Quadratic blending surfaces

Christoph Hoffmann and John Hopcroft

A blending surface is a surface that smoothly connects two primary surfaces. Usually such surfaces are intended to smooth edges in which the primary surfaces would intersect otherwise. Blending surfaces are important to geometric modelling since virtually all solid objects designed and manufactured have them.

The paper investigates the generality of a method for deriving blending surfaces automatically, given that the primary surfaces are quadrics. The main result is the following theorem. Given two quadric surfaces g = 0 *and* h = 0 *in general position, and given a nondegenerate, quartic space curve on each surface:*

- *There exist quartic blending surfaces* f = 0 *tangent to the quadrics* g *and* h *in the respective space curves if and only if the curves of tangency lie on a common quadric* k = 0*.*
- *All such blending surfaces satisfy the equation* $g \cdot h - \mu k^2 = 0$*, where* μ *is a number.*
- *All such blending surfaces may be derived using the derivation method described elsewhere in greater detail.*

Applications of this theorem are also discussed.

Geometry, blending surfaces, modelling

Effective use of solid modelling in the design process requires good interactive editors and automated design techiques. At the moment, existing modelling systems are limited both in the geometric shapes they are able to represent, and in the automatic capabilities with which they support the design process. Currently even the modelling of an already existing object such as a crankshaft is a major undertaking.

Presently, much of the design effort must be devoted to surfaces whose sole functional role is to smoothly connect two other surfaces and thus whose actual shape is relatively unimportant provided that certain mathematical constraints are satisfied. Ideally, one would like to concentrate on those surfaces whose shape is of critical importance to the function of the object being designed, and let the system supply the less important surfaces. An automated design system can then remove work from the designer by actually supplying many of the surfaces.

In most parts and part systems of interest, there already exists a large class of surfaces which are functionally relatively unimportant, and these surfaces, including rounds, fairings, and fillets, are often incompletely specified in blueprints. For instance, a fillet is specified by an approximate curvature and the implicit requirement to be tangent to two intersecting surfaces. It rarely matters whether the cross sections of the fillet are exactly circular — an approximately circular fillet would serve equally well. These approximately specified surfaces are collectively known as blends, and a long standing problem has been to add the capability to solid modellers to supply these blending surfaces automatically, based on intuitive and approximate specifications.

Department of Computer Science, Cornell University, Ithaca, NY 14853, USA

A blending surface is usually mathematically more complicated than the primary surfaces it must smoothly connect. But the importance of the blending problem is not only found in the wish to relieve the designer from more demanding mathematics. Another important reason for automating the design process in general and blending of surfaces in particular is the goal of making a (partially) completed design editable.

Suppose we have constructed a model of an internal combustion engine. After the design is completed, we might wish to modify the piston diameter. A large number of surfaces need to be altered, as the diameter is changed, simply to maintain the integrity of the physical object. One would like to relate the parameters of these surfaces so that, ideally, only one parameter needs to be changed, and all the other necessary changes are performed automatically. This goal involves understanding how the object might be edited, and how this will affect the various surfaces. Simply relating the parameters will involve a sufficiently large overhead so that an automated system is needed in the construction stage to correctly establish the relationships between the parameters.

We have previously provided a method for deriving and positioning blending surfaces, given only the algebraic equations of the two surfaces whose edge of intersection needs to be smoothed[1,2]. This method, referred to as the potential method, is completely general and works for algebraic surfaces of arbitrary degree. Moreover, given that surfaces of algebraic degree *m* and *n* are to be blended, a blending surface of degree max(2*m*, 2*n*) is obtained. Thus, two quadrics can always be blended with a degree 4 surface. Other serious attempts have been reported recently: Rossignac[3,4] approximates blends from toroidal and cylindrical pieces. His surfaces approximate a blending surface of constant curvature, sacrificing constant curvature and tangency to the primary surfaces for lower algebraic degree. Rockwood and Owen[5] offer a method that derives blends as a function of the two surfaces and their respective gradient functions, yielding, in the case of quadrics, surfaces of degree 8 or higher. Middleditch and Sears[6] have a method that for quadrics also delivers degree 4 surfaces. Much of their work concentrates on interfacing their method with constructive solid geometry, as does Rossignac's work.

