On Local Implicit Approximation and Its Applications

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A method is proposed for computing an implicit approximant at a point to a parametric curve or surface. The method works for both polynomially and rationally parameterized curves and surfaces and achieves an order of contact that can be prescribed. In the case of nonsingular curve points, the approximant must be irreducible, but in the surface case additional safeguards are incorporated into the algorithm to ensure irreducibility. The method also yields meaningful results at most singularities. In principle, the method is capable of exact implicitization and has a theoretical relationship with certain resultant-based elimination methods.

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1. INTRODUCTION

A recurring operation in solid modeling is the evaluation of surface intersections [24]. If both surfaces are given parametrically, the two major approaches given the greatest prominence in the literature are subdivision and substitution methods.

In the subdivision method (e.g., [12, 14–16, 22]), both surfaces are recursively subdivided in the vicinity of their intersection. The subdivision results in an adaptive piecewise linear approximation of both surfaces and their intersection. Among the advantages of the method, we mention its robustness and its potential for locating all intersection branches. A major drawback of the subdivision method is the large volume of data it creates, which slows it down in areas of high surface curvature.

In the substitution method, (e.g., [7, 17, 25, 28, 29]), one of the surfaces, S_1, is converted to implicit form F, and the parametric form of S_2 is substituted into F resulting in an implicit algebraic curve f in the parameter space of S_2. This curve

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f is in birational correspondence with the intersection of S_1 and S_2 in xyz-space, and thus serves as an accurate representation of the intersection. Major difficulties of the substitution method limit its utility in practice. There are two general methods for implicitizing a parametric surface. The first method is based on Elimination Theory [26] and does resultant computations. It is expensive and generates extraneous factors whose detection is a delicate problem; see also Section 5. The second method for implicitization is based on Gröbner Basis techniques [5]. It is also fairly expensive and requires, moreover, rational coefficients in the description of S_1. Another difficulty with the substitution method, less prominently pointed out but well known (N. M. Patrikalakis, personal communication, 1987), is that the substitution itself can be numerically unstable and is a nontrivial algorithmic task when desiring efficiency and accuracy. Some authors have suggested the use of rational arithmetic for this reason [8], thus further adding to the computational load of the approach.

In this paper, we provide a middle ground by deriving a local implicit approximation of rational or polynomial parametric curves and surfaces with low-degree implicit forms. In the context of subdivision techniques, such approximations have the potential of reducing the number of generated surface approximants because we are not restricted to planar approximants only. In the context of substitution methods, the approximations avoid the high cost of implicitizing a parametric curve or surface, and provide, moreover, irreducible approximants. In both cases a number of practical issues remain open for exploration, including the trade-off between the degree of the approximant and the accuracy with which the curve or surface has been approximated. In particular, a comparative evaluation of our method that contrasts its performance with other surface intersection methods is desirable, including the higher dimensional approach proposed in [10] and its specialization to parametric curves and surfaces explained in [9]; see also [11]. We are currently engaged in research elucidating some of these questions.

Since the distribution of a preliminary version of this paper, a number of related investigations have been developing and applying similar ideas. Bajaj and Ihm [2] apply a technique, analogous to ours, to the problem of designing blending surfaces and prove results on minimum degree blends satisfying certain constraints.

Previously, local *explicit* approximations to integral parametric curves and surfaces have been proposed in [20]. An approximant of the form

$$z = f(x, y) = \sum a_{ij} x^i y^j \quad \text{or} \quad y = f(x) = \sum a_i x^i$$

is constructed for surfaces and curves. Recurrence formulas were also derived for the coefficients of f. Bajaj [1] extends this method using power series composition and inversion techniques together with rational Padé approximations. In our experience, a local explicit approximation is less favorable than a local implicit approximation. In fact, while a quadratic explicit approximation to a curve achieves second-order contact at the point at which it is constructed, a quadratic implicit approximation achieves *fourth-order* contact. For curves, the order of contact grows linearly with the degree of the explicit approximation, whereas the order of contact of the implicit approximation has a quadratic growth in the degree. Thus, much lower degree approximations suffice.

In general, local explicit approximation can only approximate curves or surfaces *locally* no matter how high a degree of approximant is used. This is due to the asymmetry introduced by making one variable an explicit function of the other(s). For instance, a circle cannot be completely approximated with a single explicit approximant. In contrast, our approximants are capable of approximating curves or surfaces not only locally but also globally in the sense that the radius of convergence increases when the degree of approximation increases, and the exact implicitization can be finally derived when the degree of approximation is equal to the degree of the given parametric curve or surface.

After reviewing the necessary definitions and facts in Section 2, we describe the method for polynomially and rationally parameterized curves in Section 3. Section 4 presents the surface case. In Section 5, we comment briefly on some theoretical connections between the method we propose here and several resultant formulations found in the literature.

2. PRELIMINARIES

A polynomial of degree n in the variables $x_1, x_2, \ldots, x_k$ is denoted $f^n(x_1, \ldots, x_k)$ whenever we wish to stress the degree. The *gradient* of f at the point $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is the vector $\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_k})$, where the partials are evaluated at $\mathbf{x}$.

A rational plane curve $\mathbf{r}(t)$ can be given as the pair $(x(t), y(t))$, where $x(t)$ and $y(t)$ are rational functions of t. The curve points are all points $(x(t), y(t))$ on the plane. The curve is *properly parameterized* if for all but finitely many curve points p we have $p = (x(t), y(t))$ with a *unique* value of t. When a parametric curve is not properly parameterized, then there exists a rational nonlinear function $s(t)$ such that $x(t) = x^*(s(t))$ and $y(t) = y^*(s(t))$. We assume in this paper that all parametric curves are properly parameterized and note that a parametric curve is always irreducible. For methods to detect improper parameterization, see [27].

The degree of a rational parametric curve is the highest degree of the numerator or the denominator polynomial, assuming both $x(t)$ and $y(t)$ have been written with a common denominator. The implicit equation $f(x, y)$ of the rational curve $\mathbf{r}(t)$ is a lowest degree polynomial in x and y satisfying $f(x(t), y(t)) \equiv 0$. It is unique up to a multiplicative constant. If $\mathbf{r}(t)$ has degree m, then so does $f(x, y)$; see for example, [18] and [21].

As with parametric curves, a parametric surface

$$\mathbf{P}(s, t) = (x(s, t), y(s, t), z(s, t))$$

can be *improperly parameterized* if there are nonlinear rational functions $u(s, t)$ and $v(s, t)$ such that

$$\mathbf{P}(s, t) = (x^*(u(s, t), v(s, t)), y^*(u(s, t), v(s, t)), z^*(u(s, t), v(s, t)))$$

In that case, there is a many-to-one correspondence between the parameter values and the surface points. $\mathbf{P}(s, t)$ is *properly parameterized* if this correspondence is one-to-one except, possibly, on a one-dimensional set of points. We also assume that all parametric surfaces are properly parameterized. For a parametric surface described by rational functions of total degrees m, there always exists an irreducible implicit equation $f(x, y, z) = 0$ satisfying $f(x(s, t), y(s, t),$

$z(s, t)) \equiv 0$, and f is unique within a constant factor. Moreover, f has a degree at most m^2.

A point $p = (x, y)$ is *regular* on a plane curve $f(x, y) = 0$ if the gradient of f at p is not null; otherwise the point is *singular*. Likewise, $p = (x, y, z)$ is regular on $f(x, y, z) = 0$ if the gradient of f is not null at p; otherwise p is singular.

3. LOCAL IMPLICIT APPROXIMATION OF PARAMETRIC PLANE CURVES

We seek an implicit curve $g(x, y) = 0$ that approximates the parametric curve $\mathbf{r}(t) = (x(t), y(t))$ at the origin to a specified order of contact. The idea is to set up the polynomial $g(x, y)$ of sufficiently high degree with symbolic coefficients e_{ij}. Then, a system of linear equations with unknowns e_{ij} is formulated and solved.

The linear system is obtained by substituting $\mathbf{r}(t)$ into $g(x, y)$. The result is

$$g(x(t), y(t)) = \sum \alpha_k t^k$$

where the α_k are linear combinations of the e_{ij}. We require that a certain number of the α_k vanish. With

$$\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_s = 0$$

for some s, an implicit approximation is obtained that has contact of order s with $\mathbf{r}(t)$ at the origin. The approach depends on the following details:

(1) There is a recurrence for deriving the linear system directly from $\mathbf{r}(t)$ without explicit substitutions. This recurrence is derived in Section 3.1.

(2) Assume that the degree n of the approximation is smaller than m, the degree of $\mathbf{r}(t)$. There is a function $\varphi(n)$ that determines the order of contact that $g(x, y)$ can achieve. This function is obtained by analyzing the rank of the linear system in Section 3.2.

(3) In Section 3.3 we discuss the error behavior of the implicit approximation, and in Section 3.4 we present several experiments.

Let

$$\mathbf{r}(t) = (x(t), y(t)) = \left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right)$$

be a properly parameterized rational curve of degree m containing the origin, where

$$p(t) = \sum_{i=1}^{m} a_i t^i, \qquad q(t) = \sum_{i=1}^{m} b_i t^i, \qquad w(t) = \sum_{i=0}^{m} c_i t^i$$

We assume that a_m and b_m are not both zero, and that $c_0 \neq 0$. There exists an irreducible polynomial $f^m(x, y) = 0$ of degree m such that

$$f^m(x(t), y(t)) \equiv 0$$

Let $g^n(x, y) = \sum_{i+j=1}^{n} e_{ij} x^i y^j = 0$ be a degree n implicit curve containing the origin. Since $g^n(x, y) = 0$ and $\gamma g^n(x, y) = 0$, where $\gamma \neq 0$, are the same curve, $g^n(x, y) = 0$ has $\varphi(n) = (n^2 + 3n - 2)/2$ coefficients on which the curve depends.

Let $G^n(x, y, z)$ be the homogeneous form of $g^n(x, y)$. Substitution yields

$$g^n\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right) = \frac{G^n(p(t), q(t), w(t))}{(w(t))^n} = \frac{\sum_{i=1}^{nm} \alpha_i t^i}{(w(t))^n}$$

where the α_i are linear combinations of the e_{ij}.

We look for an implicit form $g^n(x, y) = 0$ of degree $n < m$ approximating $\mathbf{r}(t)$ at the origin, and the method of deriving should work whether $w(t) \equiv 1$ or not, that is, irrespective of whether the curve is parameterized polynomially or rationally. The following simple example demonstrates the approach.

