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Report Number:

91-015

Hoffmann, Christoph M. and Vermeer, Pamela J., "Eliminating Extraneous Solutions in Curve and Surface Operations" (1991). *Department of Computer Science Technical Reports.* Paper 864.
https://docs.lib.purdue.edu/cstech/864

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ABSTRACT

We study exact representations for offset curves and surfaces, for equal-distance curves and surfaces, and for fixed- and variable-radius blending surfaces. The representations are systems of nonlinear equations that define the curves and surfaces as natural projections from a higher-dimensional space into 3-space. We show that the systems derived by naively translating the geometric constraints defining the curves and surfaces can entail degeneracies that result in additional solutions that have no geometric significance. We characterize these extraneous solution points geometrically, and then augment the systems with auxiliary equations of a uniform structure that exclude all extraneous solutions. Thereby, we arrive at representations that capture the geometric intent of the curve and surface definitions precisely.

Keywords: geometric modeling, faithful problem formulation, offsets, blends, equidistance surfaces, extraneous solutions

1. Introduction

Geometric modeling uses a number of surface operations that are intuitively straightforward, yet pose significant mathematical problems. Examples include offset surfaces,[1,2,3,4,5,6,7] equidistant or Voronoi surfaces,[8,9] and fixed- and variable-radius blends.[8,9,10,11] In order to find closed-form solutions, elimination algorithms could be used in principle. However, the problems arising in practical settings are well beyond the current state of the art and current hardware resources.[12] In view of this, the conventional approach is to approximate these surfaces.[2,5,13]

In a number of papers it has been argued that such surface operations can be expressed conveniently as systems of nonlinear equations that are formed by expressing equationally, one by one, the geometric constraints entailed by the surface operation. That is, the desired surface can be represented exactly by a system of nonlinear equations.[8,9,12] Furthermore, uniform efficient techniques have been developed to analyze these equation systems and the surfaces they define.[12,14,15]

While the resulting equations do indeed represent the desired surfaces, they may

entail additional solutions that do not have geometric significance in the context of the operation.[12] Such additional solutions arise when, at certain points, some of the equations become interdependent or vanish outright. In this paper, we describe a method for augmenting the system of equations with additional equations that effectively exclude such extraneous solutions.

Roughly speaking, there are two sources that can generate extraneous solutions. First the equation system may have solutions unrelated to the geometric intent with which the equations were constructed. This is the problem we solve in this paper. Secondly, if the equations are subsequently processed symbolically, eliminating some or all auxiliary variables, the resulting closed-form representation may include extraneous solutions introduced by the elimination procedure. We do not address this problem here, and refer to the sizable literature on the subject.[3,16,17,18,19,20]

In the second section, we define the term extraneous solution, and characterize informally how extraneous solutions arise in offsets, equidistant curves and surfaces, and fixed- and variable-radius blends. We then describe two devices for eliminating extraneous solutions from the problem formulation, and prove that they eliminate all extraneous solutions. We also give some examples illustrating the method.

2. Geometric Operations and Extraneous Solutions

In general, the geometric operations we consider define new curves and surfaces in terms of given *base curves* or *surfaces*. Consider the offset curve. Here, the base curve is defined by a given equation $f(x,y) = 0$. The d-offset curve consists of all points which are at distance d from some *footpoint* on the base curve, where the distance is measured along the normal to the base curve at the footpoint. The point on the offset curve and the footpoint are said to *correspond* with each other if they satisfy this relationship. We translate these geometric requirements into nonlinear equations, thereby expressing the offset curve as a system of equations. The translation is shown explicitly below, and each type of curve or surface is discussed in depth.

For the cases of offset curves and surfaces and for equal-distance curves and surfaces, we will define extraneous solutions as follows:

> A solution is *extraneous* if it corresponds to a footpoint which corresponds to infinitely many solutions.

Blending surfaces require a different definition, which we will give later. Since we are interested in physical interpretations of these systems of equations, we restrict our work to points in real, affine space.

2.1. Offset Curves

Offset curves provide a good starting point for looking at extraneous solutions which arise when using the higher-dimensional method for curve and surface construction because the system which defines the offset curve has a very simple structure. The d-offset to a given curve $C : f(u,v) = 0$ can be formulated by 1–3:

$$f(u,v) = 0 \tag{1}$$

$$(x - u)^2 + (y - v)^2 - d^2 = 0 \tag{2}$$

$$-f_v(x - u) + f_u(y - v) = 0 \tag{3}$$

For convenience of notation, here and throughout the paper all partials are assumed to be evaluated at their respective points, i.e., $f_u = f_u(u,v)$ in general, or $f_u = f_u(\hat{u},\hat{v})$ when we are considering a particular point $(\hat{u},\hat{v})$ on C.

If (x,y,u,v) is a solution to 1–3 then $\mathbf{p} = (u,v)$ is on the curve C, and $\mathbf{x} = (x,y)$ is both on the circle centered at $\mathbf{p}$ of radius d and on the normal to f at $\mathbf{p}$. Hence $\mathbf{x}$ is a point on the offset curve which corresponds to footpoint $\mathbf{p}$ on the base curve. However, if $\mathbf{p}$ is a singular point of f, then 3 vanishes. Thus all of the points on the circle centered at $\mathbf{p}$ of radius d will correspond with $\mathbf{p}$, and so that circle will be extraneous. It is also clear that only if $\mathbf{p}$ is singular will the third equation vanish independently of x and y. Moreover, the second equation will never vanish independently of x and y. Thus for offset curves, extraneous solutions arise only when the base curve has singularities, and the extraneous solutions all correspond to the singular points.