Given the availability of these alternatives, one must ask what relationship, if any, is there between the different methods? Moreover, are there other methods, undiscovered as yet, that offer better alternatives? These questions pose difficult mathematical problems which we explore in this paper for quadric surfaces. In particular, we prove that given two quadrics, the potential method gives the only degree 4 surfaces accomplishing the blend. Thus, if two quadrics are to be blended, and the surfaces obtained by the potential method are for some reason not suitable, then surfaces of degree higher than 4 are needed. In particular, the class of all degree 4 surfaces tangent to two quadrics in prescribed curves of tangency is a one dimensional vector space and has a particularly simple and intuitive structure. As further implications of the result, we can state positively that the blending method of Middleditch and Sears[6] is an equivalent formulation of the potential method, and that those surfaces of Rockwood and Owen[5], which simplify to degree 4 surfaces, ie the simple blends for cylinders and spheres, make unnecessary use of the gradient functions.

To carry out the proof requires certain results from the theory of ideals and from algebraic geometry. Where these are not proved, we give references to where they may be found in the literature, and attempt throughout to accompany the results with geometric intuitions further clarifying their nature. We distinguish between a surface $S(F)$ and the polynomial F (in x, y and z) defining it via the implicit equation $F = 0$. This distinction is necessary to avoid confusing geometric arguments with algebraic ones. Both lines of reasoning are needed.

The paper is structured as follows: the next section reviews the potential method for blending two intersecting surfaces. The third section outlines the ideal-theoretic results needed from algebraic geometry and explains their geometric significance. In particular, we show how requiring tangency to a given surface imposes specific constraints on the surface equation. The fourth section gives explicit bounds on the degree of certain coefficient polynomials needed to represent all surfaces F that intersect, or are tangent to, a given quadric surface G in a specified space curve. The fifth section, contains the main result that the potential method delivers the only degree 4 surfaces to smooth the intersection of two quadrics in general.

As must be expected, there are a number of special situations arising when the curve in which the blending surface is to be tangent, degenerates in certain ways. These special cases are discussed in the final section, and for them, other blending methods become possible. Indeed, the homotopy method[1] is one such special case.

POTENTIAL METHOD

The potential method[1] smoothes the intersection of two algebraic surfaces $S(G)$ and $S(H)$ whose implicit equations are $G = 0$ and $H = 0$. In the simplest version of the method we pick two constants a and b, and consider the surfaces specified by $G' = G - a = 0$ and $H' = H - b = 0$. These surfaces are similar to $S(G)$ and $S(H)$ but are entirely on their outside or inside, depending on the sign of a and of b. The intersections of $S(H')$ with $S(G)$ and $S(G')$ with $S(H)$, define two space curves. We construct a family of blending surfaces that are tangent to $S(G)$ and $S(H)$ in the respective space curves.

Intuitively, one may think of these blending surfaces as being obtained by sweeping a space curve in a specific manner. The two space curves in which tangency is obtained are specific positions of the sweeping space curve. To this end, we consider the space curve obtained by intersecting $S(G - s)$ with $S(H - t)$, and let it sweep through space by relating the values of s and t through a curve $f(s, t) = 0$. Since the intersections of $S(G)$ with $S(H')$ should be an instance of the sweeping curve, we require that $f(0, b) = 0$. Similarly, we require that $f(a, 0) = 0$. In this manner, a surface $S(F)$ is defined whose equation is

$$F = f(G, H) = 0$$

If we require that f be tangent to the $s = 0$ axis in the point $(0, b)$, and tangent to the $t = 0$ axis in the point $(a, 0)$, then $S(F)$ will be tangent to $S(G)$ and $S(H)$ in the respective curves, as proved in Hoffmann and Hopcroft[1,2].