Example 3.1 Consider $\mathbf{c}_0(t) = (x(t), y(t)) = (p(t)/w(t), q(t)/w(t))$, where $p(t) = 2t^3 + t^2 - 3t$, $q(t) = t^3 - t^2 - 2t$, and $w(t) = t^2 + 4t + 5$. $\mathbf{c}_0(t)$ is a properly parameterized plane curve containing the origin. Let $g^2(x, y) = e_{10}x + e_{01}y + e_{20}x^2 + e_{11}xy + e_{02}y^2$ be a degree 2 curve containing the origin with symbolic coefficients. Substituting $x(t)$ and $y(t)$ into g^2 yields $g^2(x(t), y(t)) = (\sum_{i=1}^{6} \alpha_i t^i)/(w(t))^2$ where

$$\alpha_1 = -15e_{10} - 10e_{01}$$

$$\alpha_2 = -7e_{10} - 13e_{01} + 9e_{20} + 6e_{11} + 4e_{02}$$

$$\alpha_3 = 11e_{10} - e_{01} - 6e_{20} + e_{11} + 4e_{02}$$

$$\alpha_4 = 9e_{10} + 3e_{01} - 11e_{20} - 8e_{11} - 3e_{02}$$

$$\alpha_5 = 2e_{10} + e_{01} + 4e_{20} - e_{11} - 2e_{02}$$

$$\alpha_6 = 4e_{20} + 2e_{11} + e_{02}$$

By requiring $e_{10} - 1 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, and $\alpha_4 = 0$, we can solve for the unknown coefficients e_{ij}. The resulting g^2 approximates $\mathbf{c}_0$ at the origin to the fourth order of contact.

3.1 A Recurrence for α_k

Since $g^n(x, y) = g^{n-1}(x, y) + \sum_{i+j=n} e_{ij}x^i y^j$, the homogeneous form of $g^n(x, y)$ can be written as

$$G^n(x, y, z) = zG^{n-1}(x, y, z) + \sum_{i+j=n} e_{ij}x^i y^j \tag{1}$$

Let α_k^{n-1} and α_k^n denote the coefficient of t^k in $G^{n-1}(p(t), q(t), w(t))$ and $G^n(p(t), q(t), w(t))$, respectively. It is clear that α_k^n can be derived from the α_i^{n-1}, $i = 1, 2, \ldots, k$ because of (1).

We define $(a(i))_l$ and $(b(j))_l$ as in [20], setting

$$(p(t))^i = \left(\sum_{l=1}^{m} a_l t^l\right)^i = \sum_{l=i}^{im} (a(i))_l t^l$$

and similarly,

$$(q(t))^j = \left(\sum_{l=1}^{m} b_l t^l\right)^j = \sum_{l=j}^{jm} (b(j))_l t^l.$$

For a recurrence to compute the $(a(i))_l$ and $(b(j))_l$ see [20].

From (1), we therefore obtain

$$\alpha_k^n = \text{coefficient of } t^k \text{ in } \left(w(t) \sum_{j=1}^{m(n-1)} \alpha_j^{n-1} t^j + \sum_{i+j=n} e_{ij}(p(t))^i (q(t))^j\right)$$

$$= \sum_{j=1}^{k} \alpha_j^{n-1} c_{k-j} + \sum_{i+j=n} \sum_{p+q=k} e_{ij}(a(i))_p (b(j))_q.$$

In particular, $\alpha_k^1 = e_{10} a_k + e_{01} b_k$.

For an integral parametric curve $\mathbf{r}(t)$, a straightforward computation shows that the α_k^n specialize to

$$\alpha_k^n = \begin{cases} \alpha_k^{n-1} & 1 \le k \le n-1 \\ \alpha_k^{n-1} + \sum_{i+j=n} \sum_{p+q=k} e_{ij}(a(i))_p (b(j))_q & n \le k \le (n-1)m \\ \sum_{i+j=n} \sum_{p+q=k} e_{ij}(a(i))_p (b(j))_q & (n-1)m < i \le nm \end{cases}$$

3.2 Derivation of the Method

3.2.1 *Rank of the Linear System.* Having explained how to obtain the α_k^n, we now show that the coefficient matrix of the linear system defined by setting $\alpha_k^n = 0$, $k = 1, 2, \ldots, nm$, has rank at least $\varphi(n)$. We are able to determine a nontrivial solution to unknown coefficients by setting one of the coefficients to 1 and solving the system $\alpha_1^n = 0, \alpha_2^n = 0, \ldots, \alpha_s^n = 0$ for $s \ge \varphi(n)$ chosen such that the rank is $\varphi(n)$.

Let $\mathbf{e}_n = (e_{10}, e_{01}, e_{20}, e_{11}, e_{02}, \ldots, e_{n0}, e_{(n-1)1}, \ldots, e_{1(n-1)}, e_{0n})^{\mathrm{T}}$ be the vector of unknowns and write the system of equations $\alpha_1^n = 0, \alpha_2^n = 0, \ldots, \alpha_{nm}^n = 0$ in matrix form:

$$\mathbf{A}_{mn}\mathbf{e}_n = 0 \tag{2}$$

Note that $\mathbf{A}_{mn}$ is a nm by $\varphi(n) + 1$ matrix. Furthermore, the maximum rank of $\mathbf{A}_{mn}$ is $\varphi(n) + 1$ since $m \ge n$ and $nm \ge \varphi(n) + 1$. Example 3.2 shows matrix $\mathbf{A}_{32}$ symbolically.

Example 3.2 For $m = 3$ and $n = 2$, $\mathbf{A}_{32}$ is

$$\begin{pmatrix} a_1c_0 & b_1c_0 & 0 & 0 & 0 \\ a_1c_1 + a_2c_0 & b_1c_1 + b_2c_0 & a_1^2 & a_1b_1 & b_1^2 \\ a_1c_2 + a_2c_1 + a_3c_0 & b_1c_2 + b_2c_1 + b_3c_0 & 2a_1a_2 & a_1b_2 + a_2b_1 & 2b_1b_2 \\ a_1c_3 + a_2c_2 + a_3c_1 & b_1c_3 + b_2c_2 + b_3c_1 & 2a_1a_3 + a_2^2 & a_1b_3 + a_2b_2 + a_3b_1 & 2b_1b_3 + b_2^2 \\ a_2c_3 + a_3c_2 & b_2c_3 + b_3c_2 & 2a_2a_3 & a_2b_3 + a_3b_2 & 2b_2b_3 \\ a_3c_3 & b_3c_3 & a_3^2 & a_3b_3 & b_3^2 \end{pmatrix}$$

When computing the local implicit approximation $g^n(x, y)$ of $\mathbf{r}(t)$, if the rank of $\mathbf{A}_{mn}$ is at least $\varphi(n)$, then we can select one coefficient of $g^n(x, y)$ to be 1 and determine the others by selecting the first s rows of (2) and choosing s such that the system has rank $\varphi(n)$.

Let $f^m(x, y) = 0$ be the exact irreducible implicit form of $\mathbf{r}(t)$ with $F^m(x, y, z)$ as its corresponding homogeneous form, and let f^n, $n < m$, be the degree n *initial segment* of f^m with its corresponding homogeneous form $F^n(x, y, z)$, that is, $f^m(x, y) = f^n(x, y) +$ *terms with degree* $> n$. Also, let $\sum_{i=1}^{nm} \bar{b}_i t^i \equiv F^n(p(t), q(t), w(t))$ and $\bar{\mathbf{b}}_{mn} \equiv (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{mn})^{\mathrm{T}}$.

LEMMA 3.1 1. *$f^n(x, y)$ is the zero polynomial if and only if $\bar{\mathbf{b}}_{mn} = \mathbf{0}$. 2. If $f^n(x, y)$ is a nonzero polynomial, then $\bar{b}_1 = \bar{b}_2 = \cdots = \bar{b}_n = 0$ and $\bar{\mathbf{b}}_{mn} \neq \mathbf{0}$.*

PROOF. Part 1: "$\Rightarrow$" trivial. "$\Leftarrow$" Suppose $f^n(x, y)$ is a nonzero polynomial. Since $\bar{\mathbf{b}}_{mn} = \mathbf{0}$,

$$F^n(p(t), q(t), w(t)) = 0 = (w(t))^n f^n\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right)$$

for all t and then $f^n(p(t)/w(t), q(t)/w(t)) = 0$, for all t with possibly finitely many exceptions, where $w(t) = 0$. Thus $f^n(x, y)$ with $n < m$ also represents $\mathbf{r}(t)$, which contradicts the irreducibility of $f^m(x, y)$.

Part 2: Since $f^m(x, y) = 0$ is the implicit form of $\mathbf{r}(t)$,

$$f^m\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right) = 0$$

for all t except finitely many t where $w(t) = 0$. Thus

$$F^m(p(t), q(t), w(t)) = (w(t))^m f^m\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right) = 0$$

for all t. From

$$F^n(p(t), q(t), w(t)) + \sum_{i+j=n+1}^{m} e_{ij}(p(t))^i(q(t))^j(w(t))^{m-i-j} = 0$$

for every t, we have

$$\sum_{i=1}^{nm} \bar{b}_i t^i = -\sum_{i+j=n+1}^{m} e_{ij}(p(t))^i(q(t))^j(w(t))^{m-i-j} = \sum_{i=n+1}^{nm} \bar{b}_i' t^i$$

for every t. By comparison, we have $\bar{b}_1 = \bar{b}_2 = \cdots = \bar{b}_n = 0$. The rest of Part 2 follows from Part 1. ☐

Since $f^n(x, y)$ is an initial segment of $f^m(x, y)$, it could be either a zero polynomial or a nonzero polynomial with zero or nonzero $\bar{\mathbf{b}}_{mn}$ vectors respectively. If $\bar{\mathbf{b}}_{mn}$ is known beforehand, the coefficient vector $\mathbf{e}_n$ of $f^n(x, y)$ is uniquely determined by $\mathbf{A}_{mn}\mathbf{e}_n = \bar{\mathbf{b}}_{mn}$, which is an overdetermined linear system. Note that, for a fixed n, the elements of the matrix $\mathbf{A}_{mn}$ depend only on the coefficients of $p(t)$, $q(t)$, and $w(t)$. The following results characterize the rank of $\mathbf{A}_{mn}$.

LEMMA 3.2 *If* $\mathbf{r}(t)$ *is a properly parameterized rational plane curve of degree* m*, then for* $n < m$*, we have*

$$rank(\mathbf{A}_{mn}) = \varphi(n) + 1$$

PROOF. Suppose, knowing $\overline{\mathbf{b}}_{mn}$, we want to determine the coefficients of $f^n(x, y)$ by solving the overdetermined linear system $\mathbf{A}_{mn}\mathbf{e}_n = \overline{\mathbf{b}}_{mn}$, where $\mathbf{e}_n$ is the coefficient vector of the general degree n polynomial.

If $\overline{\mathbf{b}}_{mn} = \mathbf{0}$, then $\mathbf{A}_{mn}\mathbf{e}_n = \mathbf{0}$ is a homogeneous system. If $rank(\mathbf{A}_{mn}) < \varphi(n) + 1$, there will be infinitely many nontrivial solutions as well as the trivial solution for this linear system. This cannot be true by Lemma 3.1. Thus, $rank(\mathbf{A}_{mn}) = \varphi(n) + 1$ if $\overline{\mathbf{b}}_{mn} = \mathbf{0}$.