We will present methods that eliminate extraneous solutions by augmenting the equation system. In some cases, these methods will eliminate finitely many meaningful points as well. For example, consider the curve given by $v^2 - u^3 = 0$ and its 1-offset described by the system

$$v^2 - u^3 = 0$$

$$(x - u)^2 + (y - v)^2 - 1 = 0$$

$$-2v(x - u) + 3u^2(y - v) = 0$$

At $(0,0)$ the base curve is singular, so the points $(x,y,0,0)$ with $x^2 + y^2 = 1$ also satisfy the system, although almost all of these points do not lie on the 1-offset of the base curve. By augmenting the system suitably, we succeed in removing the points $\{(x,y,0,0) \mid x^2 + y^2 = 1\}$, including the points $(1,0,0,0)$ and $(-1,0,0,0)$ which define the offset points $(1,0)$ and $(-1,0)$.

For numerical processing[14] of the augmented system, this is generally not a problem, since the points that have been lost are a lower-dimensional set than the solution we want. Moreover, should we derive closed-form solutions by variable elimination, then these points would be reinstated by continuity.

2.2. Offset Surfaces

Offset surfaces are defined analogously to offset curves: a point is on the d-offset of a surface if it is distance d along the normal to some footpoint on the base surface. The normal condition for surfaces is expressed as two equations linear in x, y, and z, and at some points on the base surface these two equations may be dependent. That is, while neither equation vanishes outright, it is possible that the two equations can be reduced to a single equation at certain points, thus reducing the number of constraints on the system and introducing extraneous solutions. When this happens, we say that the normal is *degenerate*; otherwise, the normal is *well-defined*. A system for the offset surface is given by 4–7:

$$f(u, v, w) = 0 \tag{4}$$
$$(x - u)^2 + (y - v)^2 + (z - w)^2 - d^2 = 0 \tag{5}$$
$$-f_v(x - u) + f_u(y - v) = 0 \tag{6}$$
$$-f_w(y - v) + f_v(z - w) = 0 \tag{7}$$

If (x, y, z, u, v, w) is a solution of this system of equations, then $\mathbf{p} = (u, v, w)$ lies on the base surface given by f, and $\mathbf{x} = (x, y, z)$ is distance d from $\mathbf{p}$ along the normal to f at $\mathbf{p}$. As in the case of offset curves, if $\mathbf{p}$ is singular, 6 and 7 vanish independently of $\mathbf{x}$. In this case, all points on the sphere of radius d centered at $\mathbf{p}$ will be extraneous solutions corresponding to $\mathbf{p}$.

Additionally, the normal may exist, but it may not be well-defined by the system. In the particular definition given, this can happen just in case $f_v = 0$. Then 6 and 7 are multiples of each other, and geometrically, they no longer define the normal line but only a plane in which the normal lies. This is not enough information to completely specify a unique point on the offset surface corresponding to the footpoint, thus we have as extraneous solutions the points on the circle obtained by intersecting the sphere of radius d centered at $\mathbf{p}$ with this plane, forming a tube whose spine is the space curve $(f = 0) \cap (f_v = 0)$.

We could actually choose any two of

$$-f_v(x - u) + f_u(y - v) = 0$$
$$-f_w(y - v) + f_v(z - w) = 0$$
$$-f_u(z - w) + f_w(x - u) = 0$$

to specify the normal condition. Choosing a different pair would merely alter the curve along which the tube of extraneous solutions lies. For example, using the second and third equations, extraneous solutions would arise along the curve which is the intersection of $f = 0$ and $f_w = 0$.

Equal-distance curves can be defined in terms of offset curves. A point is on the equal-distance curve of two curves f and g if it is on the d-offset curve of both f and g, for some d.

Equal-distance curves have an additional layer of complexity. This additional complexity comes from having to consider the geometric relationship between the two base curves and their respective normal directions. There are two footpoints to consider, one on each curve, and different results depending on whether both of them are regular, both are singular, or one is singular while the other is not. Also, we must consider what happens when the two footpoints coincide, that is, when the curves intersect each other. In certain situations, this yields extraneous solutions, while in others it does not.

The algebraic counterpart to these geometric considerations involves the dependence between equations related to one of the curves with equations related to the other curve. Using the system for offset curves given above, we can write the equal-distance system 8–13:

$$f(u_1, v_1) = 0 \tag{8}$$

$$(x - u_1)^2 + (y - v_1)^2 - d^2 = 0 \tag{9}$$

$$-f_{v_1}(x - u_1) + f_{u_1}(y - v_1) = 0 \tag{10}$$

$$g(u_2, v_2) = 0 \tag{11}$$

$$(x - u_2)^2 + (y - v_2)^2 - d^2 = 0 \tag{12}$$

$$-g_{v_2}(x - u_2) + g_{u_2}(y - v_2) = 0 \tag{13}$$

From the discussion of offset curves above, if $(x, y, u_1, v_1, u_2, v_2, d)$ is a solution to this system, (x, y) is on the d-offset of f and on the d-offset of g, and hence it is a point on the equal-distance curve of f and g.

However, as in the previous cases, singularities in the footpoints cause extraneous solutions. If one of the footpoints is singular, say $\mathbf{p} = (u_1, v_1)$, then for every point $\mathbf{q}$ on g there is a point $\mathbf{x} = (x, y)$ and a d such that $\mathbf{x}$ corresponds to the footpoints $\mathbf{p}$ and $\mathbf{q}$. Geometrically, this means that we get as extraneous solutions the curve of points which are at equal distance to the singular point $\mathbf{p}$ and the curve defined by g. See Figure 1. When $\mathbf{q}$ is also singular, the perpendicular bisector of $\overline{\mathbf{pq}}$ is extraneous as well, since in this case both 10 and 13 vanish. Then the only equations depending on x and y are the two circles 9 and 12, and as d varies, the bisector is obtained.

