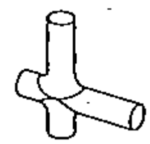

The Potential Method for Blending Surfaces and Corners

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Abstract. The potential method for blending implicit algebraic surfaces is summarized, summarizing and extending work previously reported. The method is capable of deriving blends for pairs of algebraic surfaces, and is guaranteed to produce blending surfaces of lowest possible degree for two quadrics in general position.

Two paradigms are given by which to understand the method. The first paradigm views the blends as surfaces swept out by a family of space curves. The second, more general paradigm considers the surfaces as a result of deformation of a parameter space effected by substitution. The method has a general formulation based on projective parameter spaces.

The deformation by substitution paradigm is extended to blend blending surfaces at solid vertices without a degree penalty, under the assumption that the vertex valence has been reduced to three. It may also lead to a general solution for blending patches of algebraic surfaces that meet tangentially. A special case of this problem is solved and illustrated.

1. Introduction. Mechanical parts have primary surfaces whose shapes are functionally important and secondary surfaces whose purpose it is to smoothly connect the functional, primary surfaces. Usually, the secondary surfaces are only approximately specified in the engineering drawings of the part, and their exact shape, within bounds, is irrelevant. Surfaces of this type are called *blending surfaces* or *blends,* and we consider here how to derive them in a simple and systematic manner.

Specifying a blend for a computer based geometric modeling system should be simple; after all, the surfaces need to conform only approximately to the designer's shape requirements. Yet this is not the case. The principal difficulty is in shaping and positioning these surfaces so as to achieve tangency to the primary surfaces. As a consequence, blending surfaces have a higher algebraic degree, and are mathematically more complicated to derive than the primary surfaces they connect. Thus, a long term goal of our research into the existence and properties of blending surfaces is automating the derivation of blending surfaces, by computer, from the adjacent primary surfaces and a few, spatially intuitive parameters such as width and approximate curvature.

This paper surveys the mathematical aspects of deriving blending surfaces, given the primary surfaces to which tangency is to be achieved. Throughout, the surfaces are assumed to be algebraic, and are specified by their implicit equation, i.e., by an equation $F=0$, where F is a polynomial in x, y, and z. Specifically, we address the *potential* method, introduced in Hoffmann and Hopcroft [1], [2], which has the advantage of deriving, in a very simple manner, a rather extensive and flexible class of blending surfaces of low degree. Other work on blending implicit algebraic surfaces is described in Middleditch and Sears [3] and in Rockwood and Owen [4], and these approaches are related to the potential method in § 3.

Although the blending problem has been formulated in terms of surfaces, the techniques can be integrated into a constructive solid geometry (CSG) based modeling system. Even Rockwood and Owen [4], reporting on work intended for a new version of the boundary representation based modeling system ROMULUS, was first experimentally tested in a CSG based modeler according to A. Rockwood.

Throughout our work, we stress the importance of obtaining blending surfaces of low degree. This is a practical consideration. Both the size of the surface representation, as well as the difficulty encountered by e.g., root finding algorithms, grow quickly with increasing degree. Fortunately, we not only derive low degree blends, we can also show rigorously that they are of the lowest degree possible for quadratic surfaces in general position.

This paper describes work in progress. Consequently, a number of results have a preliminary character, and topics in need of further investigation are indicated throughout. We have attempted to portray the material in as intuitive a way as possible, hoping to make it accessible to a large audience. The Uniqueness Theorem has a very technical proof that draws on algebraic geometry and the theory of ideals, and is omitted here. The interested reader is referred to Hoffmann and Hopcroft [2] for a complete derivation. However, many of the technical aspects are readily accessible to the nonspecialist, and where this is the case, we did not hesitate to go into full detail.

Sections 2–4 deal with blending two surfaces. Although written for intersecting surfaces, one can also blend nonintersecting surfaces. In the case of quadrics, the corollary to the Uniqueness Theorem (§ 4) explains how this can be done. Sections 5–7 deal with blending corners and patched algebraic surfaces, a comparatively less developed area. Here the main point is that special geometric properties of quadrics can be lifted to higher degree algebraic surfaces, by a simple intuitive approach.

2. Blending Two Intersecting Surfaces. We explain a method for blending two intersecting algebraic surfaces, initially developed in Hoffmann and Hopcroft [1], called the *potential method*. All polynomials are assumed to be in x, y and z, with real coefficients, unless stated otherwise. To avoid confusion, we distinguish between the polynomial F and the surface $S(F)$ whose implicit equation is $F=0$. In general, the intersection of two surfaces $S(G)$ and $S(H)$ is a space curve denoted $S(G, H)$, and the intersection of three surfaces is a set of points denoted $S(G, H, K)$. Note that the point $(1, 2, 3)$ can also be written as the intersection of three planes $S(x-1, y-2, z-3)$.

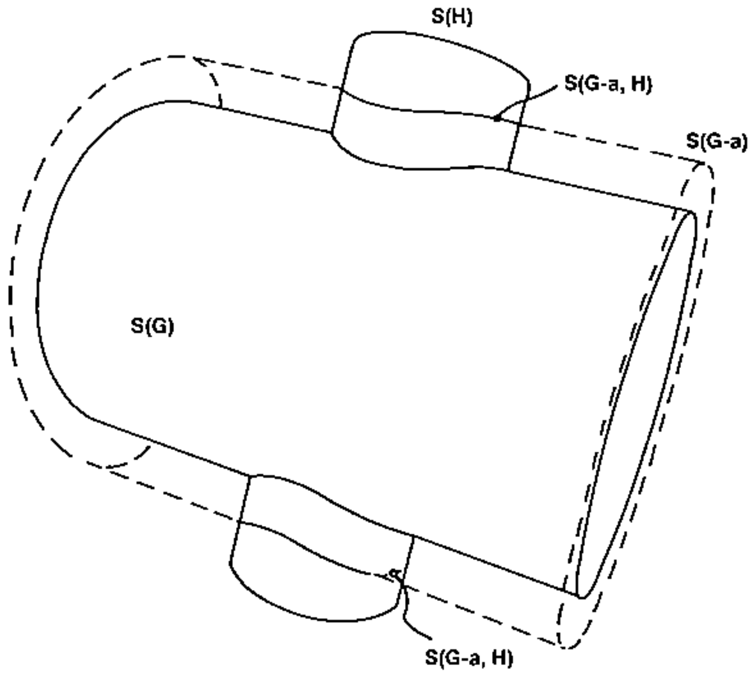

Figure 2.1. The curve $S(G-a, H)$ where the blending surface meets $S(H)$ tangentially.

A surface $S(F)$ partitions space into three sets: all points (a, b, c) such that $F(a, b, c) > 0$, called the *outside* of $S(F)$; all points such that $F(a, b, c) < 0$, called the *inside* of $S(F)$; and all points such that $F(a, b, c) = 0$, i.e., $S(F)$. Note that in general the inside of $S(F)$ is equal to the outside of $S(-F)$, and vice versa.

