, e.g., [2, 6, 11, 14, 18, 22, 20, 21].

CSG implicitly represents a solid as an algebraic expression. The operators are regularized set operations, union, intersection and difference, and rigid-body motions. The operands are primitive solids, classically block, sphere, cylinder, cone and torus, instantiated to specific dimensions.

Brep explicitly represents the boundary of a solid as a data structure. Topologically, the surface is a quilt of vertices, edges and faces, where the adjacencies are represented. Geometrically, a face is a (well-behaved) subset of a surface. The surface could be a parametric surface or patch, an implicit algebraic surface, or a procedurally represented surface. The face boundaries are recursively represented as lower-dimensional boundary representations. Operations on solids in Brep first were the operations from CSG. However, soon other operations were introduced and implemented.

From the outset, research sought to give precise mathematical foundations to these solid representations and the operations on them. A proposed criterion by which to judge the representations was *informational completeness*. This was meant that there should be an algorithm that could, in principle, decide unambiguously whether any point in 3-space was inside, on the boundary of, or outside a given solid. Semantic work in CSG also paid attention to a finitary condition imposed to exclude pathological solids, for instance solids with a fractal boundary. The operations of (regularized) union, intersection, and difference were then defined in mathematical terms.

The task for defining a mathematical semantics for CSG and its modeling operations was simplified by the algebraic structure of the representation. The parallel task of giving Brep modeling a precise semantics turned out more difficult. Some research efforts formalized the topological validity of the representation, see, e.g., [18]. The interaction between topology and geometry, however, is a subject that continues to attract research; e.g., [10, 17, 23]

Over time, new operations were introduced into solid modeling that were difficult to fit into the well-established semantic framework of CSG. For example, consider a cube on which we round some edges and vertices. Conceptually, we can think of the construction as beginning with the cube, and modifying the shape by performing rounding operations. To accomplish this using only cubes, cylinders and spheres along with the Boolean operations of union, intersection and difference is not natural, hence a "rounding" operation was introduced in solid modelers.

The importance of such new operations to applications, and the apparent difficulty of reducing them conveniently to the repertoire of classical CSG is one of the factors that contributed, over time, to the decline of pure CSG modeling. While the conceptual legacy of CSG is very much present in many of today's solid modeling systems, Breps are used in virtually all of them. Moreover, the introduction of new operations has accelerated while needed semantic foundations are underdeveloped or absent. Two example areas follow that illustrate the situation, one with the classical operation of blending, the other with emerging design practices that stress ease of editing designs.

## 2 The Semantics of Blending

Rounding a convex edge or vertex, filleting a concave edge or vertex, are operations that are collectively called *blending* operations. Their precise semantics has a geometric and a topological aspect.

Geometrically, a blending surface is a surface that has to be in tangential contact with two or more given surfaces, along prescribed link curves, and whose geometric shape should conform to qualitative expectations. The geometric problem has been isolated and treated with precision by many researchers; see, e.g., [9, 11]. A blending surface qualitatively should have a rounded shape, and for this reason, spherically-derived blending surfaces are frequently chosen in practice. There are different ways to give meaning to the vague term spherically-derived. For example, in the case of blending two surfaces the following possibilities exist:

1. The blending surface is the envelope of the volume swept by a sphere that maintains simultaneous contact with the two primary surfaces; e.g. [12]. The *spine* of such surfaces is the trajectory of the center of the sphere, and it must be defined correctly in order to define the blending surface unambiguously.
2. A circle is swept in space such that, at each point, contact with the primary surfaces is maintained, thus defining a surface. Here, an addtional difficulty is to prescribe the spatial orientation of the circle as its center moves along a suitable curve in 3-space.
3. The surfaces to be blended are systematically deformed and pairwise intersected. The intersection curves lie on a blending surface. The circular rule might be that the intersection curves must pass through a fixed circle, in 3-space or in an abstract space; see, e.g., [13].

Other approaches, used for parametric primary surfaces, are derived from properties of the parametric representations; see, e.g., [9]. Figure 1 shows a constant-radius rolling-ball blending surface between two cylinders.