Moreover, when $\mathbf{p} = \mathbf{q}$, further extraneous solutions can arise. If both points are regular, but the curves meet tangentially at the footpoints, then the common normal line is extraneous. Similarly, if one of the curves is regular at the footpoint while the other is singular, the normal line to the curve which is regular is an extraneous
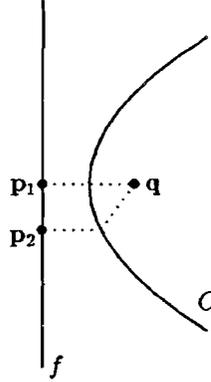
Fig. 1. C is the extraneous equal-distance curve between f and the singular point q.

solution. Finally, if both points are singular, then 9 and 12 sweep out the entire xy-plane as d varies, hence the whole plane is extraneous.

2.4. Equal-Distance Surfaces

Equal-distance surfaces are defined analogously to equal-distance curves: a point is on the equal-distance curve of two surfaces f and g if it is on the d-offset surface of both f and g, for some d. A system of equations which defines the equal-distance surface between two surfaces f and g can be given by 14–21:

$$f(u_1, v_1, w_1) = 0 \tag{14}$$

$$(x - u_1)^2 + (y - v_1)^2 + (z - w_1)^2 - d^2 = 0 \tag{15}$$

$$-f_{v_1}(x - u_1) + f_{u_1}(y - v_1) = 0 \tag{16}$$

$$-f_{w_1}(y - v_1) + f_{v_1}(z - w_1) = 0 \tag{17}$$

$$g(u_2, v_2, w_2) = 0 \tag{18}$$

$$(x - u_2)^2 + (y - v_2)^2 + (z - w_2)^2 - d^2 = 0 \tag{19}$$

$$-g_{v_2}(x - u_2) + g_{u_2}(y - v_2) = 0 \tag{20}$$

$$-g_{w_2}(y - v_2) + g_{v_2}(z - w_2) = 0 \tag{21}$$

From the discussion of offset surfaces, if $(x, y, z, u_1, v_1, w_1, u_2, v_2, w_2, d)$ is a solution to the system, then $\mathbf{x} = (x, y, z)$ is on the equal-distance surface between f and g. And, as in the case of equal-distance curves, we have extraneous solutions depending both on the regularity or singularity of the footpoints and on whether the footpoints coincide or not.

If both footpoints $\mathbf{p} = (u_1, v_1, w_1)$ and $\mathbf{q} = (u_2, v_2, w_2)$ are regular, then extraneous solutions arise when the normal lines are not well-defined by the system. If both normal lines are well-defined, extraneous solutions can arise only when the

footpoints and their respective normals coincide. Otherwise, the two normal lines intersect in at most one point, so if a solution exists, it must be unique.

If one normal line is well-defined while the other is degenerate, then, again, there is at most one solution unless p and q are identical and the well-defined normal lies in the plane defined by the degenerate normal conditions. However, the point which is a solution of the algebraic system may not satisfy the geometric criteria of lying on the normals to both surfaces, since there is no guarantee that the point lies on the normal to the surface of the point with the degenerate normal condition. We call such algebraically valid but geometrically meaningless solutions *spurious*.

Similarly, when both normal conditions are degenerate, an extraneous line exists when p $\neq$ q. This line is the intersection of the plane M defined by 15 and 19, which is the bisector of $\overline{pq}$, and the plane P defined by the degenerate normal conditions. When p = q, if any solution at all exists, the entire plane P satisfies the system, and therefore is extraneous.

When one footpoint is singular, again suppose it is p, the two linear equations 16 and 17 vanish. As in the case of equal-distance curves, for every point q on g, there is a point x and a distance d such that p and q are footpoints which correspond to x. This means that the surface of points at equal distance from p and g is extraneous. When the two remaining planar equations 20 and 21 are dependent, they define a single plane P, and the intersection of P with M is an extraneous line. If in addition the footpoints coincide, then the entire line or plane defined by 20 and 21 is also extraneous.

Finally, when both footpoints are singular, the plane M is extraneous. If the two footpoints coincide in this case, then the entire xyz-space is swept out by the spheres and is considered extraneous.

2.5. Constant-Radius Blending Surfaces

Constant-radius blending surfaces can be defined in terms of fixed-radius offsets to the two surfaces being blended. First, locate a point offset from both f and g by a distance d. Then join the two footpoints p and q on f and g, respectively, by the circle of radius d which lies in the plane spanned by the normals of f and g at p and q. This circle is determined by the intersection of a sphere of radius d centered at the common offset point and the plane spanned by the normals at the footpoints. This technique gives more than desired, since for each corresponding pair of footpoints, it generates a full circle. The surface must later be trimmed in order to get only those points which lie "close to" the base surfaces. In terms of equations, the constant-radius blending surface of radius d between f and g can be given by 22–31:

$$f(u_1, v_1, w_1) = 0 \qquad (22)$$

$$(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2 - d^2 = 0 \qquad (23)$$

$$-f_{v_1}(u - u_1) + f_{u_1}(v - v_1) = 0 \qquad (24)$$

$$-f_{w_1}(v - v_1) + f_{v_1}(w - w_1) = 0 \qquad (25)$$

$$g(u_2, v_2, w_2) = 0 \qquad (26)$$

$$(u - u_2)^2 + (v - v_2)^2 + (w - w_2)^2 - d^2 = 0 \qquad (27)$$

$$-g_{v_2}(u - u_2) + g_{u_2}(v - v_2) = 0 \qquad (28)$$

$$-g_{w_2}(v - v_2) + g_{v_2}(w - w_2) = 0 \qquad (29)$$

$$(x - u)^2 + (y - v)^2 + (z - w)^2 - d^2 = 0 \qquad (30)$$

$$(x - u, y - v, z - w) \cdot (N_f(u_1, v_1, w_1) \times N_g(u_2, v_2, w_2)) = 0 \qquad (31)$$

where N_f and N_g are the normals of f and g, respectively. In this definition, if the point $(x, y, z, u, v, w, u_1, v_1, w_1, u_2, v_2, w_2)$ is a solution to the system of equations, then $\mathbf{p} = (u_1, v_1, w_1)$ lies on f, $\mathbf{q} = (u_2, v_2, w_2)$ lies on g, and $\mathbf{m} = (u, v, w)$ is the common d offset to f and g corresponding to footpoints $\mathbf{p}$ and $\mathbf{q}$, respectively. The point $\mathbf{x} = (x, y, z)$ is the point which is actually on the blending surface.