For example, consider the cylinders $G = y^2 + z^2 - 8^2 = 0$ and $H = x^2 + y^2 - 4^2 = 0$. Here $S(G)$ and $S(H)$ intersect at a right angle, and $S(G)$ has radius 8 while $S(H)$ has radius 4. We choose $a = 36$ and $b = 20$. Then $G - 36 = 0$ is the equation of a cylinder of radius 10, whereas $H - 20 = 0$ is the equation of a cylinder of radius 6. We pick an ellipse

$$f(s, t) = \frac{(s-a)^2}{a^2} + \frac{(t-b)^2}{b^2} - 1 = 0$$

that is tangent to the s-axis at $(a, 0)$ and to the t-axis at $(0, b)$. Now the surface

$$F = f(G, H) = \frac{(G-36)^2}{36^2} + \frac{(H-20)^2}{20^2} - 1 = 0$$

is tangent to $S(G)$ in the curve of intersection of $S(G)$ with $S(H - 20)$, and to $S(H)$ in the intersection of $S(G - 36)$ with $S(H)$. The surface is of degree 4 and suitably clipped, blends the intersection of the two cylinders $S(G)$ and $S(H)$. In a similar manner, we may blend any pair of intersecting quadrics with a quartic surface.

In Hoffmann and Hopcroft[1], we advocate using a degree 2 curve for $f(s, t)$. This is merely a matter of keeping the degree of the resulting blending surface low. Higher degree functions may well be used, either to achieve osculation in place of mere tangency, or when the degrees of G and H differ.

In general, one is not required to use an ellipse as the base curve of the blend: any (nondegenerate) conic tangent to the coordinate axes in $(a, 0)$ and $(0, b)$ may be used. With the required points of tangency, one may write f of degree 2 in terms of a, b and a free parameter λ as follows

$$f = b^2 s^2 + 2\lambda st + a^2 t^2 - 2ab^2 s - 2ba^2 t + a^2 b^2 = 0$$

In the above example, $f(s, t)$ has $\lambda = 0$. Accordingly, there is a family of blending surfaces given by

$$F = b^2 G^2 + \lambda GH + a^2 H^2 - 2ab^2 G - 2ba^2 H + a^2 b^2 = 0$$

For quadric surfaces, F is evidently of degree 4. The constants a, b and λ have an intuitive meaning. Loosely speaking, the magnitudes of a and b control the distance the curves of tangency have from the intersection curve, and λ controls the curvature distribution across the blend.

In the general formulation of the potential method the polynomials G' and H' may be specified by the more complicated scheme

$$G' = G - aW$$

$$H' = H - bW$$

where W is a polynomial, not simply 1, as we have used above. In general, then, the blending surface is swept out by the intersection of the surface families $S(G - sW)$ and $S(H - tW)$, where s and t are related by the function $f(s, t) = 0$, as above. Here the blending surface is the result of substituting the rational functions $s = G/W$ and $t = H/W$ into $f(s,t)$ With f and W of degree 2, we obtain the degree 4 blending surfaces $S(F)$ from

$$\frac{F}{W^2} = f\left(\frac{G}{W}, \frac{H}{W}\right)$$

With $W = 1$ we recover the simple method.

There exists an important relationship between the curves $S(G', H)$ and $S(G, H')$ when the potential method is used. Both G' and H' may be replaced by a single polynomial $\overline{G}$ that has degree 2 if G, G', H and H' have degree 2.

Theorem 1

If F, G' and H' are specified by the potential method, then there exists a polynomial $\overline{G}$ such that $S(\overline{G}, H) = S(G', H)$ and $S(G, \overline{G}) = S(G, H')$.