If $\overline{\mathbf{b}}_{mn} \neq \mathbf{0}$, then $\mathbf{A}_{mn}\mathbf{e}_n = \overline{\mathbf{b}}_{mn}$ is a *consistent* nonhomogeneous system since there is always a solution, that is, the coefficient of $f^n(x, y)$. If $rank(\mathbf{A}_{mn}) < \varphi(n) + 1$, this system will have infinitely many solutions. Let $\mathbf{e}_n^*$ be one of the infinitely many solutions and $\mathbf{e}_n^* \neq \mathbf{e}_n$, where $\mathbf{e}_n$ is the coefficient vector of $f^n(x, y)$. Let also $h^n(x, y)$ be the corresponding polynomial of $\mathbf{e}_n^*$ and

$$h^m(x, y) \equiv h^n(x, y) + \text{terms of } f^m(x, y) \text{ with degree} > n$$

Let $H^m(x, y, z)$ and $H^n(x, y, z)$ be the homogeneous polynomials of $h^m(x, y)$ and $h^n(x, y)$, respectively. Since $\mathbf{A}_{mn}\,\mathbf{e}_n = \mathbf{A}_{mn}\mathbf{e}_n^* = \overline{\mathbf{b}}_{mn}$,

$$H^n(p(t), q(t), w(t)) = F^n(p(t), q(t), w(t))$$

for every t, and thus

$$H^m(p(t), q(t), w(t)) = F^m(p(t), q(t), w(t)) = 0$$

for all t. We then have

$$f^m\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right) = h^m\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right) = 0$$

for all but finitely many t. Hence $f^m(x, y)$ and $h^m(x, y)$ represent the same algebraic curve. Since $f^n(x, y) \neq h^n(x, y)$, $f^m(x, y)$ and $h^m(x, y)$ differ by more than a constant factor, which contradicts the fact that the equation of an irreducible curve is unique to within a constant factor. Therefore $rank(\mathbf{A}_{mn})$ must be $\varphi(n) + 1$ if $\overline{\mathbf{b}}_{mn} \neq \mathbf{0}$. □

LEMMA 3.3 *If* $\mathbf{r}(t)$ *is a properly parameterized rational plane curve of degree* m*, then for* $n = m$*, we have*

$$rank(\mathbf{A}_{mm}) = \varphi(m).$$

PROOF. If $n = m$, $f^m(p(t)/w(t), q(t)/w(t)) = 0$ for all t except finitely many t where $w(t) = 0$, and then $F^m(p(t), q(t), w(t)) = 0$ for all t, we thus have $\mathbf{A}_{mm}\mathbf{e}_m = \mathbf{0}$, which is an overdetermined linear homogeneous system. Since we have only a trivial solution if $\text{rank}(\mathbf{A}_{mm}) = \varphi(m) + 1$, $\text{rank}(\mathbf{A}_{mm})$ must be less than or equal to $\varphi(m)$.

Suppose $r \equiv \text{rank}(\mathbf{A}_{mm}) < \varphi(m)$; then the solution space of the overdetermined homogeneous system has as its basis $p \equiv \varphi(m) + 1 - r$ linearly independent vectors, and every solution of this system is the linear combination of these p

solutions. Now suppose that $r < \varphi(m)$; then the system has a solution space spanned by $p > 2$ linearly independent vectors, say $\mathbf{e}_m^1, \mathbf{e}_m^2, \ldots, \mathbf{e}_m^p$. Let $f_i^m(x, y)$ be the corresponding polynomial with coefficient vector $\mathbf{e}_m^i$, and $F_i^m(x, y, z)$ be the homogeneous form of $f_i^m(x, y)$, $i = 1, 2, \ldots, p$. Since

$$F_i^m(p(t), q(t), w(t)) = 0 = (w(t))^m f_i^m\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right)$$

for all t, $1 \le i \le p$, we have $f_i^m(p(t)/w(t), q(t)/w(t)) = 0$ for all t with finitely many exceptions, where $w(t) = 0$, $1 \le i \le p$. Thus the irreducible curve $f^m(x, y) = 0$ can be represented by $f_i^m(x, y)$, $i = 1, 2, \ldots, p$, which is not different within only a constant factor because $\mathbf{e}_m^1, \mathbf{e}_m^2, \ldots, \mathbf{e}_m^p$ are linearly independent. By the above arguments, we can conclude that $\text{rank}(\mathbf{A}_{mm}) = \varphi(m)$. □

By assigning one variable to be 1, the existence of a nontrivial solution of $\mathbf{A}_{mm}\mathbf{e}_m = \mathbf{0}$ is guaranteed by Lemma 3.3, and it is the coefficient vector of the exact implicitization of $\mathbf{r}(t)$.

Observe that Lemmas 3.2 and 3.3 are not valid for *improperly parameterized* rational plane curves, as shown in the following example:

Example 3.3 Let $\mathbf{c}_1(t) = (x(t), y(t)) = (t^2 + 2t, t^4 + 4t^3 + 6t^2 + 4t)$. Since $x(t) = s(t)$ and $y(t) = (s(t))^2$, where $s(t) = t^2 + 2$, $\mathbf{c}_1(t)$ is improperly parameterized. For $n = 2$, the rank of $\mathbf{A}_{42}$ is 4, whereas $\varphi(2) + 1$ is 5.

We summarize Lemmas 3.2 and 3.3 in the following:

THEOREM 3.4 *If $\mathbf{r}(t)$ is a properly parameterized rational plane curve of degree m then*

$$rank(\mathbf{A}_{mn}) = \begin{cases} \varphi(n) + 1 & \text{if } n < m \\ \varphi(n) & \text{if } n = m \end{cases}$$

3.2.2 *The Algorithm.* Because of Theorem 3.4, we may compute the degree n local implicit approximation as follows:

Let

$$\mathbf{B}\mathbf{e}_n = \mathbf{0} \tag{3}$$

be the subsystem of (2) consisting of the first s equations of $\mathbf{A}_{mn}\mathbf{e}_n = \mathbf{0}$, such that $\mathbf{B}$ has rank $\varphi(n)$. If the origin is not a singular curve point, then augment the system (3) with an equation $e_{01} = 1$ or $e_{10} = 1$ according to whether $x'(0) \neq 0$ or $y'(0) \neq 0$. If the origin is a singular curve point, on the other hand, then (3) is augmented by the equation $e_{ij} = 1$ where the indices i and j are selected by inspecting the system. In this way, a linear system

$$\mathbf{C}\mathbf{e}_n = \mathbf{b} \tag{4}$$

is obtained that has a nontrivial solution for $\mathbf{e}_n$. Since $\alpha_1 = 0$, $\alpha_2 = 0, \ldots, \alpha_s^n = 0$, the curve g^n so defined must have contact of degree at least s to $\mathbf{r}(t)$ at the origin.

There may be cases in which the system (4) is inconsistent, that is, the augmented matrix $[\mathbf{C}, \mathbf{b}]$ is of rank $\varphi(n) + 2$, whereas $\mathbf{C}$ has rank $\varphi(n) + 1$. In this case, the linear system can be modified to ensure consistency. For instance, when computing g^2 of $\mathbf{c}_2(t) = (t, t^9)$, α_9^2 should be removed from, and $\alpha_{18}^2 = 0$ should be added to the system (4), resulting in $e_{01} - 1 = 0$, $\alpha_1^2 = 0$, $\alpha_2^2 = 0, \ldots,$ $\alpha_8^2 = 0$, $\alpha_{10}^2 = 0$, $\alpha_{18}^2 = 0$. In this way, a $g^2(x, y) = y$ is obtained that has an eighth order of contact and is irreducible.

3.2.3 *Irreducibility of Implicit Approximations.* When the origin is a regular curve point we show that the implicit approximation $g^n(x, y)$ of $\mathbf{r}(t)$ at the origin is irreducible whenever the linear system (4) is consistent. Note that the local implicit approximations of different degrees have the same linear terms if the equations augmented to (3) are the same. In the following lemma and proposition, we assume that (4) is a consistent system. Also, let $s(n)$ be the order of contact made by the degree n implicit approximation $g^n(x, y)$.

LEMMA 3.5 *If $g^n(x, y)$ and $g^{n-1}(x, y)$ of $\mathbf{r}(t)$ at a nonsingular point* (0, 0) *are computed by augmenting the same $\alpha = 0$ to the system* (3), *where $\alpha = 0$ is either $e_{10} - 1 = 0$ or $e_{01} - 1 = 0$, then we have $s(n-1) < s(n)$.*

PROOF. Let $\bar{g}^n(x, y) = \frac{1}{2}(g^n(x, y) + g^{n-1}(x, y))$. Suppose $s(n-1) \geq s(n)$; the $\bar{g}^n(x, y)$ so defined has the following four properties: (1) $\bar{g}^n$ is of degree n; (2) $\bar{g}^n \neq g^n$; (3) $\bar{g}^n$ has the same linear terms as g^n since g^n and g^{n-1} have the identical linear terms; (4) $\bar{g}^n$ has the order of contact larger than or equal to $s(n)$ since $s(n-1) \geq s(n)$ is assumed. From properties 1, 3, and 4, the coefficients of $\bar{g}^n$ satisfy the linear system that is used to compute the coefficients of g^n, but property 2 contradicts the uniqueness of the solution of a nonsingular linear system. Thus $s(n-1) < s(n)$. □

By induction and Lemma 3.5, we can show that $s(n)$ is strictly monotone.

PROPOSITION 3.6 *At the nonsingular point* (0, 0), *the degree n local implicit approximation $g^n(x, y)$ of the degree $m > n$ properly parameterized parametric curve $\mathbf{r}(t) = (x(t), y(t))$ is irreducible.*

PROOF. Suppose $g^n(x, y)$ is reducible, and $g^n = g^k g^l$, where $n = k + l$ and k, $l > 0$. Since g^n contains linear terms, one of the g^k and g^l must have a constant term. Let $g^k(x, y) = \sum_{i+j=1}^{k} p_{ij} x^i y^j$, where p_{10} and p_{01} are not both zero, and $g^l(x, y) = \sum_{i+j=0}^{l} q_{ij} x^i y^j$, where $q_{00} \neq 0$. Let also $g^n(x(t), y(t)) = \sum_{i=1}^{mn} \alpha_i t^i$, $g^k(x(t), y(t)) = \sum_{i=1}^{mk} \beta_i t^i$, and $g^l(x(t), y(t)) = \sum_{i=0}^{ml} \gamma_i t^i$, where $\gamma_0 = q_{00}$. We thus have

$$\sum_{i=1}^{mn} \alpha_i t^i = \left(\sum_{i=1}^{mk} \beta_i t^i\right)\left(\sum_{i=0}^{ml} \gamma_i t^i\right)$$

The coefficients of $g^n(x, y)$ are computed by solving the nonsingular system as (4) for some $s \geq \varphi(n)$. Moreover, $s(n)$, the order of contact of g^n, is greater than or equal to s. Thus the coefficients of g^n satisfy the linear system $\alpha = 0$, $\alpha_1^n = 0$, $\alpha_2^n = 0, \ldots, \alpha_{s(n)}^n = 0$, where, without loss of generality, we assume that $\alpha = e_{10} - 1 = 0$. The above linear system can be represented in terms of β_i and γ_i

as follows:

$$q_{00}p_{10} = 1$$
$$q_{00}\beta_1 = 0$$
$$q_{00}\beta_2 + \beta_1\gamma_1 = 0$$
$$q_{00}\beta_3 + \beta_2\gamma_1 + \beta_1\gamma_2 = 0$$
$$\vdots$$
$$q_{00}\beta_{s(n)} + \beta_{s(n)-1}\gamma_1 + \cdots + \beta_1\gamma_{s(n)-1} = 0,$$

which implies $q_{00}p_{10} = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_{s(n)} = 0$. Thus g^k has either an order of contact larger than or equal to $s(n)$ if $s(n) < km$, or $g^k(x(t), y(t)) = 0$ for all t if $s(n) \geq km$. The first result contradicts the fact that $s(n)$ is strictly monotone, and the second contradicts the irreducibility of the exact implicitization of $\mathbf{r}(t)$. Thus g^n is irreducible. □

When the origin is a singular curve point, the implicit approximation is not always irreducible. For example, the degree n implicit approximation of $\mathbf{c}_3(t) = (t^2, t^5)$, with implicit form $x^5 - y^2 = 0$, is $y^n = 0$ when $n < 5$. Note that $y = 0$ is the curve tangent at the origin.