Originally we defined extraneous to mean that a footpoint on one of the surfaces corresponded to infinitely many points on the surface being defined. This definition no longer suffices, since every pair of footpoints corresponds to a curve on the blend. Instead, a solution is extraneous if either footpoint has extraneous offset points or if for the given footpoints and offset point, there is a sphere of points which satisfy the system.

From the discussion of offset surfaces above, it is easy to see when extraneous solutions arise because of extraneous offset points. If both $\mathbf{p}$ and $\mathbf{q}$ are regular, and both of them have a well-defined normal, then there are no extraneous solutions, since the normals can meet in just one point. If one of the normals is degenerate, there is still at most one possible point which geometrically is offset to the footpoint whose normal is well-defined, but it may be spurious with respect to the footpoint with degenerate normal. If both normals are degenerate, then each offset subsystem defines an extraneous circle in the plane $y = v_1 = v_2$, and the number of offset solutions depends on how these two circles intersect.

If $\mathbf{p}$ is singular while $\mathbf{q}$ is regular, then if the linear equations associated with g do not degenerate at $\mathbf{q}$, the point $\mathbf{m}$ is unique. Otherwise, if the normal at $\mathbf{q}$ is degenerate, the offset subsystem gives an extraneous circle of solutions, and the number of extraneous offset points for the equal-distance surface are determined by the way in which that circle intersects the sphere of radius d centered at $\mathbf{p}$, given by 23.

Lastly, if both footpoints are singular, then $\mathbf{m}$ is a single point if the two spheres given by 23 and 26 meet tangentially. It is a point on the intersection circle if they meet transversally, and if the footpoints coincide, then the entire sphere is extraneous.

If $\mathbf{m}$ is the unique point which is on the d-offset of both f and g, then extraneous solutions can still arise when 31 vanishes independently of x, y, and z. This happens

if and only if $N_f(p) \times N_g(q) = (0,0,0)$, and results in an extraneous sphere in the blending surface, given by 30.

2.6. Variable-Radius Blending Surfaces

Circular blends of variable radius can also be defined as systems of equations. In the case of constant-radius blends, the spine of the blend is defined as the intersection of the d-offsets of f and g, where d is fixed. The circular arcs of the blending surface are then centered on the spine. For variable-radius blends, the spine must lie on the equal-distance surface between the two base surfaces so that a sphere centered on the spine will touch both f and g. Such a spine can be obtained by intersecting the equal-distance surface of f and g with a reference surface h. The variable-radius blend between f and g is then defined as the envelope of the family of spheres whose centers lie on the spine and whose radii are such that each sphere touches both f and g.

To write the variable-radius blending surface as a system of equations, we need the equations for the equal-distance surface between f and g, which we know from the previous discussion, and the equation for the reference surface h. We also require an equation which defines the family of spheres, and an equation which defines the envelope of that family. Each sphere S_d must be centered on the spine, and since it must touch f and g, its radius d must be the same as the distance of its center to the footpoints on f and g. Finally, to get the envelope of the family of spheres, the derivative of S_d in the tangent direction must be zero. For further details, see e.g. (Refs. 9, 25). The system is then given by 32–42:

$$f(u_1, v_1, w_1) = 0 \tag{32}$$

$$(u - u_1)^2 + (v - v_1)^2 + (w - w_1)^2 - d^2 = 0 \tag{33}$$

$$-f_{v_1}(u - u_1) + f_{u_1}(v - v_1) = 0 \tag{34}$$

$$-f_{w_1}(v - v_1) + f_{v_1}(w - w_1) = 0 \tag{35}$$

$$g(u_2, v_2, w_2) = 0 \tag{36}$$

$$(u - u_2)^2 + (v - v_2)^2 + (w - w_2)^2 - d^2 = 0 \tag{37}$$

$$-g_{v_2}(u - u_2) + g_{u_2}(v - v_2) = 0 \tag{38}$$

$$-g_{w_2}(v - v_2) + g_{v_2}(w - w_2) = 0 \tag{39}$$

$$h(u, v, w) = 0 \tag{40}$$

$$(x - u)^2 + (y - v)^2 + (z - w)^2 - d^2 = 0 \tag{41}$$

$$\left(\frac{\partial S_d}{\partial u}, \frac{\partial S_d}{\partial v}, \frac{\partial S_d}{\partial w}\right) \cdot (N_h \times N_V) = 0 \tag{42}$$

where S_d is given by 41, N_h is the normal to h at (u, v, w), and N_V is the normal to the equal-distance surface at (u, v, w).

As with the constant-radius blends, extraneous solutions arise if extraneous solutions exist in the equal-distance surface. However, in the variable-radius case,

such points will be extraneous only if they are also contained in h. By eliminating all extraneous solutions to the subsystem 32–39, these extraneous points can be eliminated.

Also similar to constant-radius blends, extraneous solutions can arise for variable-radius blends when 42 vanishes independently of x, y, and z. This can only occur if $N_h \times N_V = (0,0,0)$, and results in an extraneous sphere, given by 41.

3. Generic Elimination Strategies

A generic strategy for removing extraneous solutions is to exclude those footpoints with which the extraneous solutions are associated. We will give a precise geometric characterization of these points, and show that the addition of certain inequalities to the system excludes them. These inequalities are actually expressed equivalently by additional equations with additional variables.