Proof

Since F is derived from the potential method, we have

$$G' = G - aW$$

and

$$H' = H - bW$$

Now $S(G, H) = S(uG + vH, H)$ for $u \neq 0$. We let

$$\overline{G} = bG + aH - abW$$

Then $\overline{G} = bG' + aH = aH' + bG$, from which the result follows.

As we shall see in the fifth section, in general the converse of Theorem 1 is true, and these surfaces are the only degree 4 blending surfaces for intersecting quadrics.

ALGEBRAIC GEOMETRY

We now explore some of the algebraic properties of the equation F that describes an algebraic surface intersecting or tangent to a given quadric surface in a specified curve. These properties are derived from classical results of algebraic geometry (see, for example, Fulton[7]). The surface equations are considered polynomials over the ground field C of complex numbers, since most results needed from algebraic geometry are only valid for algebraically closed ground fields and the field R of real numbers is not algebraically closed.

Let $S(G)$ be a nondegenerate quadric surface, ie it does not consist of two planes and so corresponds to the irreducible degree 2 polynomial G in $C[x, y, z]$. Let $S(G, H)$ be the space curve on the surface $S(G)$ defined as the complete intersection of $S(G)$ with another quadric surface $S(H)$, in turn specified by the degree 2 polynomial H. Under certain circumstances, the intersection curve $S(G, H)$ splits into a number of components. This introduces complications that must be dealt with as special cases.

Definition

An ideal I is a subset of polynomials in $C[x, y, z]$ closed under addition and closed under multiplication with every polynomial in $C[x, y, z]$. That is, for A and B in I, $A + B$ is in I, and for A in $C[x, y, z]$ and B in I, AB is in I.

Consider the ideal (G, H) generated by polynomials G and H. The ideal is the set of all polynomials of the form $AG + BH$, where A and B are arbitrary polynomials in $C[x, y, z]$. Intuitively, the ideal (G, H) contains only polynomials defining algebraic surfaces that contain the intersection curve $S(G, H)$, since G and H vanish simultaneously at every point on the curve. G and H will also vanish at other points, but not simultaneously.

In general, the ideal (G, H) will not contain all polynomials F that vanish on the intersection curve $S(G, H)$. The relationship between the set of all such polynomials and the ideal (G, H) is explained in the following theorem.

Theorem 2 (Hilbert Nullstellensatz)

If $S(F)$ contains $S(G, H)$, then F^k is in (G, H), for some integer k.

Intuitively, the space curve $S(G, H)$ does not reflect the algebraic multiplicity of the intersection. For example, the plane $x^2 = 0$ intersects the plane $y = 0$ in a line, yet the plane $x = 0$ which contains this line is not in the ideal (x^2, y). This is one of the reasons why F may have to be raised to a power greater than 1 in the Nullstellensatz.

When the ideal (G, H) contains all polynomials vanishing on the intersection curve, then the exponent of F is always 1. This happens when the intersection of G and H is an irreducible space curve. Both the geometric notion of irreducibility as well as its algebraic equivalent will now be explained.

Definition

An algebraic set $S(I)$ is the set of all points satisfying $A = 0$ for all polynomials A in an ideal I. The algebraic set is reducible if it is the union of two different algebraic sets, otherwise it is irreducible.

Definition

An ideal $I \subset C[x, y, z]$ is prime if, for all polynomials A and B in $C[x, y, z]$, AB in I implies that either A or B is in I.

Note that if (G, H) is a prime ideal, then (G, H) contains all polynomials F such that $S(F)$ contains $S(G, H)$. The concept of prime ideals and of irreducible algebraic sets are linked as follows (for example, see p 15 of Fulton[7]).

Theorem 3

If I is a prime ideal then $S(I)$ is irreducible. Conversely, if $S(I)$ is irreducible, then there is a prime ideal J such that $S(J) = S(I)$ and $I \subset J$.

Now let F specify an algebraic surface that intersects a given surface $S(G)$ in the curve $S(G, H)$, specified as the complete intersection of $S(G)$ with $S(H)$. The preliminary, algebraic characterization of F is given by the following standard result.