3.3 Error Analysis

3.3.1 *Quality of the Approximation.* Given ϵ, let $T(\epsilon, n) > 0$ be such that for all $|t| < T(\epsilon, n)$ the orthogonal distance $d(t, n)$ between point $(x(t), y(t))$ and the degree n approximation $g^n(x, y) = 0$ is less than ϵ, assuming that $(x(t), y(t))$ is a regular curve point. The distance $d(t_p, n)$ from a point $P = (x_p, y_p) = (x(t_p), y(t_p))$ on the curve $\mathbf{r}(t)$ to the degree n approximation $g^n(x, y) = 0$ is the solution of a difficult nonlinear system. A reasonable estimate of $d(t_p, n)$ would be the distance to the $g^n(x, y) = 0$ in a direction orthogonal to the level curve $g^n(x, y) = c$, where $c = g^n(x_p, y_p)$, denoted by $d'(t_p, n)$. Note that $d'(t, n) \geq d(t, n)$ since $d(t, n)$ is the shortest distance from the point to the curve. Let $P' = (x'_p, y'_p)$ be the point on $g^n(x, y) = 0$ on which $g^n(x, y) = 0$ intersects the line orthogonal to level curve $g^n(x, y) = c$ at P; see Figure 1. The Taylor series on $P' = (x'_p, y'_p)$ with respect to P is

$$g^n(x'_p, y'_p) = g^n(x_p, y_p) + d'(t_p, n) \cdot \nabla g^n(x_p, y_p) + \text{higher order terms}$$

Taking the linear term, since $g^n(x'_p, y'_p) = 0$, $d'(t_p, n)$ can be approximated by $d''(t_p, n)$ where

$$d''(t_p, n) = \frac{g^n(x_p, y_p)}{\|\nabla g^n(x_p, y_p)\|} = \frac{g^n(x(t_p), y(t_p))}{[(g^n_x(x(t_p), y(t_p)))^2 + (g^n_y(x(t_p), y(t_p)))^2]^{1/2}}$$

Note that $d''(t, n)$ may be less than, greater than, or equal to $d(t, n)$, although $d'(t, n)$ is always greater than or equal to $d(t, n)$.

We have found no method for computing $T(\epsilon, n)$ analytically. However, in practice we only need a method of obtaining a reasonably good estimate of

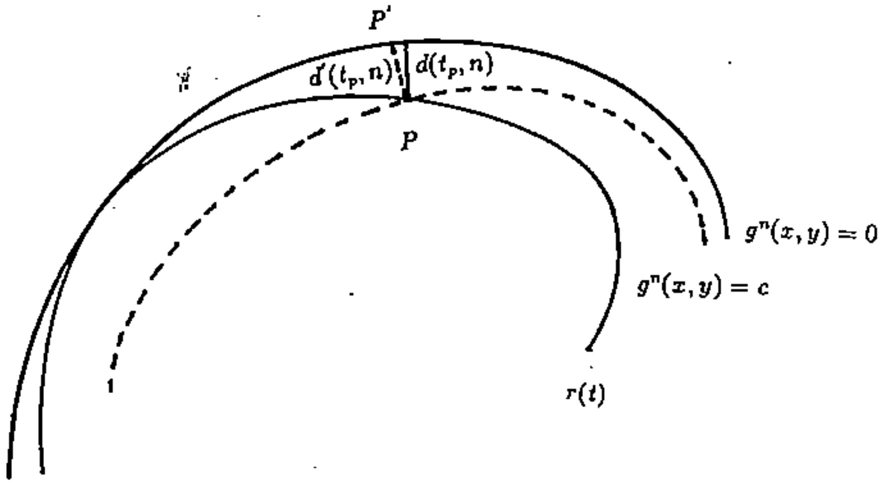

Figure 1

$T(\epsilon, n)$. Thus it is desirable to determine $T'(\epsilon, n)$, for given ϵ and n, such that $d''(t, n) \le \epsilon$ for $|t| \le T'(\epsilon, n)$.

Since $2ab \le a^2 + b^2$ for any a and b, we have $(|a|\,|b|)^{1/2} \le (|a| + |b|)/2 \le ((a^2 + b^2)/2)^{1/2}$, so that

$$d''(t, n) \le \hat{d}(t, n) \equiv \frac{\sqrt{2} g^n(x(t), y(t))}{|g_x^n(x(t), y(t))| + |g_y^n(x(t), y(t))|}.$$

When tracing $\mathbf{r}(t)$, we can detect the first value of t such that $\hat{d}(t, n) \le \epsilon$ and $\hat{d}(t + \Delta t, n) > \epsilon$, where Δt is the step distance for t.

3.3.2 *Curve Translation to the Origin.* In the derivation of the approximant we assume that $\mathbf{r}(0) = (0, 0)$, that is, we require that $\mathbf{r}(t)$ be translated to the origin and reparameterized. Since this may incur additional inaccuracies we comment on it now.

Translation of $\mathbf{r}(t)$ to the origin is a simple operation that incurs a minimum of error. For, with $p = (u, v)$ as the curve point to be brought to the origin, the translated curve is simply

$$x_1(t) = x(t) - u = (p(t) - uw(t))/w(t)$$

$$y_1(t) = y(t) - v = (q(t) - vw(t))/w(t).$$

So, we have to subtract two polynomials in order to bring p to the origin.

Now assume that $\mathbf{r}(t_0) = p$, and consider reparameterization such that p not only is moved to the origin but that also $t_0 = 0$ for the reparameterized curve. Here we need to substitute $\bar{t} + t_0$ for t, that is,

$$x_2(\bar{t}) = x_1(\bar{t} + t_0)$$

$$y_2(\bar{t}) = y_1(\bar{t} + t_0)$$

is the final curve. As observed in the introduction, although substitution is conceptually simple, it nonetheless may introduce numerical errors that could be significant. According to experiments by Prakash and Patrikalakis [23], Kahan's method described in [13] exhibits good numerical stability and offers one method of implementing the needed reparameterization.

A second method would be to avoid reparameterization altogether by reformulating the derivation of the approximant given before. That is, we consider $\mathbf{r}(t)$ containing the origin at which t is not necessarily 0, seeking again an implicit approximant at the origin. Clearly this is possible and requires only straightforward modifications of our method. In fact, even translation of the point to the origin can be avoided by such modifications. The details are routine.

3.4 Experiments

A good local approximation of a curve provides a more accurate local approximation and a larger interval $T(\epsilon, n)$ of approximation, for a given ϵ, when the degree n of the approximation increases. The local implicit approximation $g^n(x, y) = 0$ of $\mathbf{r}(t)$ is determined by $\varphi(n)$ linear conditions imposed on its coefficients, where $\varphi(n)$ is the degree of freedom of $g^n(x, y)$. Thus, as n increases, more conditions can be satisfied, and finally the exact implicitization is obtained when $n = m$. Hence our local implicit approximation is capable of approximating a given curve not only locally but also globally in the sense that $T(\epsilon, n)$, for a given ϵ, is larger when n increases. On the other hand, a local explicit approximation is limited because of the asymmetry introduced by making one variable an explicit function of the other. Thus, a local explicit approximation can only approximate the given curve locally for $|x| < R$, where R is its radius of convergence, no matter how high the degree of approximant is.

We give as an example the approximation of several parametric curves that are not singular at the origin, showing both implicit and explicit approximations.

Example 3.4 We present four curve examples

$$\mathbf{c}_4(t) = (t^6 + t^5 - 2t^3 + 3t^2 + 12t, t^6 - t^5 + t^4 - 4t^3 - 2t^2 + 24t)$$

$$\mathbf{c}_5(t) = (3t^6 - 4t^5 - 8t^3 + 6t^2 + 3t, -3t^6 + 4t^5 + 5t^4 - 6t^3 - 8t^2 + 3t)$$

$$\mathbf{c}_6(t) = (3t^6 + t^5 - 2t^4 + 38t^3 - 5t^2 - 14t, t^6 - 12t^5 - 2t^4 + 2t^3 - 7t^2 + 13t)$$

$$\mathbf{c}_7(t) = ((t^6 + 3t^5 - 6t^4 + 4t^3 - 36t^2 + 36t)/w(t),$$
$$(3t^6 + t^5 - 2t^4 + 39t^3 - 69t^2 + 33t)/w(t)),$$

where

$$w(t) = 7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7.$$

The curves of $\mathbf{c}_4(t)$, $\mathbf{c}_5(t)$, $\mathbf{c}_6(t)$, and $\mathbf{c}_7(t)$ with t in $[-1, 1]$ and their local implicit approximations and local explicit approximations are shown in Figures 2, 3, 4, and 5. Note the good quality of local implicit approximation. Tables I, II, and III, for $\mathbf{c}_4(t)$, $\mathbf{c}_5(t)$, and $\mathbf{c}_6(t)$, respectively, list the y-values of a sequence of x-values to quantify how accurately the low-degree local implicit forms approximate the original curves. The corresponding values of local explicit forms are also listed for comparison. For such examples, we observe that

(a) For local implicit approximation, $d(t, n + k) < d(t, n)$ for t in $[-1, 1]$ and $k \geq 1$. In addition, $T(\epsilon, n) < T(\epsilon, n + k)$ for $k \geq 1$.