Consider the case of the offset curve, where extraneous solutions arise just in case the footpoint is singular. If we could eliminate the possibility of such points being footpoints, we would eliminate all solutions which correspond to those footpoints. A point is singular if and only if its normal vector is the zero-vector. So we add an equation which eliminates all points whose normal vector is identically zero. This will eliminate from the solution set all points for which the footpoint on the base curve is singular, and so will eliminate the extraneous solutions associated with the singular points. The equation we add is

$$(\alpha f_u - 1)(\alpha f_v - 1) = 0$$

Whenever $f_u = f_v = 0$, this equation reduces to

$$-1 = 0$$

and then it will not have a solution. Otherwise, if $f_u \neq 0$ or $f_v \neq 0$, there is a value for α, the new variable, which will solve the added equation. Therefore this equation eliminates all and only those solutions which are extraneous. Note that the equation

$$\alpha x - 1 = 0$$

effectively expresses the inequality $x \neq 0$. This technique is used extensively in geometry theorem proving.[21,22,23,24]

It is sometimes convenient to use a second device for excluding extraneous solutions. When defining surfaces, perpendicularity conditions may have to be expressed by two equations of the form:

$$\mathbf{u} \cdot \mathbf{t}_1 = 0$$
$$\mathbf{u} \cdot \mathbf{t}_2 = 0$$

Here, $\mathbf{u}$ is a vector that is to be perpendicular to two linearly independent tangent directions $\mathbf{t}_1$ and $\mathbf{t}_2$. When the surface f to which $\mathbf{t}_1$ and $\mathbf{t}_2$ should be tangent is

given implicitly, the tangents may be chosen as

$$t_1 = (-f_y, f_x, 0)$$
$$t_2 = (f_z, 0, -f_x)$$

where the subscripts denote partial differentiation. But for points on the intersection $f_x \cap f$ the tangent vectors are not linearly independent, and this causes extraneous solutions. They can be excluded by introducing an additional equation

$$\mathbf{u} \cdot (0, -f_z, f_y) = 0$$

which is not redundant at the regular points of $f \cap f_x$. That is, to express perpendicularity, we include the three equations

$$\mathbf{u} \cdot (-f_y, f_x, 0) = 0$$
$$\mathbf{u} \cdot (f_z, 0, -f_x) = 0$$
$$\mathbf{u} \cdot (0, -f_z, f_y) = 0$$

Then, at every regular point of f, at least two linearly independent tangent directions have been included, so the normal is well-defined.

4. The Details

For each curve and surface discussed above, we prove what extraneous solutions exist and how to eliminate them. The basic strategy in each proof is to assume that the footpoints are fixed and then to consider what points on the defined curve or surface could correspond to those footpoints.

4.1. Offset Curves

Theorem 1 *Let $F(x, y, u, v)$ be the d-offset curve to $C : f(u, v) = 0$ defined by 1–3. Suppose $\hat{x} = (\hat{x}, \hat{y}, \hat{u}, \hat{v})$ satisfies $F(\hat{x}) = 0$. Then, either $\mathbf{p} = (\hat{u}, \hat{v})$ is a regular point of C and $(\hat{x}, \hat{y})$ is one of the two offsets to C corresponding to $\mathbf{p}$, or $\mathbf{p}$ is a singular point of C and $(\hat{x}, \hat{y})$ lies on the circle of radius d centered at $(\hat{u}, \hat{v})$.*

Moreover, the addition of the equation

$$(\alpha f_u - 1)(\alpha f_v - 1) = 0$$

to the system F removes all and only such extraneous solutions.

Proof.

In the system F, fix $u = \hat{u}$ and $v = \hat{v}$, and consider F as a function of x and y. Then $\mathbf{p} = (\hat{u}, \hat{v})$ lies on the base curve, since $f(\hat{u}, \hat{v}) = 0$. Also, if $\mathbf{p}$ is a regular point of f, then at least one of f_u and f_v is not zero. So F becomes the system

$$(x - \hat{u})^2 + (y - \hat{v})^2 - d^2 = 0$$
$$-f_v(x - \hat{u}) + f_u(y - \hat{v}) = 0$$

and neither equation is identically zero. By Bezout's theorem, there are exactly two solutions to this reduced system.

If p is a singular point of C, then both $f_u = 0$ and $f_v = 0$, so the system degenerates to the single equation

$$(x - \hat{u})^2 + (y - \hat{v})^2 - d^2 = 0$$

Thus p corresponds to all points on this circle, and therefore the points on the circle are extraneous solutions.

We now add the equation

$$(\alpha f_u - 1)(\alpha f_v - 1) = 0$$

to the system of equations. If $(\hat{u}, \hat{v})$ is a regular point, then for every solution $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ of the old system, $(\hat{x}, \hat{y}, \hat{u}, \hat{v}, \hat{\alpha})$ is a solution of the augmented system, where $\hat{\alpha} = 1/f_u$ or $\hat{\alpha} = 1/f_v$. Conversely, if $(\hat{x}, \hat{y}, \hat{u}, \hat{v}, \hat{\alpha})$ solves the augmented system, then $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ solves the original system. Now, if $(\hat{u}, \hat{v})$ is a singular point, then $f_u = f_v = 0$, and hence the augmented system has no solution. ⊔

4.2. Offset Surfaces

Theorem 2 *Let $F(x, y, z, u, v, w)$ be the d-offset surface to $S : f(u, v, w) = 0$ defined by 4-7. Suppose $\hat{x} = (\hat{x}, \hat{y}, \hat{z}, \hat{u}, \hat{v}, \hat{w})$ satisfies $F(\hat{x}) = 0$. Let $\mathbf{p} = (\hat{u}, \hat{v}, \hat{w})$. Then one of the following holds:*

1. *p is a regular point of the surface S and $(\hat{x}, \hat{y}, \hat{z})$ is one of the two offsets to S at p*

2. *p is a singular point of S and $(\hat{x}, \hat{y}, \hat{z})$ lies on the sphere of radius d centered at p*

3. *p is a regular point at which the normal as given by F is not well-defined and $(\hat{x}, \hat{y}, \hat{z})$ lies on a circle of radius d centered at p in the plane with normal direction $(0, 1, 0)$.*

The addition of the equation

$$(\alpha f_u - 1)(\alpha f_v - 1)(\alpha f_w - 1) = 0$$

to the system F removes all and only such extraneous solutions as arise in case 2, while the addition of the equation

$$(\beta f_v - 1) = 0$$

removes those associated with case 3.