2.1. The affine method. Given the surface $S(G)$, we define a family of surfaces, parameterized by s, via $G - s = 0$. For a particular value of $s \neq 0$, $S(G-s)$ is always entirely on the inside or the outside of $S(G)$, depending on the sign of s. For example, consider a circular cylinder $S(G)$. Then G may always be chosen such that for positive s, $S(G-s)$ is a circular cylinder of larger radius. In a like manner the family $S(H-t)$ of surfaces based on the surface $S(H)$ is defined. For our example we pick $S(H)$ as another cylinder. A maximum radius is fixed by picking a constant a and intersecting $S(G-a)$ with $S(H)$, as shown in Fig. 2.1. We choose a such that $S(G-a)$ intersects $S(H)$ in a nondegenerate space curve $S(G-a, H)$. The space curve $S(G-a, H)$ is associated with the point $(a, 0)$ in s-t parameter space. Now reduce the value of s from a to 0, while simultaneously increasing the value of t from 0 to some other constant b. Each intermediate pair (u, v) of s-t values corresponds to a space curve $S(G-u, H-v)$. If the values for s and t lie on a curve $f(s, t) = 0$, then the corresponding space curves lie on a surface whose equation is $F(x, y, z) = f(G, H) = 0$. Figure 2.2 shows the correspondence of an arc of the curve f to a segment of the surface $S(F)$. Here the radius of $S(G)$ is 8, and the radius of $S(H)$ is 4.

As shown in Hoffmann and Hopcroft [1], if $f = 0$ is tangent to the s-axis at $(a, 0)$ then $S(F)$ is tangent to $S(H)$ in the curve $S(G-a, H)$. Likewise, tangency of $f = 0$ to

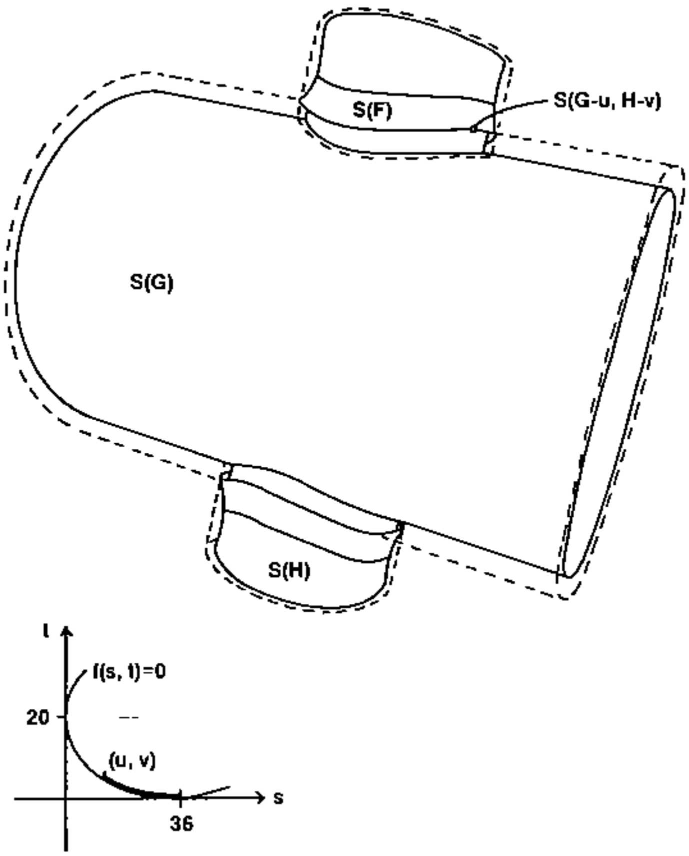

Figure 2.2. Correspondence of an arc of the parameter curve to a part of the blending surface.

the t-axis at $(0, b)$ implies tangency of $S(F)$ to $S(G)$ in $S(G, H-b)$. Higher order continuity of f with the coordinate axes gives higher order continuity of the surfaces.

Essentially, the procedure just described is the simple version of the potential method and we call it the *affine* potential method. It is not fully general, as we shall see below.

Although all subsequent examples concentrate on blending quadrics, the method applies to blending arbitrary algebraic surfaces. However, for higher degree surfaces the intrinsic surface geometry is more complicated. For example, $S(G-a)$ may split into components. While the method is robust and very intuitive for many surfaces, and especially for all quadrics, much exploration of its general behavior needs to be done.

2.2. Significance of the parameters when blending with a conic. The above procedure does not depend on the degrees of G, H and f, but as we are interested in low degree blending surfaces, we choose f of as low a degree as possible, i.e., as a conic. With the required tangency conditions f can be written as

$$f(s, t) = b^2s^2 + a^2t^2 + a^2b^2 - 2ab^2s - 2a^2bt + 2\lambda st.$$

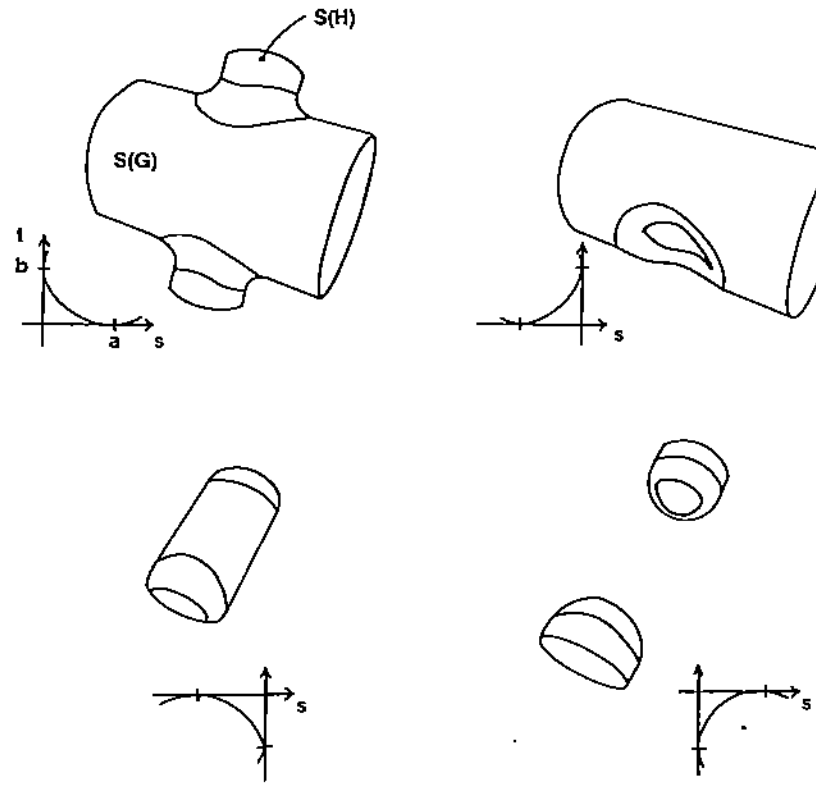

Figure 2.3. Positional correspondence of the blending surface and the parameter curve.

The corresponding blending surface $S(F)$ is seen to possess degree $\max(2m, 2n)$, assuming that G has degree m and H has degree n. For quadrics, therefore, we obtain quartic blending surfaces.