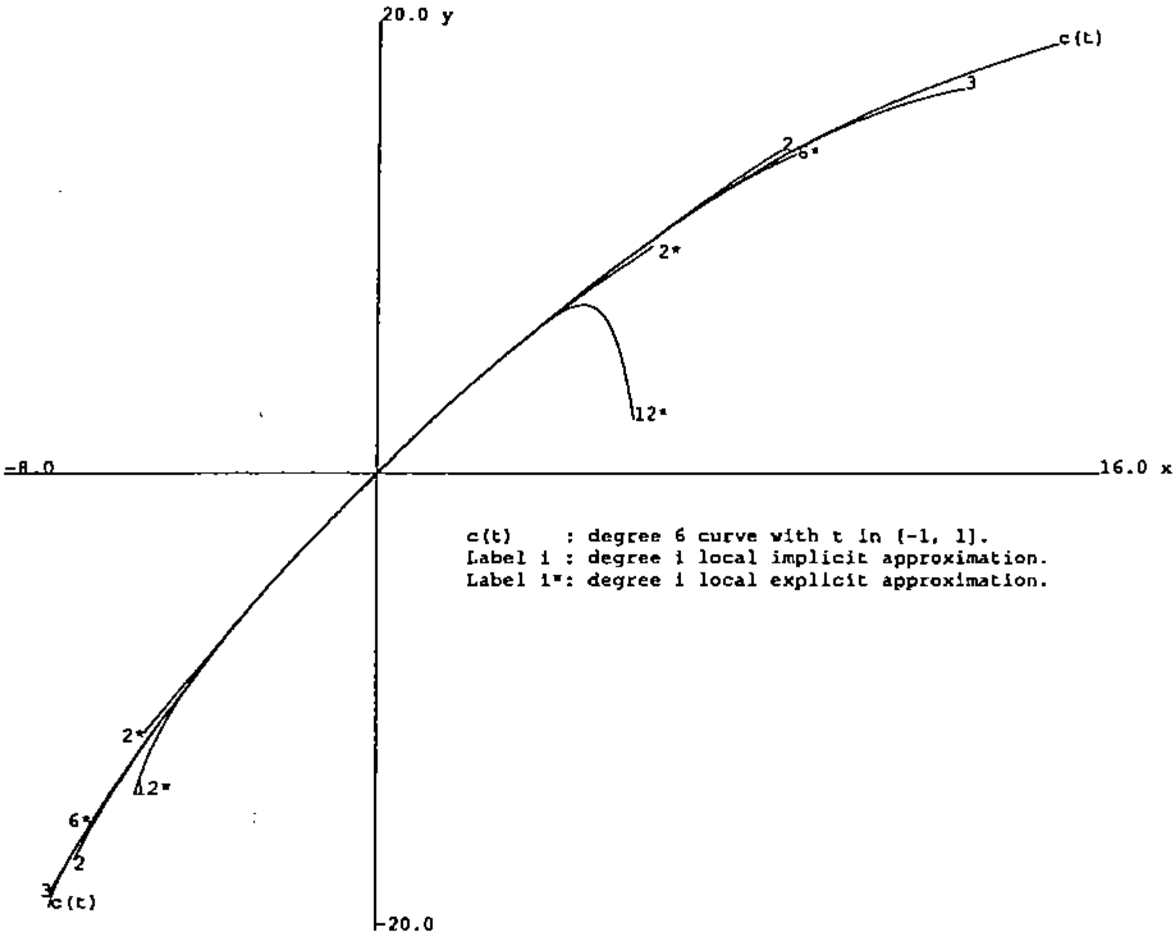

Fig. 2. $c_4(t) = (t^6 + t^5 - 2t^3 + 3t^2 + 12t, t^6 - t^5 + t^4 - 4t^3 - 2t^2 + 24t)$.

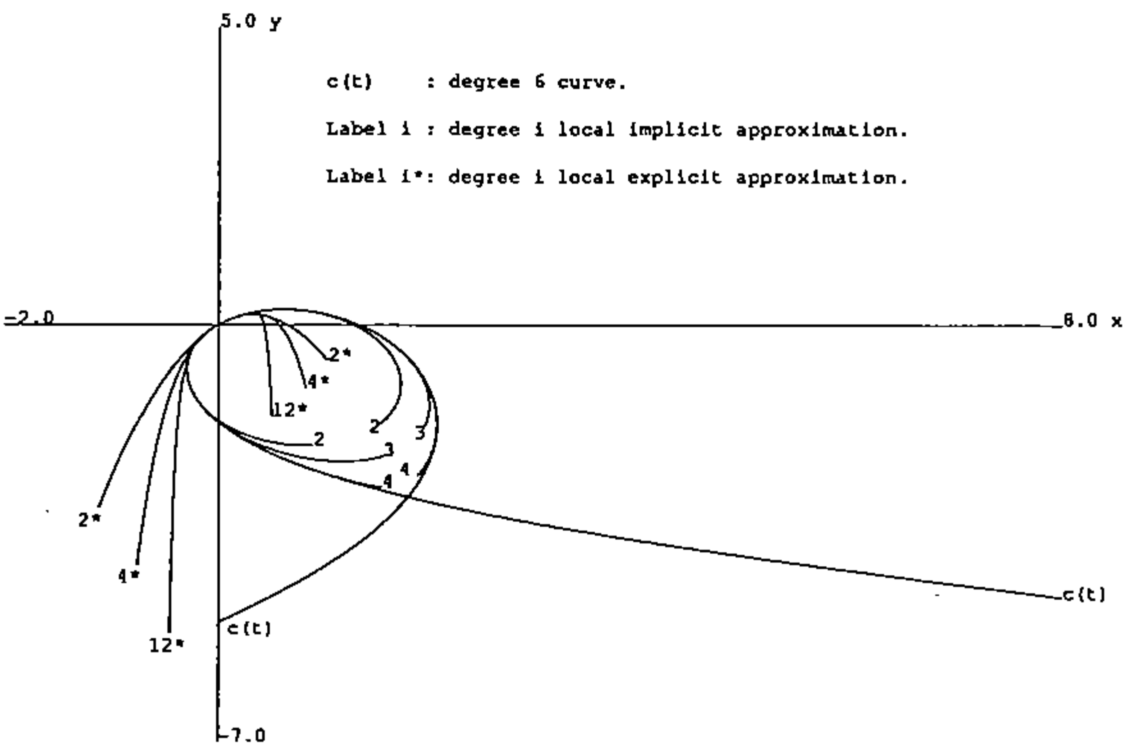

Fig. 3. $c_6(t) = (3t^6 - 4t^5 - 8t^3 + 6t^2 + 3t, -3t^6 + 4t^5 + 5t^4 - 6t^3 - 8t^2 + 3t)$.

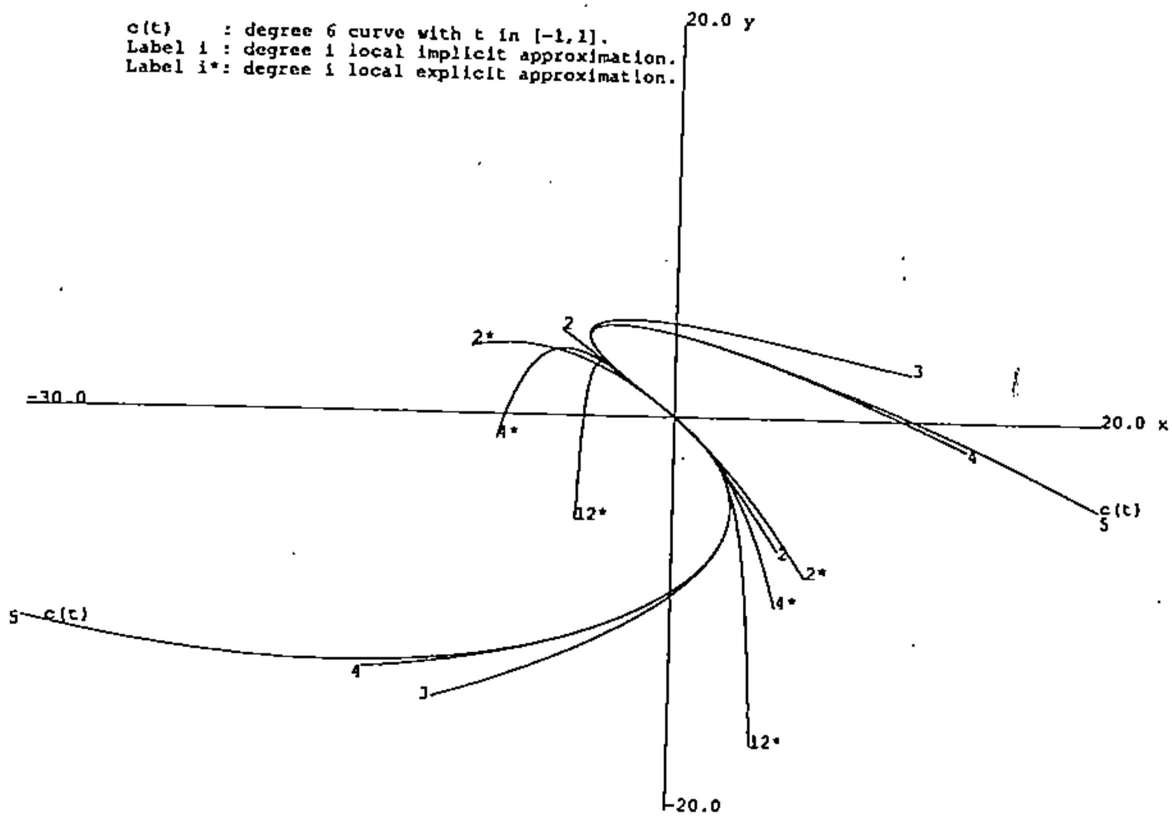

Fig. 4. $c_6(t) = (3t^6 + t^5 - 2t^4 + 38t^3 - 5t^2 - 14t,\ t^6 - 12t^5 - 2t^4 + 2t^3 - 7t^2 + 13t)$.

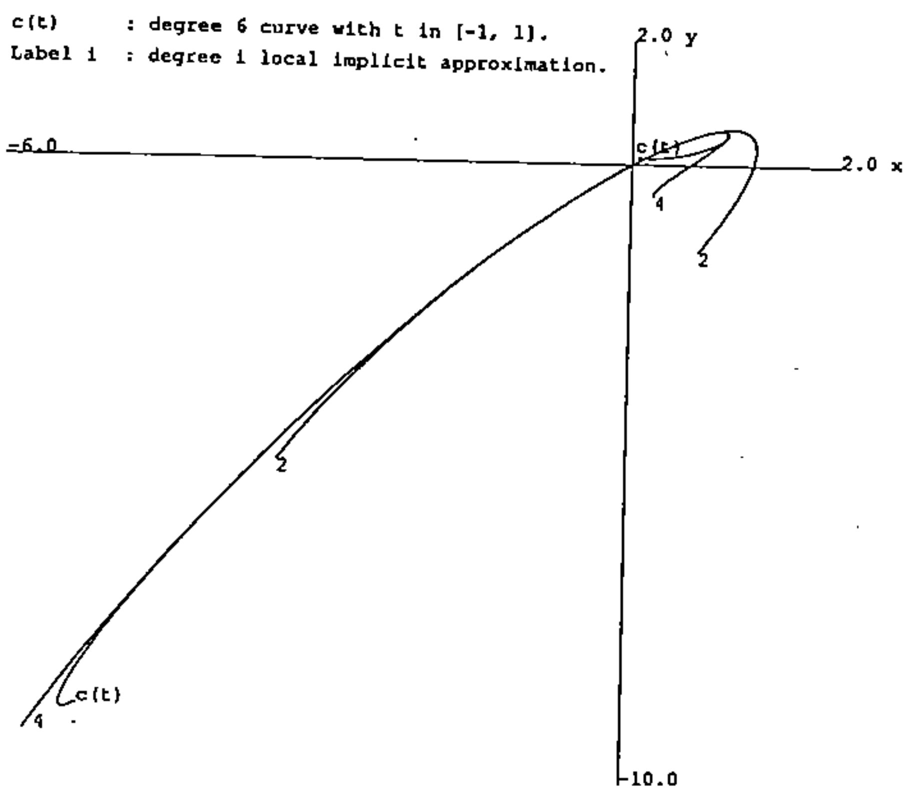

Fig. 5. $c_7(t)$

$$= \left(\frac{t^6 + 3t^5 - 6t^4 + 4t^3 - 36t^2 + 36t}{7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7}, \frac{3t^6 + t^5 - 2t^4 + 39t^3 - 69t^2 + 33t}{7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7}\right).$$

Table I

	x value								
	−5.0	−3.0	−2.0	−1.25	0	1.25	2.0	4.0	7.0
exact	−11.845579	−6.5811195	−4.2442303	−2.5918415	0	2.4172442	3.793271	7.213065	11.663555
li 4	−11.845584	−6.5811195	−4.2442300	−2.5918415	0	2.4172442	3.793271	7.213066	11.663522
li 3	−11.843186	−6.5811080	−4.2442300	−2.5918415	0	2.4172442	3.793271	7.213022	11.658528
li 2	−11.910466	−6.5846540	−4.2446346	−2.5918765	0	2.4172716	3.793540	7.220462	11.761091
le 2	−23.889015	−11.000046	−6.2222424	−3.3680634	0	1.6319366	1.777758	−0.8889694	−13.222469
le 4	−24.304024	−11.078828	−6.2439766	−3.3730750	0	1.6359663	1.793057	−0.79231834	−12.91141
le 6	−24.336723	−11.081027	−6.2442436	−3.3730989	0	1.6359847	1.793233	−0.7881178	−12.878107
le 12	−25.297968	−11.082955	−6.2442613	−3.3730989	0	1.6359840	1.793167	−0.9260575	−103.35114

li n: local implicit form of degree n. le n: local explicit form of degree n.