Proof.

The proof for cases 1 and 2 and for the validity of the equation which removes the extraneous points which arise from case 2 is directly analogous to the proof for curves given above.

For case 3, if the normal is not well-defined, then f_v must be zero. Hence, the last two equations of F are multiples of each other, and therefore degenerate to one equation. Since p is a regular point, at least one of f_u and f_w is not zero, so the equations do not both vanish. Thus, considering F as a system only in (x, y, z), the remaining two equations are

$$(x - \hat{u})^2 + (y - \hat{v})^2 + (z - \hat{w})^2 - d^2 = 0$$
$$y = \hat{v}$$

which clearly is the circle

$$(x - \hat{u})^2 + (z - \hat{w})^2 - d^2 = 0$$

in the plane $y = \hat{v}$.

Now, in order to remove these extraneous points, the points at which $f_v = 0$ must be removed, and only those points. This can be done by the equation

$$(\beta f_v - 1) = 0$$

Another alternative for removing the solutions which arise from case 3 is to include the equation

$$-f_w(x - u) + f_u(z - w) = 0$$

in the original system, which ensures that the normal is well-defined at all regular points. ⊔

4.3. Equal-Distance Curves

In section 2.3, the equal-distance curve between base curves f and g is defined as the intersection of d-offsets, where d is variable. By augmenting equations 8–13 as in Theorem 1, extraneous solutions associated with singularities of f and g are immediately excluded.

In order to classify the types of solutions of the resulting system, we show for which pairs of footpoints on f and g extraneous solutions exist. Moreover, we must prove that for a fixed footpoint on one curve, say f, there are only finitely many footpoints on g which correspond to a solution. We show this assuming that the base curves are algebraic. The nonalgebraic case will be discussed in section 6.

Theorem 3 *Let $F(x, y, u_1, v_1, u_2, v_2, \alpha, \beta, d)$ be the equal-distance curve between $f(u_1, v_1) = 0$ and $g(u_2, v_2)$, where both f and g are algebraic, defined by 8–13*

augmented with

$$(\alpha f_u - 1)(\alpha f_v - 1) = 0 \tag{43}$$

$$(\beta g_u - 1)(\beta g_v - 1) = 0 \tag{44}$$

Then extraneous solutions arise if and only if f and g meet tangentially, and the common normal is the extraneous component. These extraneous solutions can be eliminated by augmenting the system with

$$(\alpha(u_1 - u_2) - 1)(\alpha(v_1 - v_2) - 1)(\alpha(f_{v_1} g_{u_2} - f_{u_1} g_{v_2}) - 1) = 0 \tag{45}$$

Proof.

Since the system is augmented with 43 and 44, we only need to consider regular points of the base curves. Suppose that $p = (\hat{u}_1, \hat{v}_1)$ is a regular point on f. Then equations 9 and 11–13 together define the equal-distance curve C^* between p and g. Since g is algebraic, C^* is algebraic, and by Bezout's theorem, the normal line to f at p either intersects C^* at a finite number of points or is a component of C^*. The latter can occur only if f and g intersect tangentially at p. For all other nonsingular points on f, there are at most finitely many footpoints on g which can correspond to a solution of F.

Now suppose that $q = (\hat{u}_2, \hat{v}_2)$ is a footpoint on g which, with p on f, corresponds to a solution to F. Then, since both points are regular, their normals are well-defined and, if these normals are distinct, they can meet in at most one point. If the normals are identical and $p \neq q$, then the midpoint of $\overline{pq}$ is the only possible solution. However, if $p = q$ and the normals coincide, then every point on the common normal will satisfy the system, and the normal will be extraneous. Equation 45 eliminates these extraneous solutions since it cannot be satisifed at tangential intersections of f and g. ⊔

4.4. Equal-Distance Surfaces

As in the case of equal-distance curves, equal-distance surfaces are defined as the intersection of d-offsets, with d variable. Using Theorem 2, we augment the system 14–21 with equations which eliminate singular points on both base surfaces and with equations which ensure that the surface normals are well-defined at all regular points. Then the only possible degeneracies in the system arise from the interaction between the two subsystems defining the d-offsets of f and g.

Theorem 4 *Let $F(x, y, z, u_1, v_1, w_1, u_2, v_2, w_2, \alpha, \beta, d)$ be the equal-distance surface between $f(u_1, v_1, w_1) = 0$ and $g(u_2, v_2, w_2)$, where f and g are algebraic, defined by 14–21 augmented with*

$$(\alpha f_{u_1} - 1)(\alpha f_{v_1} - 1)(\alpha f_{w_1} - 1) = 0 \tag{46}$$

$$(\beta g_{u_2} - 1)(\beta g_{v_2} - 1)(\beta g_{w_2} - 1) = 0 \tag{47}$$

$$-f_{w_1}(x - u_1) + f_{u_1}(z - w_1) = 0 \tag{48}$$

$$-g_{w_2}(x - u_2) + g_{u_2}(z - w_2) = 0 \tag{49}$$

Then extraneous solutions arise if and only if f and g meet tangentially, and the common normal is extraneous. All such extraneous solutions can be eliminated by the addition of the equation

$$(\alpha(u_2 - u_1) - 1)(\alpha(v_2 - v_1) - 1)(\alpha(w_2 - w_1) - 1)(\alpha D - 1) = 0 \tag{50}$$

where D is given by

$$D = \begin{vmatrix} 1 & 1 & 1 \\ f_{u_1} & f_{v_1} & f_{w_1} \\ g_{u_2} & g_{v_2} & g_{w_2} \end{vmatrix}$$

Proof.