Theorem 4

If (G, H) is a prime ideal and $S(F)$ any algebraic surface containing $S(G, H)$, then $F = AG + BH$.

When $S(F)$ not only intersects $S(G)$ in the irreducible curve $S(G, H)$ but is also tangent to the surface, then more can be said about the coefficient polynomial B.

Theorem 5

Let (G, H) be a prime ideal with $S(G)$ and $S(H)$ intersecting transversally in $S(G, H)$. If $S(F)$ is tangent to $S(G)$ in the curve $S(G, H)$, then F can be written as $F = AG + BH^2$.

Proof

Requiring that F be tangent to G along the curve $S(G, H)$ implies

$$F = AG + BH \qquad (1)$$

$$F^x G^y - G^x F^y = 0 \bmod (G, H) \qquad (2)$$

$$F^x G^z - G^x F^z = 0 \bmod (G, H) \qquad (3)$$

Differentiating equation (1) with respect to x, y and z and substituting for F^x, F^y and F^z in equation (2) and equation (3) yields

$$B(H^x G^y - G^x H^y) = 0 \bmod (G, H)$$

$$B(H^x G^z - G^x H^z) = 0 \bmod (G, H)$$

Thus either B or both $H^x G^y - G^x H^y$ and $H^x G^z - G^x H^z$ are in (G, H). However, the latter would imply that G and H were tangent along $S(G, H)$ contrary to the hypothesis. Therefore B must be in (G, H) and hence F can be written $A'G + B'H^2$.

We conclude this section by explaining when the intersection of two quadrics is an irreducible curve and when their defining polynomials form a prime ideal.

It is well known (see, for example, Salmon and Fiedler[8]), that all space curves of degree 1 are lines, and all of degree 2 are planar conics. By Bezout's theorem, the complete intersection of two quadrics in projective space is a curve of degree 4. The type of curve that arises as the complete intersection of two quadrics is one of the following (see, for example, Snyder and Sisam[9]):

- an irreducible, nonplanar curve of degree 4
- a single line plus an irreducible, nonplanar curve of degree 3 that passes through infinity
- a pair of conics
- a pair of lines and a conic
- four lines

So, if the quadrics are defined by the forms G and H, the ideal (G, H) is prime for the first of the above listed cases, and is not otherwise. This leads to the following theorem.

Theorem 6

Let G and H be two homogeneous polynomials of degree 2. Then the ideal (G, H) is prime if and only if no plane contains more than 4 points of the intersection $S(G, H)$, ie if and only if $S(G, H)$ does not have a planar component.

In the affine case, we must consider whether some of the components are at infinity. For example, the two hyperbolic cylinders $xy + w^2 = 0$ and $yz + w^2 = 0$ have an intersection contained in the pair of planes $w(z - y)$, but the plane $w = 0$ is at infinity. Since we wish to determine the algebraic form of all surfaces $S(F)$ containing a space curve given as the complete intersection of $S(G)$ with $S(H)$, it makes sense to require that the curve be specified in the simplest way. This means that in the above example, we should replace one of the quadrics, say $yz + 1 = 0$, with the plane $z - y = 0$. Note that the ideal $(yx + 1, z - y)$ is prime, as the intersection curve, a hyperbola, is irreducible of degree 2.

Theorem 7

Let G and H be of degree 2. If $S(G)$ and $S(H)$ are tangent to each other, then (G, H) is not a prime ideal.

Proof

The curve of tangency $S(G, H)$ is the limit of two separate curves infinitesimally apart and therefore has algebraic multiplicity 2. Hence $S(G, H)$ is reducible, ie (G, H) is not prime.