Table II

	x value							
	−0.25	−0.1	0.0	0.1	0.25	1.0	1.25	1.5
exact	−0.4291671 −1.1245294	−0.11828928 −1.4897133	0 −1.6495429	0.08622920 −1.7816917	0.17517897 −1.9479930	0.17769729 −2.5130520	0.043922530 −2.6506336	−0.16892694 −2.7737687
li 4	−0.4291671 −1.1245286	−0.11828928 −1.4897133	0 −1.6495163	0.08622920 −1.7816285	0.17517897 −1.9478126	0.17769726 −2.5069103	0.043921884 −2.6342275	−0.168934 −2.7259927
li 3	−0.4291676 −1.1233613	−0.11828928 −1.4840809	0 −1.6388266	0.08622920 −1.7638947	0.17517897 −1.9152321	0.17769955 −2.2900155	0.043849964 −2.2978785	−0.16974114 −2.2328222
li 2	−0.4286827 −1.1285945	−0.11828820 −1.4734732	0 −1.6147509	0.08622874 −1.7239690	0.17514941 −1.8473737	0.16556032 −2.0102053	0.008065359 −1.91018380	−0.27427490 −1.685317
le 12	−0.42683282 —	−0.11828927 —	0 —	0.0862292 —	0.17486227 —	−9,337.529 —	−143,129.16 —	−1,323,236.8 —
le 6	−0.41469532 —	−0.11827987 —	0 —	0.0862238 —	0.17268895 —	−18.46853 —	−75.29226 —	−234.37852 —
le 4	−0.39822853 —	−0.11815020 —	0 —	0.0861461 —	0.16890110 —	−2.872428 —	−7.885883 —	−17.354166 —
le 2	−0.34722220 —	−0.11555555 —	0 —	0.0844444 —	0.15277778 —	0.55555556 —	−1.1805556 —	−2.0 —

li n: local implicit form of degree n. le n: local explicit form of degree n.

Table III

	x value								
	−3.5	−2.5	−0.5	−0.25	0	0.25	0.5	2.0	2.5
exact	3.1174486	2.1534863	.45135155	.22867906	0	−.23612773	−.48143223	−2.3418179	−3.4202664
li 4	3.1174495	2.1534863	.45135155	.22867906	0	−.23612773	−.48143226	−2.3418179	−3.4202664
li 3	3.1159813	2.1534630	.45135158	.22867908	0	−.23612775	−.48143230	−2.3418021	−3.4196750
li 2	2.9099880	2.1148510	.45132570	.22867821	0	−.23612662	−.48139048	−2.2394760	−2.0944711
le 12	2.7423260	2.1480120	.45135152	.22867905	0	−.23612773	−.48143230	−2.3401886	−3.3552332
le 6	2.8504105	2.1270490	.45135110	.22867908	0	−.23612775	−.48143163	−2.3187237	−3.2120173
le 4	2.7878397	2.0934718	.45132630	.22867824	0	−.23612681	−.48139983	−2.2753644	−3.0687234
le 2	2.5223215	1.9501640	.44943514	.22843021	0	−.23585550	−.47913630	−2.0947520	−2.6926932

li n: local implicit form of degree n. le n: local explicit form of degree n.

Table IV

degree	$r = 0.25$	$r = 0.50$	$r = 0.75$	$r = 1.25$	$r = 1.50$	$r = 1.75$
li 3	0.000000	0.000004	0.000035	0.000837	0.004964	0.019612
li 2	0.000289	0.002531	0.009630	0.066627	—	—

li n: local implicit form of degree n. le n: local explicit form of degree n.

(b) $T(\epsilon, 2)$ of local implicit approximation is greater than $T(\epsilon, 6)$ of local explicit approximation.

(c) Degree 2 and 3 local implicit approximations give very accurate approximations on a reasonable range of t.

(d) Degree 5 local implicit approximation approximates the original curve very precisely at least for $-1 \leq t \leq 1$.

When computing an explicit approximation $y = h(x)$ directly from the curve $\mathbf{r}(t)$, we first compute the degree n power series $t = \sum_{i=1}^{n} d_i x^i$ from $x = x(t)$, and then substitute it for t in $y = y(t)$. As a result, only the first n coefficients of $h(x)$ are exact and the remaining coefficients obtained in the computation should be discarded. Moreover, substituting $t = \sum_{i=1}^{n} d_i x^i$ for t in $y = y(t)$ is not a cheap computation, especially for high-degree local explicit approximations. Hence, the computation of local explicit approximations of a parametric curve directly from $\mathbf{r}(t)$ is more costly than the implicit form. In general, the computation of local implicit approximation involves generating the α_i^n and solving the linear system, which is fairly efficient for low-degree approximation.

The local explicit approximation is an analytic function that does not exist at a curve singularity. In contrast, a local implicit approximation always exists.

Example 3.5 Local implicit approximations can be derived at singularities, including cusps, where local explicit approximation fails. Let $\mathbf{c}_8(t) = (5t^3 + 2t^2, t^4 - 3t^3 + 2t^2)$ with the implicit form $f^4(x, y) = -x^4 + 55x^3 + 683x^2y + 1325xy^2 + 625y^3 - 336x^2 + 672xy - 336y^2$. The origin is a cusp of $\mathbf{c}_8(t)$ with tangent $x - y = 0$. The degree 2 local implicit approximation is a double line $(x - y)^2$, which is the best degree 2 approximation one can derive at cusp. The degree 3 local implicit approximation is $x^2 - 2xy + y^2 - 0.16259766x^3 - 2.0356445x^2y - 3.940918xy^2 - 1.8608398y^3$, which shows very nice approximation to the $\mathbf{c}_8(t)$ with t in $[-1, 1]$; see Figure 6. As a next example, we consider $\mathbf{c}_9(t) = ((5t^5 - 16t^4 + 10t^3 + 4t^2)/w(t), (t^5 + t^4 + 2t^3 - 16t^2)/w(t))$, where $w(t) = 0.1t^3 + 0.1t^2 - 2t + 12.5$. The $\mathbf{c}_9(t)$ is a singular curve with a cusp at the origin and a self-intersection as well, as shown in Figure 7. Figure 8 shows the degree 3 and degree 4 local implicit approximations of $\mathbf{c}_9(t)$. The degree 4 local implicit approximation shows remarkable performance.

4. LOCAL IMPLICIT APPROXIMATION OF PARAMETRIC SURFACES

We derive an implicit surface $g(x, y, z) = 0$ that approximates the parametric surface $\mathbf{P}(s, t) = (x(s, t), y(s, t), z(s, t))$ at the origin to a specified order of

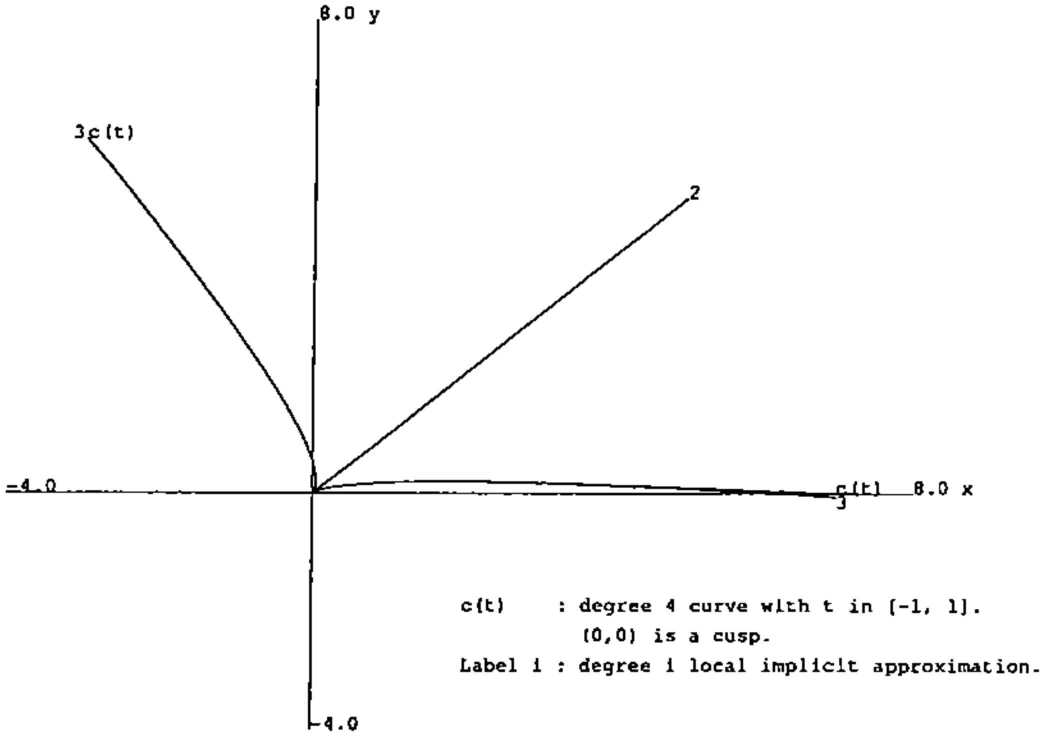

Fig. 6. $c_8(t) = (5t^3 + 2t^2, t^4 - 3t^3 + 2t^2)$.