The proof is completely analogous to that for equal-distance curves.⊔

4.5. Constant-Radius Blending Surfaces

Recall that for constant-radius blending surfaces, a point is extraneous if the offset point corresponding to the two footpoints is an extraneous solution of the offsets. We eliminate all such extraneous solutions by defining the blend in terms of offset surfaces with no extraneous solutions, as done in Theorem 2 above. A point is also extraneous if its corresponding footpoints correspond to an entire sphere of solutions.

Theorem 5 *Let $F(x, y, z, u, v, w, u_1, v_1, w_1, \alpha, \beta, u_2, v_2, w_2, \gamma, \delta)$ be the constant-radius blending surface of non-zero radius between $f(u_1, v_1, w_1) = 0$ and $g(u_2, v_2, w_2) = 0$ defined by 22–31 augmented with*

$$(\alpha f_{u_1} - 1)(\alpha f_{v_1} - 1)(\alpha f_{w_1} - 1) = 0 \tag{51}$$

$$(\beta f_{v_1} - 1) = 0 \tag{52}$$

$$(\gamma g_{u_2} - 1)(\gamma g_{v_2} - 1)(\gamma g_{w_2} - 1) = 0 \tag{53}$$

$$(\delta g_{v_2} - 1) = 0 \tag{54}$$

Let $\mathbf{p} = (\hat{u}_1, \hat{v}_1, \hat{w}_1)$ and $\mathbf{q} = (\hat{u}_2, \hat{v}_2, \hat{w}_2)$ be footpoints corresponding to a solution to F, and $\mathbf{m} = (\hat{u}, \hat{v}, \hat{w})$ be the corresponding point on the spine of the blend. Then $\mathbf{m}$ is unique, and an extraneous sphere of solutions arises if and only if $N_f(\mathbf{p}) \times N_g(\mathbf{q}) = (0, 0, 0)$.

The extraneous solutions can be eliminated by augmenting the system with

$$(\eta(f_{v_1}g_{w_2} - f_{w_1}g_{v_2}) - 1)(\eta(f_{w_1}g_{u_2} - f_{u_1}g_{w_2}) - 1)(\eta(f_{w_1}g_{u_2} - f_{u_1}g_{w_2}) - 1) = 0 \tag{55}$$

Proof.

By Theorem 2, the subsystem given by 22–25, 51, and 52 defines two unique offset points to p, and the subsystem given by 26–29, 53, and 54 defines two unique offset points to q. Since for any blend of radius greater than zero there can be no solutions with p = q, the two pairs of points cannot be identical. Hence m is unique.

The system is now reduced to

$$(x - \hat{u})^2 + (y - \hat{v})^2 + (z - \hat{w})^2 - d^2 = 0 \tag{56}$$

$$(x - \hat{u}, y - \hat{v}, z - \hat{w}) \cdot (N_f(\mathbf{p}) \times N_g(\mathbf{q})) = 0 \tag{57}$$

which generates extraneous solutions if and only if 57 vanishes independently of x, y, and z. This can only happen if and only if $N_f(\mathbf{p}) \times N_g(\mathbf{q}) = (0,0,0)$. Adding equation 55 eliminates this possibility, since the equation has a solution just in case at least one component of the cross-product vector is non-zero. ⊔

4.6. Variable-Radius Blending Surfaces

Since the variable-radius blend between f and g is defined in terms of the equal-distance surface between them, we can again immediately reduce the number of extraneous solutions by augmenting the system with the equations necessary to eliminate extraneous solutions in the equal-distance surface. Because we use Theorem 4 in the proof of this theorem, we again must assume that f and g are algebraic surfaces.

Theorem 6 *Let $F(x,y,z,u,v,w,u_1,v_1,w_1,\alpha,u_2,v_2,w_2,\beta,d,\gamma)$ be the variable-radius blending surface of non-zero radius between $f(u_1,v_1,w_1) = 0$ and $g(u_2,v_2,w_2) = 0$, where f and g are algebraic, defined by 32–42 augmented with*

$$-f_{w_1}(x - u_1) + f_{u_1}(z - w_1) = 0 \tag{58}$$

$$(\alpha f_{u_1} - 1)(\alpha f_{v_1} - 1)(\alpha f_{w_1} - 1) = 0 \tag{59}$$

$$-g_{w_2}(x - u_2) + g_{u_2}(z - w_2) = 0 \tag{60}$$

$$(\beta g_{u_2} - 1)(\beta g_{v_2} - 1)(\beta g_{w_2} - 1) = 0 \tag{61}$$

$$(\gamma(u_2 - u_1) - 1)(\gamma(v_2 - v_1) - 1)(\gamma(w_2 - w_1) - 1)(\gamma D - 1) = 0 \tag{62}$$

where D is as in theorem 4. Let $\mathbf{p} = (\hat{u}_1, \hat{v}_1, \hat{w}_1)$ and $\mathbf{q} = (\hat{u}_2, \hat{v}_2, \hat{w}_2)$ be footpoints corresponding to a solution to F, and $\mathbf{m} = (\hat{u}, \hat{v}, \hat{w})$ be the corresponding point on the spine of the blend. Then $\mathbf{m}$ is unique, and an extraneous sphere of solutions arises if and only if $N_h(\mathbf{m}) \times N_V(\mathbf{m}) = (0,0,0)$.

The extraneous solutions can be eliminated by augmenting the system with

$$(\delta T_u - 1)(\delta T_v - 1)(\delta T_w - 1) = 0 \tag{63}$$

where $T = (T_u, T_v, T_w) = N_h(\mathbf{m}) \times N_V(\mathbf{m})$.

Proof.

By Theorem 4, the subsystem given by 32–39 adjoined with 58–62 defines the equal-distance surface between f and g with no extraneous or spurious solutions admitted. Thus for any two footpoints there can be at most one solution to the subsystem, so, since p and q are footpoints corresponding to a solution to F, m is unique.