In using a conic for blending, the points of tangency may be positioned by choosing a and b. Note that the signs of a and b determine in which quadrant the conic lies. The choice of λ determines the actual conic. We now explain how these choices affect the blend using as an example two circular cylinders whose axes intersect at right angles.

If a is positive, then $S(G-s)$ is on the outside of $S(G)$. Accordingly, the surface $S(F)$ is on the outside of $S(G)$. If a is negative, however, $S(F)$ must lie on the inside of $S(G)$. Similarly, $S(F)$ must lie on the outside or the inside of $S(H)$ depending on whether b is positive or negative. The quadrant positions of f, and the corresponding blending surface positions are illustrated in Fig. 2.3.

Moving $(a, 0)$ further away from the origin moves the curve of tangency $S(G-a, H)$ further away from the intersection curve $S(G, H)$ of the surfaces being blended. Similarly, the curves $S(G, H)$ and $S(G, H-b)$ are further apart when the magnitude of b is enlarged. There is no simple relationship between the magnitude of, say, a and the (mean) Euclidean distance of $S(G-a, H)$ to $S(G, H)$. In the case of quadrics, Middleditch and Sears [3] give a method for the affine potential method.

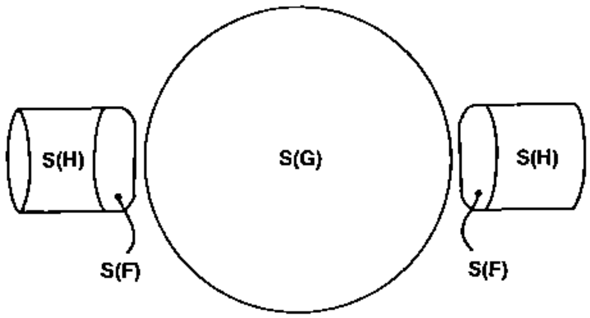

Figure 2.4. Failure to blend due to improper choice of a and b.

For certain (s, t)-values the space curve $S(G-s, H-t)$ may degenerate to a point or vanish in real three-space. Accordingly, $S(F)$ may *miss* one of the surfaces or be disconnected in real three-space, although the latter phenomenon does not happen for intersecting quadrics. For example, consider the two cylinders $G = x^2 + y^2 - 8^2 = 0$ and $H = y^2 + z^2 - 4^2 = 0$. We choose $a = 57$ and $b = -34$, and blend with $\lambda = 0$. The resulting surface is shown in Fig. 2.4, where the gap between the blending surface and $S(G)$ is approximately 0.5. The problem here is that $S(H-b)$ does not have real points, and so does not intersect $S(G)$ in a space curve. In the case of blending quadrics it is easy to avoid this situation, but little is known about it in the case of blending higher order surfaces.

The type of conic chosen for f is determined by λ. The important values and the resulting curve shapes are summarized below and illustrated in Fig. 2.5:

$\lambda = -\infty$,	a pair of lines, $s = 0$ and $t = 0$,
$-\infty < \lambda < -ab$,	hyperbola,
$\lambda = -ab$,	parabola,
$-ab < \lambda < ab$,	ellipse; a circle if $a = b$ and $\lambda = 0$,
$\lambda = ab$,	the line $bs + at - ab$, counted double.

Figures 2.6, 2.7 and 2.8 (see color insert) show the shape of the resulting blends for $G = y^2 + z^2 - 9$ and $H = x^2 + y^2 - 1$ with $a = 7$ and $b = 3$. In Fig. 2.6, λ is 20, just a little under the critical value of ab at which the surface would degenerate into the (ellipsoid) $S(bG + aH - ab)$. In Fig. 2.7, λ is 0 and in Fig. 2.8, λ is -750 and thus its magnitude is large compared to ab. The blending surface visibly begins to approximate the other degeneracy, namely the union of the two cylinders. We see that λ controls the distribution of curvature of the cross-sections of the blend.

Only a portion of the surface $S(F)$ is used in blending; the portion of $S(F)$ that corresponds to the arc of $f = 0$ lying on the inside of the line $bs + at - ab = 0$. This is the part of $S(F)$ that lies on the inside of the surface $S(bG + aH - ab)$, an ellipsoid in our example. In the illustrations all blending surfaces have been clipped accordingly.

2.3. Parameter space. We have conceptualized the blending surface as swept out by curves of intersection of two families of surfaces, controlled by a curve in

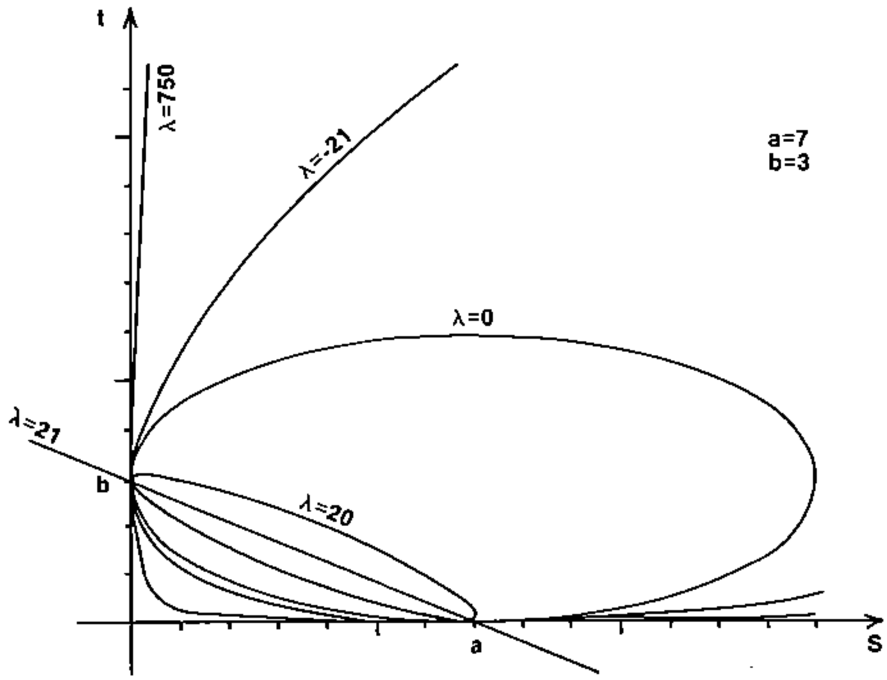

Figure 2.5. Family of quadric parameter curves for controlling the curvature of the blending surface.

two-dimensional parameter space. A different conceptualization is possible by considering a three-dimensional parameter space. The curve f is now replaced by a conic cylinder. Note that this cylinder is a blending surface for the planes $S(s)$ and $S(t)$, as shown in Fig. 2.9.