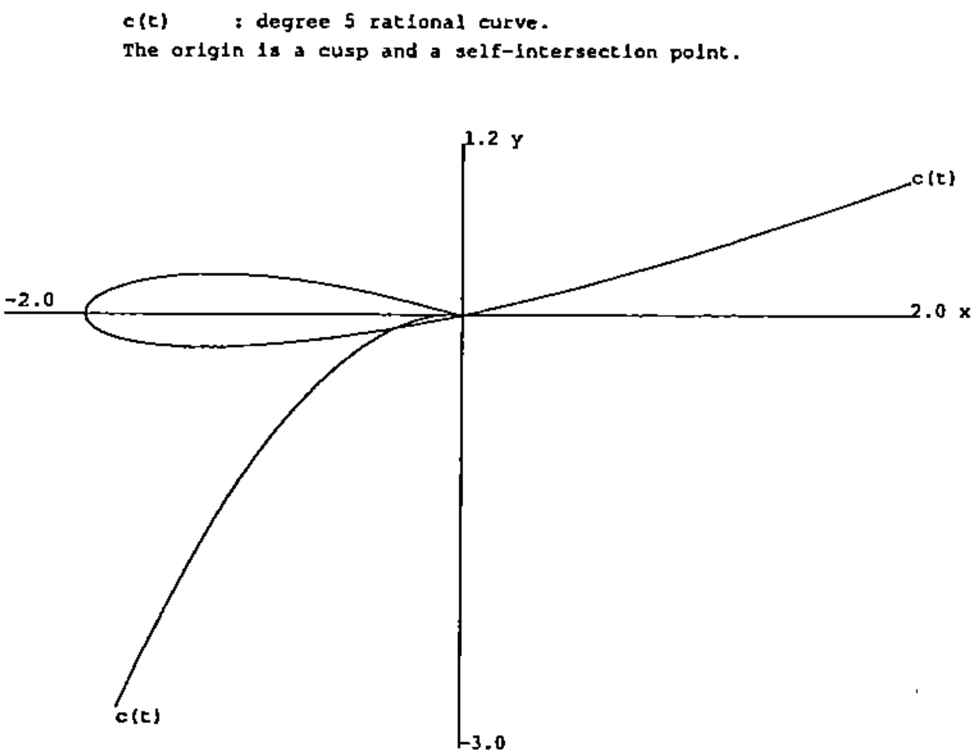

Fig. 7. $c_9(t) = \left(\frac{5t^5 - 16t^4 + 10t^3 + 4t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5}, \frac{t^5 + t^4 + 2t^3 - 16t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5}\right)$.

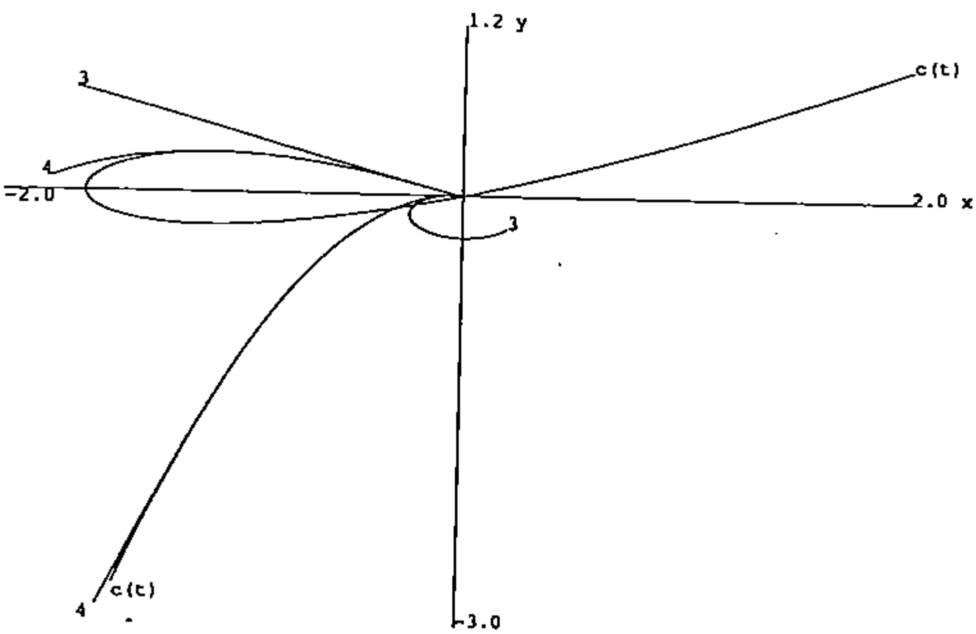

Fig. 8. $c_9(t) = \left(\dfrac{5t^5 - 16t^4 + 10t^3 + 4t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5}, \dfrac{t^5 + t^4 + 2t^3 - 16t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5}\right)$.

contact, using the method of Section 3. Let

$$\mathbf{P}(s, t) = (x(s, t), y(s, t), z(s, t)) = \left(\frac{p(s, t)}{w(s, t)}, \frac{q(s, t)}{w(s, t)}, \frac{r(s, t)}{w(s, t)}\right)$$

be a rational parametric surface of total degree m containing the origin, where

$$p(s, t) = \sum_{i+j=1}^{m} a_{ij} s^i t^j, \qquad q(s, t) = \sum_{i+j=1}^{m} b_{ij} s^i t^j,$$

$$r(s, t) = \sum_{i+j=1}^{m} c_{ij} s^i t^j, \qquad w(s, t) = \sum_{i+j=0}^{m} d_{ij} s^i t^j,$$

with a_{ij}, b_{ij}, c_{ij}, $i + j = m$, not all of these zero and $d_{00} \neq 0$. It has been shown by Macaulay [18] that a parametric surface of the above form has an irreducible implicit form $f^d(x, y, z) = 0$ of degree $d \leq m^2$.

Let $g^n(x, y, z) = \sum_{i+j+k=1}^{n} e_{ijk} x^i y^j z^k$, $n \leq m^2$ with symbolic coefficients e_{ijk} be a general implicit form of a degree n surface that contains the origin. Since $g^n(x, y, z) = 0$ is unique up to a constant factor, it has degrees of freedom $\rho(n) = ((n + 1)(n + 2)(n + 3))/6 - 2$.

When substituting $x(s, t)$, $y(s, t)$, and $z(s, t)$ into $g^n(x, y, z)$, we obtain

$$g^n(x(s, t), y(s, t), z(s, t)) = \frac{G^n(p(s, t), q(s, t), r(s, t), w(s, t))}{(w(s, t))^n} = \frac{\sum_{i+j=1}^{nm} \alpha_{ij} s^i t^j}{(w(s, t))^n}$$

where $G^n(x, y, z, w)$ is the homogeneous form of $g^n(x, y, z)$, and the α_{ij} are linear combinations of e_{ijk}. The local implicit approximation $g^n(x, y, z)$ of the parametric surface $\mathbf{P}(s, t)$ is computed as in the case of the curve approximation. The following section shows the recursive derivation of α_{ij} that obviates the need to explicitly substitute.

4.1 A Recurrence for α_{ij}

Let $G^n(x, y, z, w)$ and $G^{n-1}(x, y, z, w)$ denote the homogeneous polynomials of $g^n(x, y, z)$ and $g^{n-1}(x, y, z)$, respectively. Since

$$g^n(x, y, z) = g^{n-1}(x, y, z) + \sum_{i+j+k=n} e_{ijk} x^i y^j z^k$$

we have

$$G^n(x, y, z, w) = wG^{n-1}(x, y, z, w) + \sum_{i+j+k=n} e_{ijk} x^i y^j z^k \tag{5}$$

We define $(a(k))_{ij}$, $(b(k))_{ij}$, and $(c(k))_{ij}$ as in [20] by setting

$$(p(s, t))^k = \left(\sum_{i+j=1}^{m} a_{ij} s^i t^j \right)^k = \sum_{i+j=k}^{km} (a(k))_{ij} s^i t^j$$

and similarly for

$$(q(s, t))^k = \sum_{i+j=k}^{km} (b(k))_{ij} s^i t^j$$

and

$$(r(s, t))^k = \sum_{i+j=k}^{km} (c(k))_{ij} s^i t^j.$$

Recursive derivations for $(a(k))_{ij}$, $(b(k))_{ij}$, and $(c(k))_{ij}$ can be found in [20]. Let α_{ij}^n and α_{ij}^{n-1} be the coefficient of $s^i t^j$ in $G^n(p(s, t), q(s, t), r(s, t), w(s, t))$ and $G^{n-1}(p(s, t), q(s, t), r(s, t), w(s, t))$, respectively. From (5), α_{ij}^n can be derived from the α_{kl}^{n-1}, where $1 \le k \le i$ and $1 \le l \le j$, as shown in the following formula:

$$\begin{aligned}
\alpha_{ij}^n &= \text{coefficient of } s^i t^j \text{ in } \Bigg(w(s, t) \sum_{i+j=1}^{m(n-1)} \alpha_{ij}^{n-1} s^i t^j \\
&\quad + \sum_{k_1+k_2+k_3=n} e_{k_1k_2k_3} (p(s, t))^{k_1} (q(s, t))^{k_2} (r(s, t))^{k_3} \Bigg) \\
&= \sum_{l_1=1}^{i} \sum_{l_2=1}^{j} \alpha_{l_1 l_2}^{n-1} d_{(i-l_1)(j-l_2)} \\
&\quad + \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3} (a(k_1))_{p_1q_1} (b(k_2))_{p_2q_2} (c(k_3))_{p_3q_3}
\end{aligned}$$

Note that $\alpha_{ij}^1 = e_{100} a_{ij} + e_{010} b_{ij} + e_{001} c_{ij}$.

For an integral parametric surface $\mathbf{P}(s, t)$, since $(a(k))_{ij} = 0$, $(b(k))_{ij} = 0$, and $(c(k))_{ij} = 0$ for $i + j < k$ and $\alpha_{ij}^n = 0$ for $i + j > (n-1)m$, we have

for $1 \le i + j \le n - 1$,

$$\alpha_{ij}^n = \alpha_{ij}^{n-1}$$

for $n \le i + j \le (n-1)m$,

$$\alpha_{ij}^{n} = \alpha_{ij}^{n-1} + \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3}(a(k_1))_{p_1q_1}(b(k_2))_{p_2q_2}(c(k_3))_{p_3q_3}$$

and for $(n-1)m < i + j \le nm$,

$$\alpha_{ij}^{n} = \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3}(a(k_1))_{p_1q_1}(b(k_2))_{p_2q_2}(c(k_3))_{p_3q_3}$$

4.2 Derivation of the Method

4.2.1 *Rank of the Linear System.* Having derived α_{ij}^{n}, $i + j = 1, 2, \ldots, nm$, for the degree n implicit approximation $g^n(x, y, z)$, we write the system of linear equations $\alpha_{10}^{n} = 0$, $\alpha_{01}^{n} = 0$, $\alpha_{20}^{n} = 0$, $\alpha_{11}^{n} = 0$, $\alpha_{02}^{n} = 0, \ldots, \alpha_{(nm)0}^{n} = 0$, $\alpha_{(nm-1)1}^{n} = 0, \ldots, \alpha_{1(nm-1)}^{n} = 0$, $\alpha_{0(nm)}^{n} = 0$ in matrix form

$$\mathbf{A}_{mn}\mathbf{e}_n = \mathbf{0} \tag{6}$$

where $\mathbf{e}_n = (e_{100}, e_{010}, e_{001}, e_{200}, e_{110}, e_{101}, e_{020}, e_{011}, e_{002}, \ldots, e_{00n})^T$ is the vector of unknowns. $\mathbf{A}_{mn}$ so defined is of dimension $((nm+1)(nm+2))/2 - 1$ by $\rho(n) + 1$ and has a rank of at most $\rho(n) + 1$. As in the curve case, the rank of $\mathbf{A}_{mn}$ is critical when solving for the unknown coefficients e_{ijk}. The following theorem characterizes the rank of $\mathbf{A}_{mn}$.

THEOREM 4.1 *If $\mathbf{P}(s, t)$ is a properly parameterized rational surface of total degree m, then*

$$rank(\mathbf{A}_{mn}) = \begin{cases} \rho(n) + 1 & \text{if } n < d \\ \rho(n) & \text{if } n = d \end{cases}$$

where $d(\le m^2)$ is the degree of the implicit form of $\mathbf{P}(s, t)$.