The system is now reduced to

$$(x - \hat{u})^2 + (y - \hat{v})^2 + (z - \hat{w})^2 - d^2 = 0 \tag{64}$$

$$(\frac{\partial S_d}{\partial u}, \frac{\partial S_d}{\partial v}, \frac{\partial S_d}{\partial w}) \cdot (N_h(\mathbf{m}) \times N_V(\mathbf{m})) = 0 \tag{65}$$

which generates extraneous solutions if and only if 65 vanishes independently of x, y, and z. This can happen if and only if $T = N_h \times N_V = (0,0,0)$, and can be eliminated by adding equation 63 since this equation can only be satisfied when some component of T is non-zero. ⊔

5. Examples

Example 1

We compute the 1-offset of the equation $x^2 - y^2 = 0$, which has a singularity at the origin. The system of equations is

$$u^2 - v^2 = 0$$
$$(x - u)^2 + (y - v)^2 - 1 = 0$$
$$2v(x - u) + 2u(y - v) = 0$$

Using Gröbner basis techniques, the variables u and v are eliminated, producing the equation

$$(x^2 + y^2 - 1)^2(x^2 - 2xy + y^2 - 2)(x^2 + 2xy + y^2 - 2) = 0$$

which has as extraneous solution the circle $x^2 + y^2 = 1$. When the system is augmented with

$$(-2u\alpha - 1)(2v\alpha - 1) = 0$$

the extraneous component vanishes. Gröbner basis elimination now yields

$$(x^2 - 2xy + y^2 - 2)(x^2 + 2xy + y^2 - 2) = 0$$

Example 2

We compute the equal-distance curve between a parabola and a line which meet tangentially at the origin. The system of equations is

$$u_1 - v_1^2 = 0$$

$$(x - u_1)^2 + (y - v_1)^2 - d^2 = 0$$
$$2v_1(x - u_1) + (y - v_1) = 0$$
$$u_2 = 0$$
$$(x - u_2)^2 + (y - v_2)^2 - d^2 = 0$$
$$y - v_2 = 0$$

When the variables u_1, v_1, u_2, v_2, and d are eliminated from this system of equations, the closed-form solution obtained is

$$y^2(16y^4 - 32x^2y^2 - 40xy^2 + y^2 + 16x^4 - 24x^3 + 12x^2 - 2x) = 0$$

As expected, the normal line at the origin, $y = 0$, appears as an extraneous factor in the solution. When we augment the system with the equation

$$(\alpha(u_1 - u_2) - 1)(\alpha(v_1 - v_2) - 1)(\alpha 2v_1 - 1) = 0$$

the extraneous factor is eliminated. Elimination of u_1, v_1, u_2, v_2, d, and α using Gröbner basis techniques now yields

$$16y^4 - 32x^2y^2 - 40xy^2 + y^2 + 16x^4 - 24x^3 + 12x^2 - 2x = 0$$

6. Conclusion

The goal of this work is to clarify the relationship between conceptual geometric curve and surface operations on the one hand, and the translation of the operations into a system of nonlinear equations on the other. Approaching this goal requires giving an exact definition of the geometric intent of every curve and surface operation, and we have done this indirectly through our definition of extraneous solution. That is, we have characterized what is *not* the geometric intent.

All curve and surface operations we have considered require formalizing the minimum distance of a point from a curve or surface. We have consistently used a local distance function, where p has minimal distance from f at q on f if the line $\overline{p, q}$ is perpendicular to the tangent (plane) to f at q. If a global distance function were used, inequalities and quantification would be necessary, and it would no longer be possible to express the surface operation equationally.

In most applications, our definition of extraneous solution is intuitively correct. However, it is not wholly satisfactory in all situtaions: the translation into equations does not require any assumptions about the base curves and surfaces other than that they must be once continuously differentiable. For offset curves and surfaces, and for constant-radius blending surfaces which are based on offsets, this is sufficient. However, for equal-distance curves and surfaces, and for variable-radius blends which are based on equal-distance surfaces, we have imposed the additional requirement that the base curves and surfaces be algebraic (Theorems 3, 4, and 6).
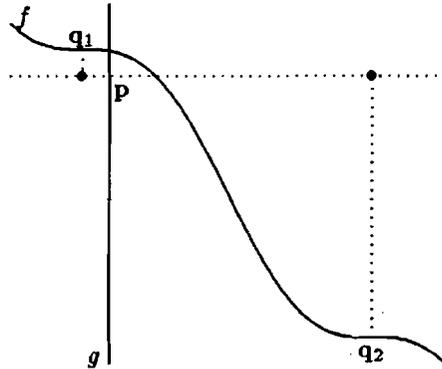
Fig. 2. p corresponds to infinitely many points on the equal-distance curve between f and g, with footpoints q_k.

This requirement is necessary because for nonalgebraic base curves and surfaces, there may be a point on one base component which corresponds to infinitely many geometrically valid solutions. According to our definition of extraneous, all of those solutions would be invalid.

For example, consider the equal-distance curve between f: $\cos(x) - x - y - 1 = 0$ and g: $x - 1 = 0$. At values of x where $\cos(x) = 0$, f has the solution $(x, -x - 1)$. The normal to f is $(-\sin(x) - 1, -1)$, which is a vertical line at

$$x = \ldots, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \cdots$$

So at these values of x, the point $(x, 0)$ is at distance $x + 1$ from both f and g. According to our definition, all of these solutions are extraneous, because they all correspond to the same point $p = (-1, 0)$ on g. However, because we use a local distance function, they are geometrically meaningful. See Figure 2.

Clearly, our techniques for eliminating extraneous solutions apply more broadly than to algebraic base curves and surfaces alone. Thus far we have not found a concise definition of extraneous solution that subsumes the definition given here and extends Theorems 3, 4 and 6 to the nonalgebraic case at the same time. With such a definition, our technique for faithfully representing these curve and surface operations achieves full generality.

Acknowledgements

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