Consider variable-radius rolling-ball blends as a specific example. Such surfaces are obtained by rolling a ball whose diamater varies along the path, and using the surface of the swept volume as in the case of constant-radius rolling-ball blends. The major difficulty is to define precisely how the sphere's diameter

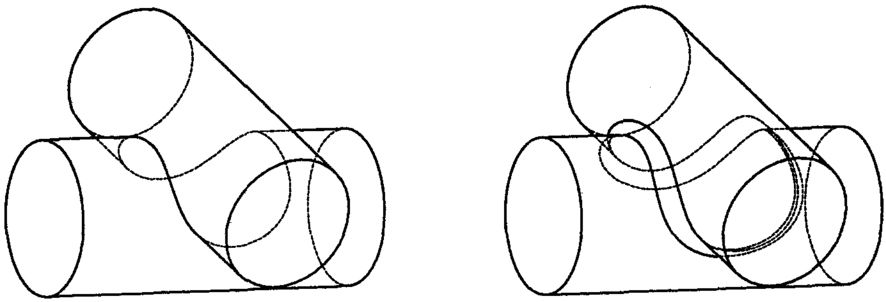

**Fig. 1.** Two cylinders and a rolling-ball blend between them

varies. The algorithmic techniques proposed in [19] are insufficient because they include an iterative step that traces an unspecified path on a two-dimensional manifold.

In the case of constant-radius rolling ball blends, the contact requirements imply that the spine of the blend must maintain a distance to the primary surfaces equal to the ball's radius. Thus, the spine is the intersection of the offset surfaces, of the primary surfaces, by the radius of the ball. Evidently, this is a necessary condition. Note, however, that it does not exclude global self-intersections of the blend.

In the case of variable-radius blends, we must determine the spine curve on the *equi-distance surface* of the primary surfaces, a generalization of the Voronoi cell boundaries familiar from computational geometry; e.g. [12]. The equidistance surface of two given surfaces $f$ and $g$ is the locus of all points $p$ that are at equal distance from both $f$ and $g$. Only in simple situations are such surfaces represntable with traditional mathematics. For example, the equidistance surface of a sphere and a (nonintersecting) plane is a paraboloid of revolution; the equidistance surface of two parallel planes is another plane; and the equidistance surface of two nonintersecting cylinders of equal radius and skew axes is a hyperbolic paraboloid. In general, the representaiton and analysis of equidistance surfaces requires higher-dimensional manifolds and projections; [7, 8].

While there is considerable work on the geometry of blending surfaces, work on the global topology and requirements for blending solid models is largely absent; [3]. A requirement of solid modeling is that blending surfaces be constructed based on the selection of edges and vertices, plus a few attributes. From this information, the blending operations proceed unassisted and in ways that are not even mapped out conceptually. For example, are the surfaces to be constructed sequentially or simultaneously? If they are to be serialized, in what order? For example, the two variants shown in Figure 2 have been obtained by blending the two edges in different order. A simultaneous blend might create a third variant. For a wide variety of such questions and how they might be answered see [3].

Clearly, such decisions, largely made automatically, affect the possible result of blending. Complete algorithms for blending of solid surfaces are of great practical importance, but there is little published work addressing the questions of topology and ordering of the possible variants.

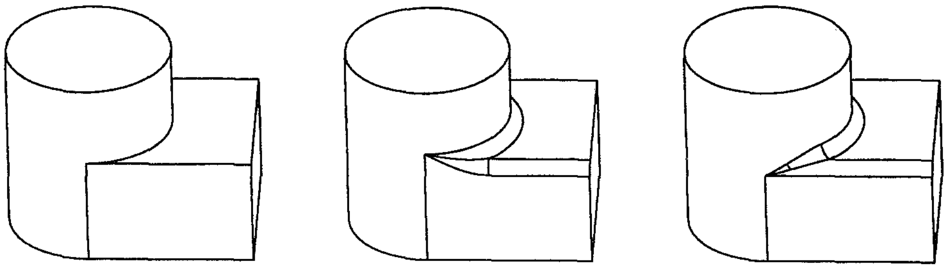

**Fig. 2.** Two variants when blending two adjacent edges

## 3 The Problem of Variational Design

Many solid modeling systems implement a design paradigm in which no longer an specific solid shape is designed, but a parameterized class of potential shapes in which a specific shape is then instanced. Figure 3 illustrates this for two-dimensional shapes: A quadrilateral with a rounded corner has been defined.