The Cartesian coordinate system of this parameter space is based on the three principal planes, $S(r)$, $S(s)$, and $S(t)$, that intersect pairwise in the three coordinate axes. Every line parallel to the plane $S(r)$ is the intersection of the planes $S(s-u)$ and

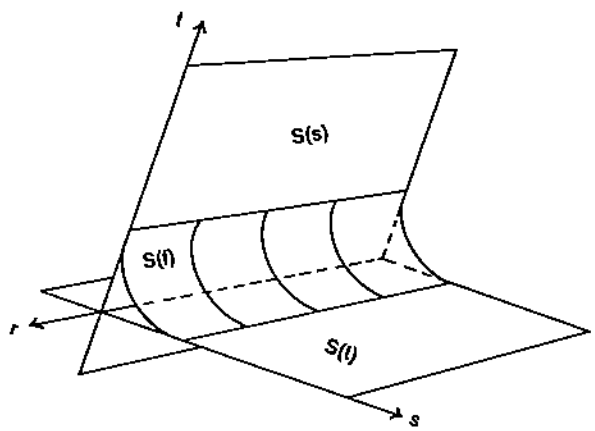

Figure 2.9. Three-dimensional parameter space configuration for blending two surfaces.

$S(t-v)$. Suppose we replace the planes $S(s)$ and $S(t)$ with the curved surfaces $S(G)$ and $S(H)$, where G and H are polynomials in x, y, and z. We view these surfaces as two of three principal surfaces defining a curved coordinate system of xyz-space, without regard to whether in this system there is a one-to-one correspondence between coordinate values and points in Euclidean space. Then to the line $S(s-u, t-v)$ corresponds in general the space curve $S(G-u, H-v)$. Moreover, if the line $S(s-u, t-v)$ lies on the cylinder $S(f(s, t))$ in parameter space, then the curve $S(G-u, H-v)$ lies on the surface $S(f(G, H))$ in xyz-space.

Just as the surface $S(f)$ is tangent to $S(s)$ and $S(t)$ in parameter space, so $S(F)$ is tangent to $S(G)$ and $S(H)$ in xyz-space. Specifically, since $S(f)$ is tangent to $S(s)$ in the curve $S(s, t-b)$, so $S(F)$ is tangent to $S(G)$ in $S(G, H-b)$. Likewise, $S(F)$ is tangent to $S(H)$ in $S(G-a, H)$ since $S(f)$ is tangent to $S(t)$ in $S(s-a, t)$. In parameter space, only the portion on the inside of the plane $S(bs+at-ab)$ is of interest as blend. Similarly, in xyz-space, only the portion inside $S(bG+aH-ab)$ is of interest.

We have just described an equivalent blending procedure in which, by substitution of G for s and H for t, the rst coordinate system is replaced by a curved coordinate system with two of its principal surfaces being $S(G)$ and $S(H)$. We can think of this process as a warping of space in which the blend $S(f)$ is deformed into the corresponding blend $S(F)$. Note that this notion of deformation cannot be thought of as a continuous process as in, e.g., topology, since $S(f)$ and $S(F)$ may have different genera. Nevertheless, the paradigm is useful when blending corners of solids, and may lead to a successful procedure for blending patches of implicit algebraic surfaces. In particular, it can be thought of as reducing the blending of algebraic surfaces to blending planes. From now on we shall drop the sweep paradigm in favor of the substitution paradigm.

2.4. The projective method. The affine formulation of the potential method given above is not fully general. Consider blending a circular cylinder $S(G)$ with a sphere $S(H)$, say $G=x^2+z^2-4$ and $H=x^2+y^2+(z-3)^2-1$. No matter how a is chosen, the curve of tangency $S(G-a, H)$ lies on $S(G-a)$, a concentric cylinder. Therefore, the oblique blend shown in Fig. 2.10 (see color insert) cannot be derived by the affine method. It is, however, a quartic surface obtained from f by substitution, using the *projective* potential method.

In the projective potential method, the intersecting families of surfaces are defined as $G-sW=0$ and $H-tW=0$, where W is a polynomial that may be chosen arbitrarily, but must not be the zero polynomial. Again, for degree consideration, we will choose W to have at most degree 2 when blending quadrics. The difference between the two methods is merely that in the affine method W is chosen as 1.

The blend of Fig. 2.10 is obtained by substituting G/W for s and H/W for t in

$$f(r, s, t) = b^2s^2 + a^2t^2 + a^2b^2 - 2ab^2s - 2a^2bt + 2\lambda st,$$

where $a=b=1$, $\lambda=\frac{1}{2}$ and $W=x^2+z^2+2y/3-2z+8/9$. Note that the procedure is equivalent to substituting G for s, H for t, and W for w in the homogeneous form

$$f(r, s, t, w) = b^2s^2 + a^2t^2 + a^2b^2w^2 - 2ab^2sw - 2a^2btw + 2\lambda st.$$

Certain projective blending surfaces derived in this way are the *projective transform* (see, e.g., Snyder and Sisam [7]) of affine blending surfaces for different surfaces G and H. Every blending surface derived from f using $W = U^2$, where U is some linear form, is the projective transform of another blending surface derived with the affine potential method.

3. A Uniqueness Theorem. We have outlined a method for deriving blending surfaces which for quadric surfaces obtains degree 4 blending surfaces. It is natural to ask how general this method is, and how it relates to other blending methods proposed in the literature, e.g., Middleditch and Sears [3], Rockwood and Owen [4] and Rossignac and Requicha [5]. The Uniqueness Theorem provides a comprehensive, but not complete, answer to this question. We develop the theorem informally, since an exact formulation requires a fair number of concepts from algebraic geometry. The interested reader is referred to Hoffmann and Hopcroft [2] for complete details and the proof of the theorem.

UNIQUENESS THEOREM. *Given two quadrics, $S(G)$ and $S(H)$, mark on each a space curve by intersecting $S(G)$ with an auxiliary quadric surface $S(H')$, and by intersecting $S(H)$ with a second auxiliary quadric surface $S(G')$. All degree 4 surfaces $S(F)$ that are tangent to $S(G)$ in the curve $S(G, H')$, and are tangent to $S(H)$ in the curve $S(H, G')$ may be derived from f using the (projective) potential method.*

In the case of the affine potential method the auxiliary surfaces are given by $G' = G - a$ and $H' = H - b$. In § 4, we replace these two auxiliary surfaces with a single, common one.

The Uniqueness Theorem has a number of hypotheses that must be satisfied. Expressed in intuitive terms, these are:

(1) The curves $S(G, H)$, $S(G, H')$ and $S(H, G')$ are not the union of algebraic curves of lower degree and are all distinct;

(2) $S(F)$ is not the union of algebraic surfaces of lower degree;

(3) the quadratic terms in G and H do not possess a common factor.

A thorough discussion of these hypotheses can be found in Hoffmann and Hopcroft [2].

In Middleditch and Sears [3] a blending method has been proposed that blends two quadrics with a degree 4 surface. Because of the Uniqueness Theorem, we know that the method is no more powerful than the potential method. In fact, it is a formulation of the affine potential method.

In Rockwood and Owen [4] a blending method has been proposed which derives blending surfaces as a function of G, H, and their gradient functions. For arbitrary quadrics, blending surfaces of degree 8 are obtained, but when blending cylinders and spheres term cancellation takes place and surfaces of degree 4 are obtained. Because of the above theorem, those degree 4 surfaces could equally well have been derived with the potential method, i.e., the gradient functions are not used in an essential way for those surfaces.