DEGREE BOUNDS

If the surface $S(F)$ is tangent to the surface $S(G)$ in the space curve $S(G, H)$, then it contains the space curve. Hence, if (G, H) is a prime ideal, then F is of the form $F = AG + BH$. Given the degrees of F, G and H, it is by no means straightforward to specify the minimum degree the polynomials A and B must have. In this section we develop such bounds for the case when G and H have degree 2, assuming that (G, H) is a prime ideal. For the remainder of the paper we use the notation A_k to denote the homogeneous polynomial consisting of all degree k terms of a polynomial A. It may be remembered that homogeneous polynomials are also called forms in the literature.

The minimum degree of A and B depends on whether the polynomial G_2 consisting of all degree 2 terms of G has a factor in common with the polynomial H_2, consisting of all degree 2 terms of H. If these two polynomials are coprime, then A and B need not have a degree higher than $\deg(F) - 2$. If G_2 and H_2 have a common factor but (G, H) is prime, then the degrees of A and B may be as high as the degree of F.

Let G and H be degree 2 polynomials specifying the quadric surfaces $S(G)$ and $S(H)$, respectively. We assume that neither $S(G)$ nor $S(H)$ degenerate into planes, so that both G and H are irreducible polynomials. Moreover, we assume that $S(G)$ and $S(H)$ intersect in a nonempty irreducible space curve $S(G, H)$, or, equivalently, that the ideal (G, H) generated by G and H is a prime ideal. Let $F = AG + BH$ be a polynomial defining the surface $S(F)$.

Lemma 1

If $F = AG + BH$ has degree m, and the degree 2 terms G_2 of G and H_2 of H form two polynomials without a common factor, then both A and B may be assumed to have degree, at most, $m - 2$. In particular, F cannot have degree 1 or 0, unless it is the zero polynomial.

Proof

Write $G = G_2 + \bar{G}$ and $H = H_2 + \bar{H}$. By assumption, G_2 and H_2 are relatively prime. Let n be the higher of the degrees of A and of B. If $n > m - 2$, we will construct polynomials A' and B' of degree $n - 1$ such that $F = A'G + B'H$. Then the Lemma follows by induction.

Write $A = A_n + \bar{A}$ and $B = B_n + \bar{B}$, where A_n consists of all degree n terms in A, and B_n consists of all degree n terms in B. Assuming $n > m - 2$, we have

$$F = \bar{A}G + \bar{B}H + A_n\bar{G} + B_n\bar{H}$$

and

$$A_n G_2 + B_n H_2 = 0$$

Since G_2 and H_2 have no common factors, $n \leqslant 1$ is impossible and, for $n > 1$, it follows that $A_n = C_{n-2}H_2$ and $B_n = -C_{n-2}G_2$, where C_{n-2} is a form of degree $n - 2$. Substituting these identities for A_n and B_n, we obtain, after adding

$C_{n-2}\overline{G}\overline{H} - C_{n-2}\overline{G}\overline{H}$, that

$$F = \overline{A}G + \overline{B}H + C_{n-2}\overline{G}(H_2 + \overline{H}) - C_{n-2}\overline{H}(G_2 + \overline{G})$$

which is equivalent to

$$F = (\overline{A} - C_{n-2}\overline{H})G + (\overline{B} + C_{n-2}\overline{G})H = A'G + B'H$$

Note that A' and B' are of degree at most $n - 1$.

We give an example demonstrating that the coprimality of G_2 and H_2 must be assumed. Consider the hyperbolic paraboloid $G = xz + y = 0$ and the hyperbolic cylinder $H = yz + 1 = 0$. The ideal (G, H) contains the polynomial $F = yG - xH = y^2 - x$, which defines another hyperbolic cylinder. It is easy to see that there are no constants u and v such that $uG + vH = y^2 - x = F$. Hence the bound of $\deg(F) - 2$ on the coefficient polynomials A and B cannot be satisfied.

We have explored degree bounds on the coefficient polynomials A and B for prime ideals not satisfying the hypotheses of Lemma 1. These bounds are summarized in the following theorem, which we do not prove here, since we assume subsequently that the prime ideals considered satisfy the hypotheses of the Lemma.