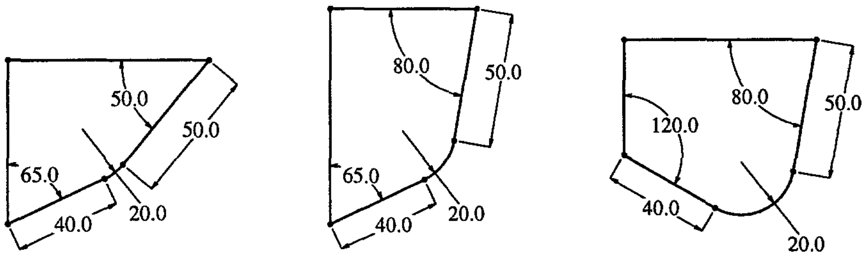


**Fig. 3.** Design variants of a quadrilateral

Some of the lengths and angles have been prescribed, as well as that two sides should be perpendicular and the rounding arc tangent to the two adjacent edges. By varying the lenghts or angles, different rounded quadrilaterals are obtained from the same underlying definition schema. Moreover, this definition is descriptive rather than procedural because there are no requirements on the sequence in which the constraints should be elaborated. This descriptive definitional style is often termed *variational* in the solid modeling literature, and the procedural definitional style is called *parametric*.

A basic semantic difficulty of variational design is that there are different ways to interpret the design constraints and the meaning of the changes. In the case of the rounded quadrilateral, the problem is the ambiguity of the constraints. For example, the two interpretations shown side-by-side in Figure 4 are both correct mathematically, but presumably one interpretation is intended while the other one is not.

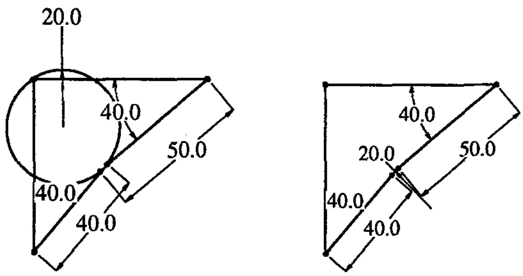


**Fig. 4.** Two interpretations of a design variant

We could argue that the multiplicity of interpretating variational geometric constraints is unavoidable. After all, the constraints will naturally correspond to a nonlinear system of equations, and such systems have multiple solutions in general. Hence, the question of "which variant" might be reformulated instead as the question of "what additional information" is needed, to unambiguously define the members of a class of designs in such a way that only the meaningful shape instances are included. Some mathematical and combinatorial difficulties arise in the investigation of this formulation; e.g., [1].

Another broad class of ambiguities in how to interpret a change in design arises from the persistent naming problem; [15, 16, 4, 5]. Consider Figure 5. Here, the part on the right was obtained by changing the position of the center of

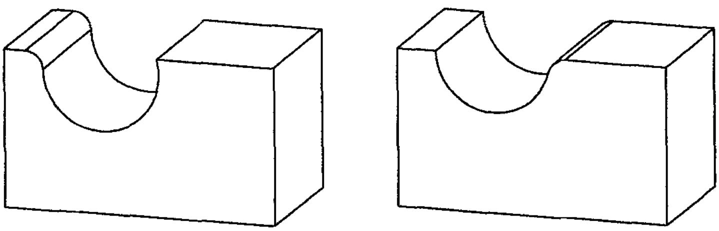

**Fig. 5.** Variant designs?

the round slot. This variation is counter-intuitive, because we probably consider such design changes conceptually as a continuous deformation of one shape into the other, here by raising or lowering the center line of the slot.

To understand why a solid modeling system could construct the variant we must consider the steps in which the part was designed: First, a block was created, with prescribed height, width, and depth. The block's shape could be varied subsequently by changing these dimensions. Next, the round slot was created by choosing a direction of the cut, a radius for the circular profile, and a position for the center, based perhaps on the distance from one of the sides and the bottom of the block. Finally, an edge round was created by selecting one of the edges bounding the slot laterally and prescribing a radius for the round. The edge selection was done visually, interacting with a specific instance.