Suppose a blending method is sought that is to deliver degree 4 surfaces of constant curvature for blending quadrics. Because the surfaces obtained by the potential method do not possess constant curvature, the theorem states that this project must fail. Higher algebraic degrees are needed. Note, however, that such surfaces can be approximated in various ways, i.e., Rossignac and Requicha [5] and Rockwood and Owen [4].

In Middleditch and Sears [3] a blend is shown for two axially intersecting circular cylinders of equal radius. The blending surface shows a bulge. By the sweep paradigm of the affine potential method, the bulge seems unavoidable. Here it is not possible to draw conclusions from the theorem: The curve of intersection of the cylinders is reducible to two ellipses, a violation of hypothesis (1) above. Indeed, in Warren [8] a degree 4 blend without a bulge is given for precisely this case, consisting of the reducible surface $S(F)$ that is the union of two one-sheeted hyperboloids. The surface may also be derived with the projective potential method.

4. The Projective Potential Method for Quadrics. In the projective form of the potential method, the constants a and b no longer have the direct interpretation given to them in § 2.2. A different approach to controlling the blending surface is needed and is provided by the following corollary that is a consequence of the proof of the Uniqueness Theorem:

COROLLARY. *Given two quadrics $S(G)$ and $S(H)$. There is a degree 4 blending surface $S(F)$ tangent to both $S(G)$ and $S(H)$ if and only if the respective curves of tangency lie on a common quadric $S(\bar{G})$. Moreover, every such surface is derivable from the potential method.*

What is this surface $S(\bar{G})$? Recall that the curves of tangency of $S(f)$ to $S(s)$ and $S(t)$ are $S(s-a, t)$ and $S(s, t-b)$, in parameter space. The plane through these two lines is given by $bs + at - ab = 0$. Hence the surface $\bar{G}$ is just $S(bG + aH - ab)$ in the affine formulation, and $S(bG + aH - abW)$ in the projective formulation of the method. In § 2.2 this surface was used for clipping the unwanted parts of the blending surface.

As it were, the constants a and b may be replaced by 1, as W assumes their role. The projective quadric into which to substitute is then given by

$$f(r, s, t, w) = (s-w)^2 + (t-w)^2 - w^2 + 2\lambda st.$$

Given the quadrics $S(G)$ and $S(H)$ to be blended, we pick a quadric surface $S(\bar{G})$ such that it intersects $S(G)$ and $S(H)$ in the desired curves of tangency. We determine W from $\bar{G} = G + H - W$ and so obtain, by substitution into f above, the one-parameter family of blending surfaces given by

$$F = \mu GH + \bar{G}^2$$

where $\mu = 2\lambda - 2$. Here λ retains its previous interpretation as the parameter controlling the curvature distribution.

While this procedure is satisfactory mathematically, it does pose difficulties for automating the choice of blends, because the determination of $\bar{G}$ is not simple. More work is needed to give this method the practicality that its flexibility deserves.

5. Corner Blending. Edges between two faces of an object end at vertices. If one or more of the incident edges have to be smoothed by blends, then one must terminate a blend or smoothly combine several blends meeting at the vertex. Both problems have received attention in the literature, but much work remains to be done. Middleditch and Sears [3] and Rossignac and Requicha [5] can provide an entry into this problem.

Consider the problem of combining three joining blends at a vertex. In principle, a solution to this problem can be adapted to an arbitrary number of meeting blends, after reducing the vertex valence to three with the help of an auxiliary surface snubbing the vertex. Suppose we combine three blends as follows: first, blend two of them with a new blending surface; then, blend this new blending surface with the remaining third blend. The difficulty with this approach is that in each step the surface degree is doubled. Hence, if the three edge blends are degree 4 each, the two additional surfaces have degree 8 and 16, respectively. We seek alternatives which do not drive up the degree of the combining blending surfaces.

The general approach taken to obtain low-degree corner blends first solves the problem in parameter space with planes as primary surfaces and quadrics as their blending surfaces. Then this solution is lifted to the vertex of the solid at hand, by substitution. In this manner, any corner of three quadrics will be blended entirely by degree 4 surfaces. Of course, the method is not limited to quadrics as primary surfaces.

Recall the interpretation of quadrant position of the parametric base curve, as shown in Fig. 2.3. The blending surface is on the outside of $S(G)$ when $a>0$, and on the inside of $S(G)$ when $a<0$. Similarly, it is on the outside of $S(H)$ when $b>0$, and on its inside when $b<0$. In blending a three edge corner, we first examine on which side of the adjacent faces the edge blending surfaces are. Two generic cases arise:

(1) For every face of the vertex, the two adjacent edge blending surfaces are always on the same side, i.e., always on the outside or always on the inside.

(2) There is one face whose two adjacent edge blends are on the same side, and two faces such that the adjacent edge blends are on opposite sides of the same face.

No other cases are possible at vertices with three edges.

Throughout this section, we use only the affine potential method. Additional work is required to extend the techniques given here to the general potential method.

5.1. Adjacent blends always on the same face side. Assume that the corner is formed by the surfaces $S(G)$, $S(H)$, and $S(K)$. The generic situation is shown in Fig. 5.1. In parameter space, the three faces meeting at the vertex are modeled by the three principal coordinate planes, $S(r)$, $S(s)$ and $S(t)$, and the vertex is represented by the origin. In the Fig. 5.1 we have assumed that the edge blends are on the outside of every face. If this is not so, i.e., if the edge blends adjacent to the face $S(G)$ are on the inside of $S(G)$, we simply substitute $-G$ for r. This is equivalent to reformulating Fig. 5.1 in the second octant.

As an example, we blend the edges parametrically with circular cylinders, and combine them at the vertex with a sphere. The respective equations are the following:

$$\begin{aligned}
&\text{B1: } (r-1)^2+(s-1)^2-1=0,\\
&\text{B2: } (s-1)^2+(t-1)^2-1=0,\\
&\text{B3: } (t-1)^2+(r-1)^2-1=0,\\
&\text{B4: } (r-1)^2+(s-1)^2+(t-1)^2-1=0.
\end{aligned}$$

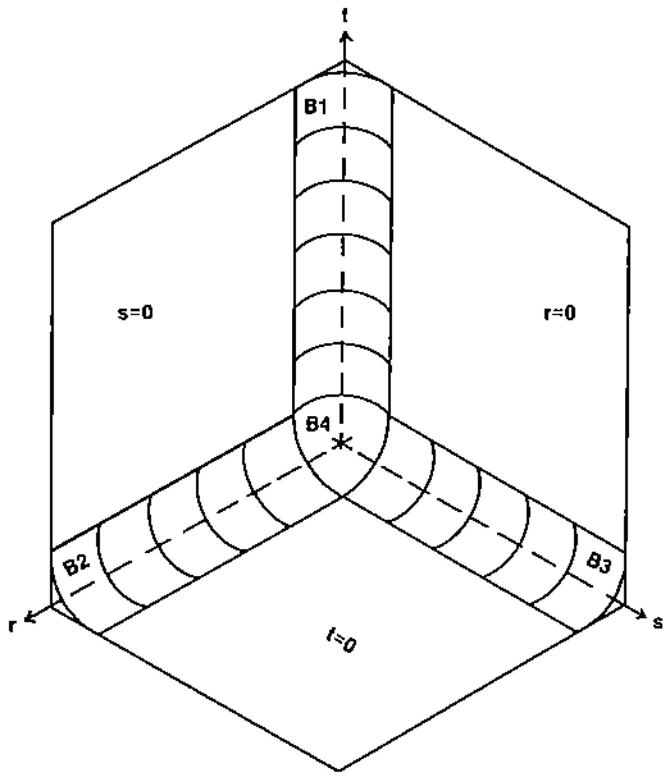

Figure 5.1. Parameter space configuration for vertex blending when adjacent blends are all on the same surface side.