Theorem 8

Let $S(G)$ and $S(H)$ be irreducible quadrics, and assume that (G,H) is a prime ideal. Let $F = AG + BH$ have degree m. If G_2 and H_2 are coprime, then the degrees of A and B may be bounded by $m - 2$. If G_2 and H_2 have a common factor Z, then Z has degree 1. Moreover, if Z does not divide $YG_1 - XH_1$ then the degrees of A and B may be bounded by $m - 1$. If Z does divide $YG_1 - XH_1$, then $G = uX^2 + vXY + vW + G_0$ and $H = uXY + vY^2 - uW + H_0$, where $u \neq v$, and X, Y and W are linearly independent forms of degree 1. In this case, the degrees of A and B cannot be bounded by $m - 1$.

We apply the theorem to surfaces $S(F)$ of degree 4 that are required to be tangent to a given quadric in a prescribed curve. Referring to Theorems 5 and 7, we have a corollary.

Corollary 1

Let $S(G)$ and $S(H)$ be irreducible quadrics, and assume that (G, H) is a prime ideal and that G_2 and H_2 are coprime. If $S(F)$ is a degree 4 surface tangent to $S(G)$ in the curve $S(G, H)$, then $F = AG + bH^2$, where A is of degree 2 and b is a constant.

TANGENCY TO TWO SURFACES

Let G and H be nondegenerate quadrics. On each of the surfaces $S(G)$ and $S(H)$, define an irreducible degree 4 curve by the complete intersection with the additional quadrics H' and G', respectively. In this section we show that the family of all degree 4 blending surfaces that are tangent to $S(G)$ in the curve $S(G, H')$ and to $S(H)$ in the curve $S(G', H)$ is precisely the family of surfaces constructed by the potential method. Throughout the section we assume that:

- The surface $S(F)$ is tangent to $S(G)$ in $S(G, H')$ and tangent to $S(H)$ in $S(G', H)$, and these curves do not coincide with the intersection of $S(G)$ with $S(H)$.
- The polynomials G, H and F are irreducible, ie the respective surfaces are nondegenerate.
- The ideals (G, H), (G, H') and (G', H) are prime.
- The quadratic terms of G and H' are coprime, likewise the quadratic terms of G' and H, and the quadratic terms of G and H.

These assumptions are justified in the last section.

We first show that if the surface $S(F)$ that is tangent to $S(G)$ at $S(G, H')$ and tangent to H at $S(G', H)$ is to be of degree 4, then there must be a linear relationship among G, G', H, and H'.

Lemma 2

Under the assumptions at the beginning of the section

$$H' = w_1G' + w_2G + w_3H$$

Proof

Since F is of degree 4 and $S(F)$ is tangent to $S(G)$ in $S(G, H')$, we may write by Corollary 1

$$F = F_1 = UG + uH'^2$$

where the u is a constant and U has degree 2. Since F is tangent to $S(H)$ in $S(G', H)$ we may write

$$F = F_2 = VH + vG'^2$$

Again, v is a constant and V has degree 2. If $u = 0$, then $F = GU$, and hence $S(F)$ is degenerate. By a symmetrical argument $v \neq 0$. Thus

$$F_1 - F_2 \equiv uH'^2 - vG'^2 \equiv 0 \bmod (G, H)$$

Since we work with the ground field of complex numbers, $w_0 = (v/u)^{1/2} \neq 0$ exists, and so $uH'^2 - vG'^2$ factors. Since (G, H) is prime, at least one of the factors is in (G, H). Hence, with $w_1 = w_0$ or $w_1 = -w_0$ we have

$$H' - w_1G' = w_2G + w_3H$$

from which the Lemma follows.

Lemma 3

Under the assumptions made at the beginning of this section, if $G' = H'$, then $F = GH + uH'^2$, and F may be derived from the potential method.