When the slot's position is changed, we must recreate the data structure of the old instance to reflect the change. In particular, the design is reverted to the block, the slot is re-created from the new dimensions, and a new round is created. This means that a description of the edge has to be given that is independent of the data structure of the first design instance. The description has to be re-evaluated for the new design.

How could the edge be described? Clearly, it is the intersection of the slot with the top of the block — but so is the other edge, and it is this ambiguity whose resolution is imperfect in the example. Various schemata have been proposed in the cited literature. Moreover, proprietary schemata exist in implemented commercial solid modeling systems. These schemata are difficult to characterize.

The combination of the naming schema, and its reinterpretation after editing a design, constitutes a procedural semantics of variational solid design that is not clearly understood and requires further exploration. The difficulty of re-interpreting includes arbitrating multiple occurrences of a named entity, for example when subdividing an edge or a face, resolving clashes when merging vertices, edges or faces, and responding to the diappearance of named entities such as the obliteration of a vertex by an expanded feature elsewhere on the solid.

Consider the design shown in Figure 6. A protrusion was added to the block, but the height was chosen such that the top face, $f_2$, merged with the top of the block, $f_1$. The height of the prism is controlled by a dimension measured agains the top face. When the protrusion is lowered, a decision must be made which part of the L-shaped top face is to be used to reconstruct the prism. For a more detailed discussion of these and other problems see, e.g., [15, 16, 4, 5].

## 4 Summary

We have sketched several operations and design paradigms in solid modeling that remain without adequate semantic foundation. To obtain a proper mathematical semantics, and to harmonize it with the intuitions and expectations applications of solid modeling require, poses some fascinating and difficult research topics worthy of sustained exploration.

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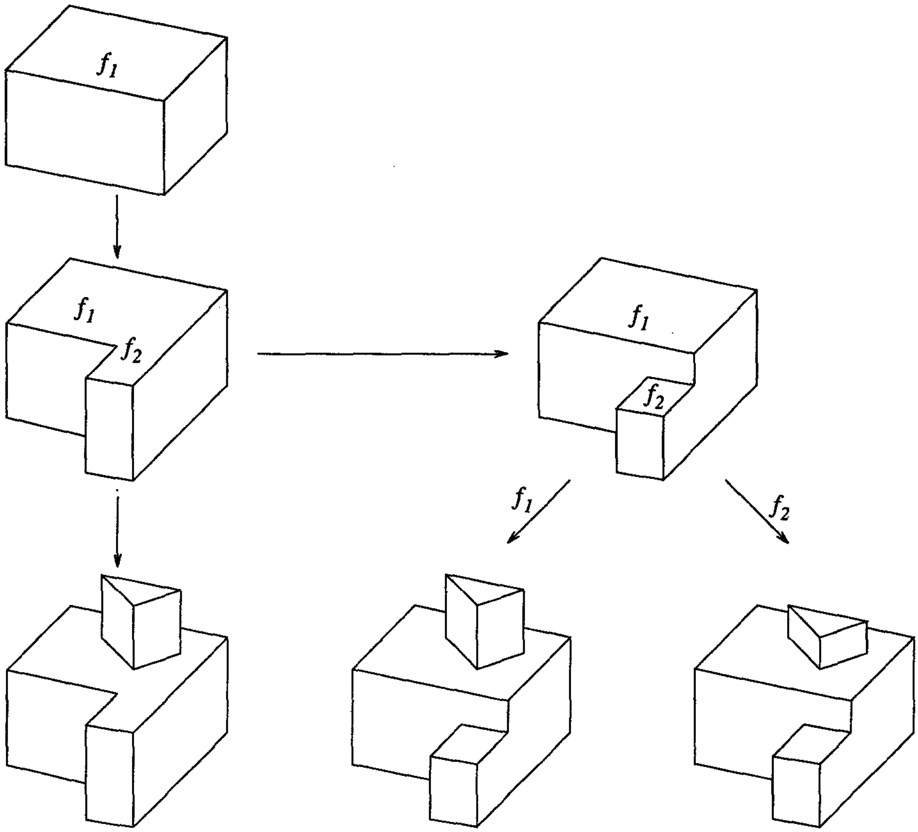


**Fig. 6.** Identification problems for merged faces and consequences

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