Clipping the unwanted parts is accomplished for each prototype blend by retaining those parts that are on the *inside* of the following planes:

$$\text{C1: } r+s-1=0 \quad \text{and} \quad 1-t=0,$$

$$\text{C2: } s+t-1=0 \quad \text{and} \quad 1-r=0,$$

$$\text{C3: } t+r-1=0 \quad \text{and} \quad 1-s=0,$$

$$\text{C4: } r-1=0 \quad \text{and} \quad s-1=0 \quad \text{and} \quad t-1=0.$$

In order to blend three intersecting cylinders, given by $G=x^2+y^2-1$, $H=y^2+z^2-1$, and $K=z^2+x^2-1$, we substitute G, H, and K for r, s, and t, respectively, and obtain

$$\text{B1}': \ (G-1)^2+(H-1)^2-1=0,$$

$$\text{B2}': \ (H-1)^2+(K-1)^2-1=0,$$

$$\text{B3}': \ (K-1)^2+(G-1)^2-1=0,$$

$$\text{B4}': \ (G-1)^2+(H-1)^2+(K-1)^2-1=0.$$

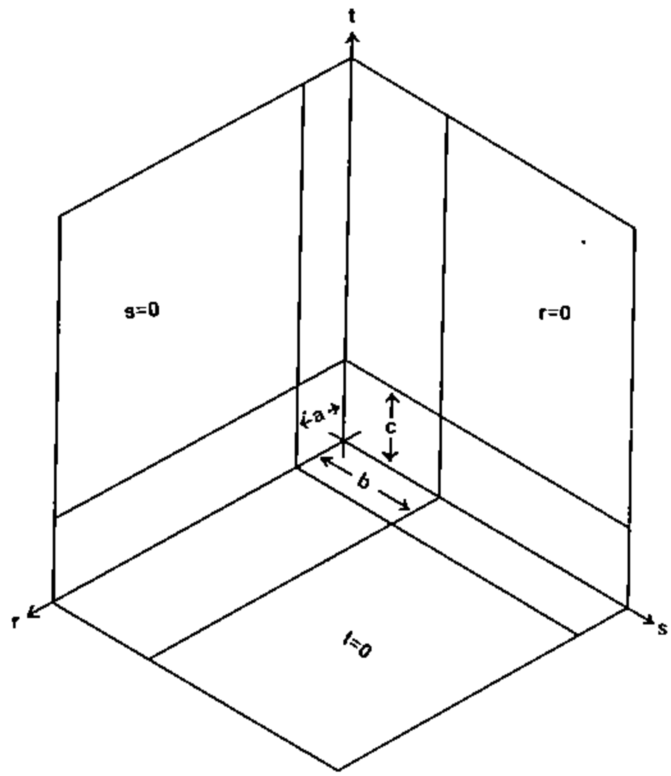

Figure 5.4. Parameters for controlling tangency position when blending vertices of valence three.

These surfaces need to be clipped. This is accomplished just as in parameter space, by retaining those parts which are on the *inside* of the following surfaces, given by:

$$\begin{aligned}
&\text{C1}'\text{:}\ G + H - 1 = 0 \quad \text{and} \quad 1 - K = 0,\\
&\text{C2}'\text{:}\ H + K - 1 = 0 \quad \text{and} \quad 1 - G = 0,\\
&\text{C3}'\text{:}\ K + G - 1 = 0 \quad \text{and} \quad 1 - H = 0,\\
&\text{C4}'\text{:}\ G - 1 = 0 \quad \text{and} \quad H - 1 = 0 \quad \text{and} \quad K - 1 = 0.
\end{aligned}$$

The resulting object is shown in Fig. 5.2 (see color insert). Here, the blends B1′ through B3′ are shown in light blue, and the blend B4′ is shown in purple. All surfaces meet C^1-continuously. Since three quadrics in general intersect in eight points, the surface B4′ has eight real components, as seen in Fig. 5.3 (see color insert). An interesting aspect of the method is that if the edge blends B1–B3 do not intersect after substitution, then the surface B4′ is either clipped away entirely or becomes imaginary.

What flexibility does this method offer? With the affine potential method, the width of the three blends is controlled by six constants a_k and b_k, but three are now dependent, so there remain exactly three independent constants, a, b, and c. Figure 5.4 shows the lines of tangency positioned in parameter space as a function of a, b, and c. Moreover, there is only one free parameter λ controlling the curvature distribution of all three edge blends simultaneously. The generic formulas are

$$\begin{aligned}
&\text{B1: } b^2r^2 + 2ab\lambda rs + a^2s^2 - 2b^2ar - 2a^2bs + a^2b^2 = 0,\\
&\text{B2: } c^2r^2 + 2ac\lambda rt + a^2t^2 - 2c^2ar - 2a^2ct + a^2c^2 = 0,\\
&\text{B3: } c^2s^2 + 2bc\lambda st + b^2t^2 - 2c^2bs - 2b^2ct + b^2c^2 = 0,\\
&\text{B4: } a^2b^2c^2(r^2/a^2 + s^2/b^2 + t^2/c^2 + 2(1+\lambda)(1 - r/a - s/b - t/c))\\
&\qquad + 2\lambda abc(rsc + rbt + ast) = 0
\end{aligned}$$

and the clipping planes are given by

$$\begin{aligned}
&\text{C1: } br + as - ab = 0 \quad \text{and} \quad c - t = 0,\\
&\text{C2: } cr + at - ac = 0 \quad \text{and} \quad b - s = 0,\\
&\text{C3: } cs + bt - bc = 0 \quad \text{and} \quad a - r = 0,\\
&\text{C4: } r - a = 0 \quad \text{and} \quad s - b = 0 \quad \text{and} \quad t - c = 0.
\end{aligned}$$

To blend a corner with a surface having the same degree as the edge blending surfaces, the shaping parameters must be coordinated in this way. More work is required to study if the projective potential method offers greater flexibility and permits, for example, to control edge blend curvature independently.

5.2. Edge blends on opposite sides. The other case to consider is a vertex, two of whose faces have their adjacent edge blending surfaces on opposite sides. Again, we may have to substitute negated face equations depending on the position of the edge blends. This will be necessary in the example below.