Proof

We have

$$F_1 - F_2 = UG + uH'^2 - VH - vG'^2 \equiv (u - v)G'^2 \equiv$$

$$\equiv 0 \bmod (G, H)$$

Now (G, H) is prime and G' cannot be in the ideal since $S(G', H)$ and $S(G, H)$ are irreducible curves that do not coincide by hypothesis. Hence $u - v$ is in (G, H) and so, by Lemma 1, $u - v = 0$. Note that $u \neq 0$ by the proof of Lemma 2. Substituting H' for G' and u for v we have

$$F_1 - F_2 = GU - HV = 0$$

Since $G \neq H$, we have $H = U$. Substituting for U in $F_1 = UG + uH'^2$ we obtain

$$F = F_1 = GH + uH'^2$$

which has the required form.

We now derive this class of surfaces from the potential method. Let

$$W = \frac{G}{a} + \frac{H}{b} + \frac{u^{1/2}}{ab} H'$$

Substituting for H' in the equation for F we thus obtain

$$F = GH + a^2b^2W^2 + b^2G^2 + a^2H^2 - 2ab^2GW -$$

$$- 2a^2bHW + 2abGH$$

Let $f(s,t) = b^2s^2 + a^2t^2 + a^2b^2 - 2ab^2s - 2a^2bt + (2ab + 1)st$. Then with

$$s = \frac{G}{W} \text{ and } t = \frac{H}{W}$$

we have

$$f(s, t) = \frac{F^2}{W^2}$$

Hence F may be derived by the potential method.

We now obtain the main result.

Theorem 9

Assume the hypotheses stated at the beginning of this section. Then every degree 4 blending surface $S(F)$ for a pair of intersecting quadrics may be derived by the potential method.

Proof

Recall Lemma 2, since $S(G, H') = S(G, H' - w_2G)$ and $S(G', H) = S(w_1G' + w_3H, H)$, we may replace H' with $\overline{H} = H' - w_2G$ and G' with $\overline{G} = w_1G' + w_3H$, hence $\overline{G} = \overline{H}$, $S(G, \overline{H}) = S(G, H')$ and $S(\overline{G}, H) = S(G', H)$. The theorem now follows from Lemma 3.

DISCUSSION

Most of the hypotheses of Theorem 9 are natural and do not limit the applicability of the result:

- Since the curves of tangency lie on a quadric, Bezout's Theorem implies that they have the same degree as F. Since F has degree 4, these curves can be specified as the intersection of two quadrics[8,9]. Moreover, if one of the curves of tangency coincides with $S(G, H)$, then the resulting surface $S(F)$ functionally does not serve as a useful blend.
- If $S(G)$ and $S(H)$ are reducible quadrics, then a solid modeller will treat them as planes, not as pairs of planes. Hence assuming that G or H factor implies that a different problem is being studied, not the blending of two quadrics.
- Two nondegenerate quadrics in general position intersect in an irreducible curve. This justifies assuming the primality of the ideal (G, H).

Most of the remaining assumptions should be understood as saying that quadrics in special relation to each other admit a greater flexibility in blending, ie give rise to special cases. These cases need to be explored further, as they include situations used in blending corners of solids. The assumptions we made excluding them required that the three ideals (G, H), (G', H) and (G, H') be prime, and that F be irreducible. Some of these assumptions may not be independent. For instance, if F is reducible and the two factors have degree 2 each, then the curves of tangency to $S(G)$ and $S(H)$ are reducible (*cf* Theorem 7). These special cases need further exploration.

There is one restriction that we only understand for its technical use. This is the fourth assumption listed in the preceding section, bounding the minimum degree of the coefficient polynomials A and B in $F = AG + BH$. Whenever this restriction is violated, every surface of the form $S(uG + vH)$, with u and v constants, is a ruled quadric. We do not know the deeper geometric significance of this case, nor why it leads to complications in the structure of the ideal (G, H).

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