The generic situation is shown in Fig. 5.5. Again, the vertex is the origin. Note that $S(r)$ and $S(s)$ have the adjacent blending surfaces on opposite sides. In parameter space, we take two of the edge blends, B1 and B2, as circular cylinders of equal radius. Since they are axially intersecting cylinders of equal radius, there is a hyperboloid of one sheet tangent to both which may be used to join B1 and B2. In the figure the hyperboloid is shown as B4. With a cylinder radius 1, we may take a hyperboloid whose major axes are $m = \sqrt{3}$, $n = 1$ and 1. Here m and n may be chosen differently, but must satisfy $m^2 - n^2 = 2$, so that the hyperboloid remains tangent to the cylinders. Finally, B3 is a hyperbolic cylinder matching the hyperboloid's cross-section in the plane $t = 1$. The exact equations are

$$\begin{aligned}
&\text{B1: } (r-1)^2 + (t-1)^2 - 1 = 0,\\
&\text{B2: } (s-1)^2 + (t-1)^2 - 1 = 0,\\
&\text{B3: } (r+1)^2 + (s+1)^2 - 1 - 4rs = 0,\\
&\text{B4: } 3(t-1)^2 - (r-1)^2 - (s-1)^2 + 4(r-1)(s-1) - 3 = 0.
\end{aligned}$$

The planes in which the hyperboloid is tangent to B1 and B2 are $s - 2r + 1 = 0$ and $r - 2s + 1 = 0$, and are used for clipping. The respective clipping equations, adjusted

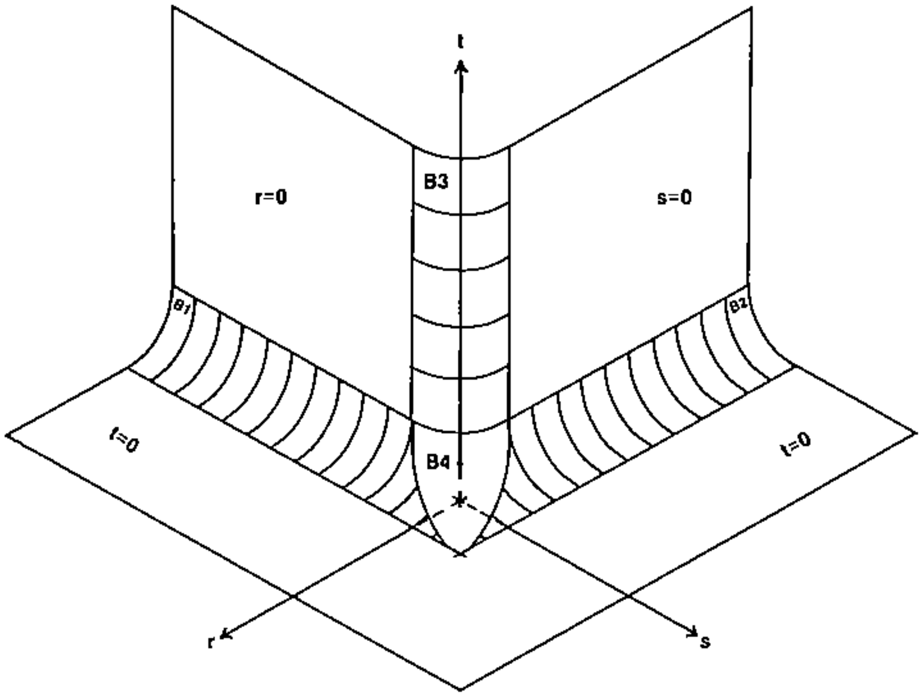

Figure 5.5. Parameter space configuration for vertex blending when not all adjacent blends are on the same surface side.

such that the wanted portion is on the surface inside, are given by

$$\text{C1: } r+t-1=0 \quad \text{and} \quad s-2r+1=0,$$
$$\text{C2: } s+t-1=0 \quad \text{and} \quad r-2s+1=0,$$
$$\text{C3: } -r-s-1=0 \quad \text{and} \quad 1-t=0 \quad \text{and} \quad r+s=0,$$
$$\text{C4: } 2r-s-1=0 \quad \text{and} \quad 2s-r-1=0 \quad \text{and} \quad t-1=0.$$

Here the third constraint on B3 is needed to remove the second branch of the hyperbolic cylinder. As previously, we substitute for r, s and t the surfaces intersecting in the vertex, observing on which side of the face the adjacent edge blends lie.

Consider blending the cylinder configuration shown in Fig. 5.6. Here $S(G)$ and $S(H)$ are two intersecting cylinders with radius $\sqrt{2}$, and $S(K)$ is the cylinder of radius 1, removed from the other two cylinders. Note that we blend the outside of $S(K)$ to the inside of both $S(G)$ and $S(H)$, whereas the outside of $S(G)$ is blended to the outside of $S(H)$. Consequently, we substituted $-G$ for r, $-H$ for s, and K for t. The result is shown in Fig. 5.7 (see color insert). Note that all surfaces meet C^1-continuously.

In parameter space, we may replace the circular cylinders B1 and B2 with elliptic ones, but their intersection must remain a pair of intersecting conics, so that the corner remains a quadric. The width of the third edge blend is controlled by the eccentricities of the corner hyperboloid. This case is more awkward than the previous case since the major axes of the hyperboloid do not lie parallel to the principal coordinate axes. As before, general formulas can be worked out.

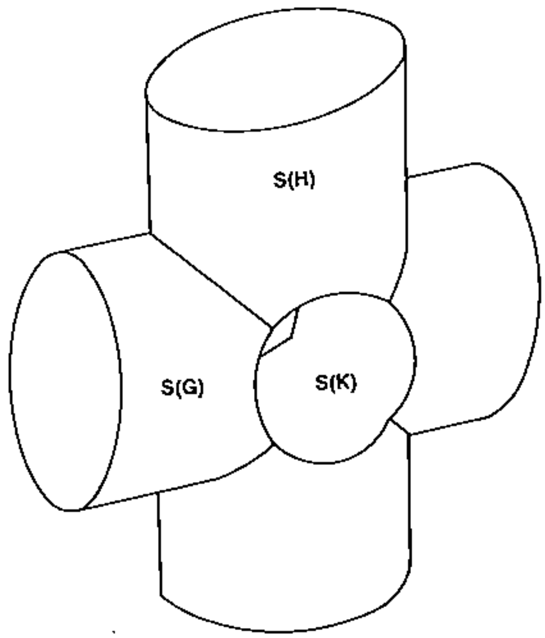

Figure 5.6. Object requiring the parameter configuration of Fig. 5.3.

6. Patched Algebraic Surfaces. The surface of an object consists, in general, of patches of algebraic surfaces. When two patches intersect transversally at an edge, then the blending methods outlined above apply. But suppose the two surfaces $S(G)$ and $S(H)$ meet tangentially in an edge and both intersect a common third surface $S(K)$ transversally in another edge that we wish to smooth, as shown in Fig. 6.1.

We may blend $S(G)$ with $S(K)$ and separately $S(H)$ with $S(K)$. Even though the curves of tangency may be correctly lined up, it is very likely that the two blending surfaces do not meet along the seam, as shown in Fig. 6.2. Again, it is our wish to provide solutions that do not raise the degree of the blending surfaces unnecessarily. For instance, the situation depicted in Fig. 6.2 may be solved with one blending surface of degree 4, the other of degree 8, but such a solution seems unsatisfactory unless we can prove that there are no lower degree surfaces with the necessary properties.

While the general problem remains unsolved, there is a situation in which degree 4 surfaces automatically match: Assume that $S(G)$ and $S(H)$ intersect tangentially in an edge e, and that both intersect a surface $S(K)$. If there is a plane h such that the edge e is the complete intersection of $S(G)$ with h and also the complete intersection of $S(H)$ with h, then the curves of intersection of the blending surfaces $S(f(G, K))$ and $S(f(H, K))$ with that plane are equal.

The situation is illustrated by Fig. 6.3 (see color insert). Here the green ellipsoid, given by $G = x^2/25 + y^2/9 + z^2/16 - 1$, intersects tangentially the yellow hyperboloid, given by $H = -x^2/25 + y^2/9 + z^2/16 - 1$. Both surfaces, in turn, transversally intersect a cylinder given by $K = x^2 + y^2 - 1$ and shown in red. Both transversal intersections are blended with the same parametric cylinder $f(r, s, t) = (s-4)^2 + 4(t-1)^2 - 4 = 0$. The

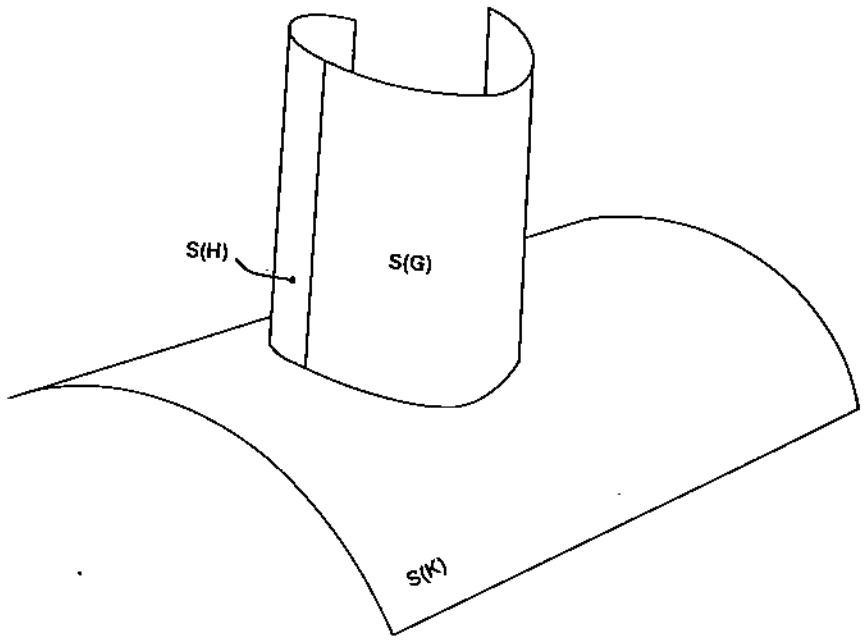

Figure 6.1. Patches of algebraic surfaces.

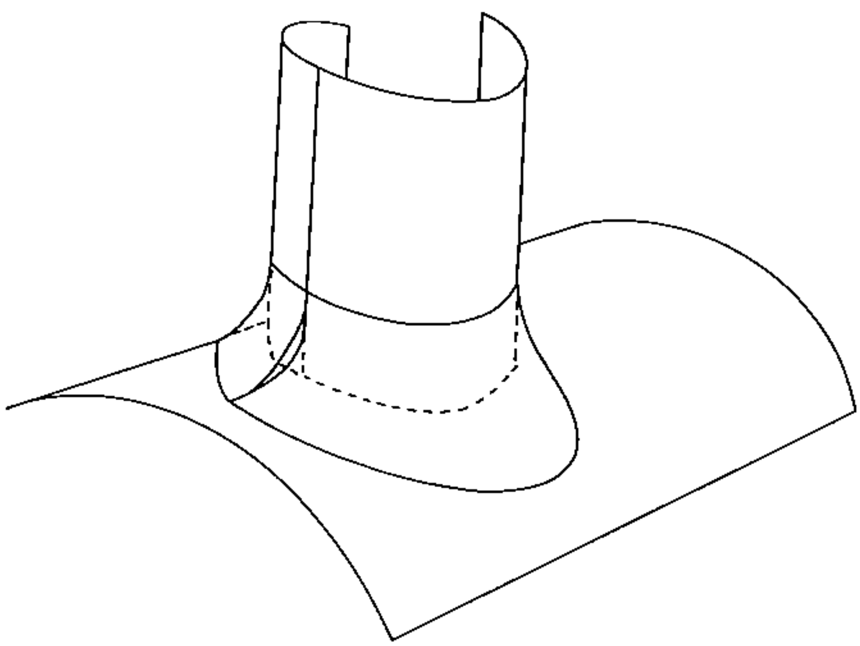

Figure 6.2. Blend discontinuity for patches of algebraic surfaces.

resulting (light blue) blending surface $S(f(G, K))$ matches the (purple) blend $S(f(H, K))$ C^1-continuously.

7. Dimensionality of Parameter Space. In our approach to low degree blending surface derivation, we have derived the actual blends from simple blends to planar surfaces in three-dimensional parameter space. Essentially we have advocated reducing the problem of blending algebraic surfaces to the problem of blending planar surfaces. This works quite well, since the deformation effected by substitution for the principal coordinate surfaces usually is not too drastic. That is, the surfaces $G - s = 0$ are usually very similar in shape to the surface $S(G)$.

In all cases considered here, the parameter space has at most three dimensions. This seems artificial, and we believe that the investigation of parameter space configurations of higher dimensionality can lead to better ways to blend complex corners than reducing vertex valence to three. It may also help to localize the shape control of edge blends at a vertex.

Blending algebraic patches is a more difficult matter. The major problem is that when two patches $S(G)$ and $S(H)$ are C^1-continuous along an edge, there is no guarantee that the surface families $G - s = 0$ and $H - t = 0$ are related in a deep way. It is possible that an approach working in higher-dimensional parameter space can provide results, or that the projective method yields the necessary tools.

Sederberg [6] works with implicit surfaces that possess rational parameterizations. This approach is interesting since it aims at a spatially intuitive procedure for deriving and placing free form surfaces. In fact, some of our blending surfaces are known to possess rational parameterization. For instance, the Steiner surface advocated by Sederberg is also a blending surface. Consider, for example, the Steiner surface $x^2y^2 + y^2z^2 + z^2x^2 - 2xyz = 0$. Since its equation may be written as

$$(x^2 + y^2 - 1)(z^2) + (xy - z)^2 = 0,$$

it is a blending surface where the quadrics blended are the cylinder $x^2 + y^2 - 1 = 0$ and the double plane $z^2 = 0$. The common quadric defining the curve of intersection is $xy - z = 0$, a hyperbolic paraboloid. The exact relationship between the class of all quartics having a rational parameterization on the one hand, and the degree 4 blending surfaces for two quadrics on the other, is not understood at this time.

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