PROOF. Similar to proofs of Lemmas 3.2 and 3.3. □

As a result of Theorem 4.1, it is clear that the exact implicitization of $P(s, t)$ is the solution of $\mathbf{A}_{mm}\mathbf{e}_m = \mathbf{0}$, with one variable set to a fixed value.

4.2.2 *The Algorithm.* We compute the degree n implicit approximation $g^n(x, y, z)$ as follows: Let

$$\mathbf{B}\mathbf{e}_n = \mathbf{0} \tag{7}$$

be the subsystem of (6) that consists of the first s equation of (6) such that $\mathbf{B}$ has rank $\rho(n)$. Augment the system (7) with $\alpha = 0$, where α is determined as follows: If the origin is a regular surface point, α is $e_{100} - 1$, $e_{010} - 1$, or $e_{001} - 1$ depending on the gradient of the surface at the origin. If the origin is a singular surface point, then $\alpha = e_{ijk} - 1$ where the indices i, j, and k are selected by inspection. Thus, a linear system

$$\mathbf{C}\mathbf{e}_n = \mathbf{b} \tag{8}$$

that has a nontrivial solution for $\mathbf{e}_n$ is obtained.

System (8) may be an inconsistent system. If this happens, some equations must be removed from (8) to ensure the consistency.

One alternative for handling inconsistencies is that we replace $\alpha = 0$ in (8) with $e_{n00} - 1 = 0$, $e_{0n0} - 1 = 0$, or $e_{00n} - 1 = 0$ and then solve it as usual. Experiments show that $g^n(x, y, z)$ computed by this method can be of the form $(ax + by + cz)^n$ for some a, b, c, that is, it degenerates to the tangent plane. To remove this degeneracy, we do the following:

(1) Solve for $g^1(x, y, z)$ and compute

$$\sum_{i+j=n} \beta_{ij} s^i t^j = (g^1(x(s, t), y(s, t), z(s, t)))^n$$

(2) Consider the linear system that consists of the first s' equations of (6) and $\alpha = 0$, where $\alpha = 0$ is $e_{n00} - 1 = 0$, $e_{0n0} - 1 = 0$, or $e_{00n} - 1 = 0$ and s' is chosen such that the coefficient matrix of the system has rank $\rho(n)$.

(3) Find a β_{ij} that is nonzero and augment the corresponding $\alpha_{ij} = 0$ to the above system; then solve it.

This computation of the local implicit approximation results in an approximant that has roughly $n^{3/2}$-th order of contact. Thus, when raising the degree of the approximant, the order of contact with the surface $\mathbf{P}(s, t)$ grows subquadratically.

Example 4.1 Consider $\mathbf{P}(s, t) = (x(s, t)/w(s, t), y(s, t)/w(s, t), z(s, t)/w(s, t))$ where

$$\begin{aligned} x(s, t) &= -200t^2 + 12st + 400t - 200s^2 - 10s \\ y(s, t) &= 15t^2 - 14st + 10t - 11s^2 + 400s \\ z(s, t) &= 200t^2 + 11st - t + 200s^2 + 2s \\ w(s, t) &= 100t^2 - 200t + 100s^2 + 200 \end{aligned}$$

We compute degree 2 and degree 3 local implicit approximations

$$\begin{aligned} g^2(x, y, z) = {} & -108.44294z^2 - 10.264638yz - 13.162097xz + 381.19047z \\ & -95.092836y^2 - 5.241114xy - 1.8809524y - 94.85476x^2 + x \end{aligned}$$

and

$$\begin{aligned} g^3(x, y, z) = {} & 1.3012126z^3 + 5.16125yz^2 - 46.69081xz^2 - 103.818164z^2 \\ & - 1.1158845y^2z + 15.622598xyz + 4.518084yz \\ & + 1.267575x^2z + 180.40466xz + 381.19047z - 3.6884814y^3 \\ & - 48.00386xy^2 - 95.16589y^2 + 5.5613696x^2y - 6.1573525xy \\ & - 1.8809524y - 44.977395x^3 - 94.34699x^2 + x \end{aligned}$$

Note that the normal of $f^4(x, y, z)$, the exact implicit form of $\mathbf{P}(s, t)$, at the origin is almost parallel to z-axis. Thus, to show the performance of the local implicit approximation, we intersect the cylinder $h(x, y, z) = x^2 + y^2 - r^2 = 0$ with the surfaces f^4, g^2, and g^3 and plot the intersection curves of $f^4 = 0 \cap h = 0$, $g^3 = 0 \cap h = 0$, and $g^2 = 0 \cap h = 0$ in one figure. Figures 9 and 10

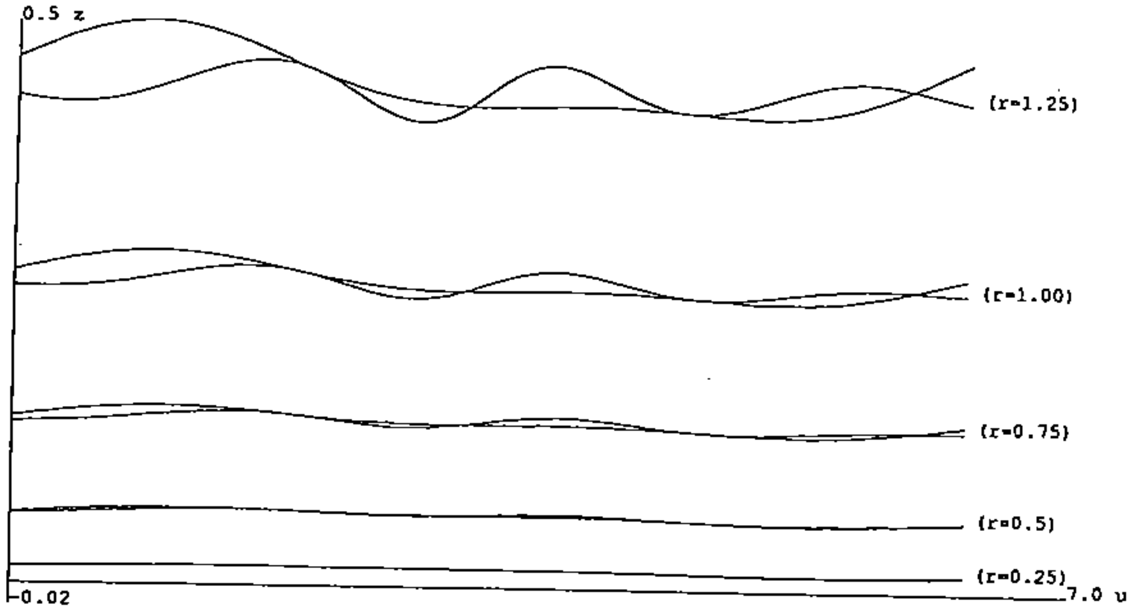

Fig. 9. $f^4 = 0 \cap h = 0$, and $g^2 = 0 \cap h = 0$.

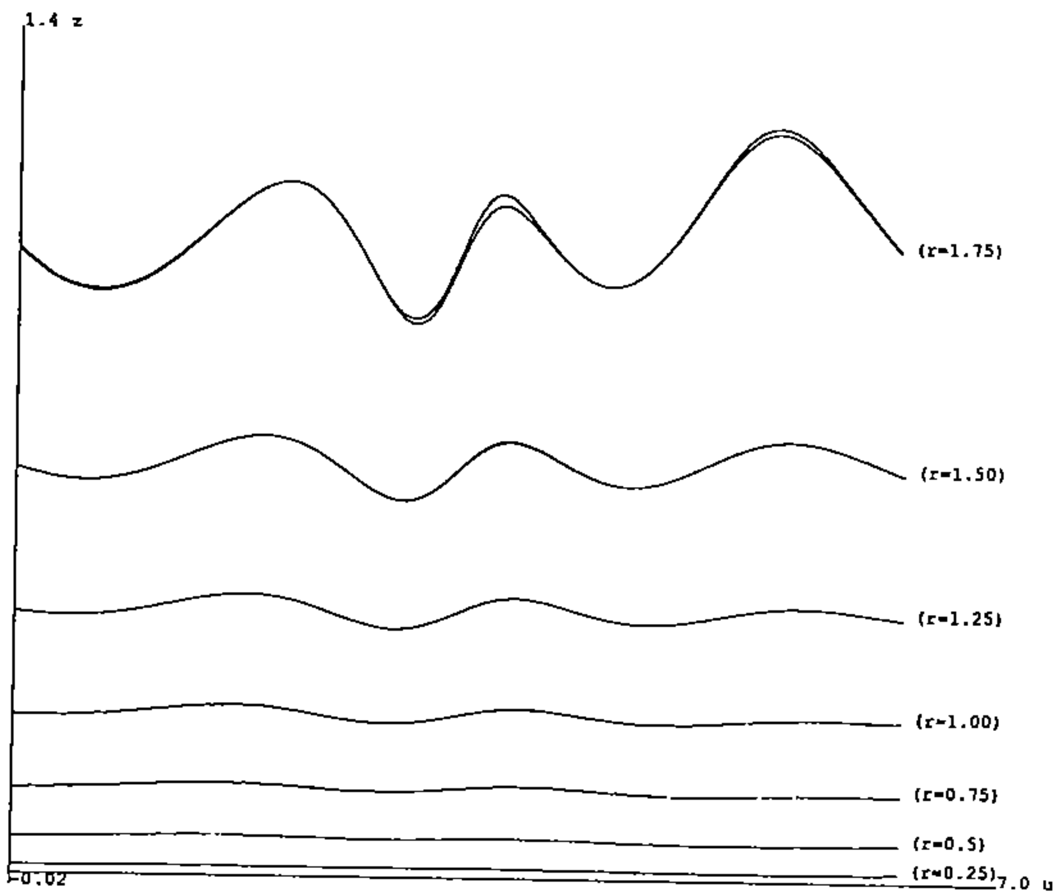

Fig. 10. $f^4 = 0 \cap h = 0$, and $g^3 = 0 \cap h = 0$.

show the intersection curves in cylindrical coordinates for $r = 0.25$, 0.5, 0.75, 1.00, 1.25, 1.50, and 1.75. Table IV lists the maximal deviations between the intersection curves $f^4 = 0 \cap h = 0$ and $g^3 = 0 \cap h = 0$, and the deviations between $f^4 = 0 \cap h = 0$ and $g^2 = 0 \cap h = 0$.

5. REMARKS ON RESULTANTS

Different resultants are formulated in the classical literature for the purpose of eliminating variables from systems of algebraic equations. Early expositions of several formulations are found in [21]. In essence, resultants constitute a projection due to which all formulations applied to surface implicitization contain extraneous factors. For example, given the parametric form of the sphere,

$$x(s, t) = \frac{1 - s^2 - t^2}{1 + s^2 + t^2}$$

$$y(s, t) = \frac{2t}{1 + s^2 + t^2}$$

$$z(s, t) = \frac{2s}{1 + s^2 + t^2}$$

elimination of s and t with the Sylvester resultant yields

$$256(x + 1)^4(x^2 + y^2 + z^2 - 1)^2$$

and with Dixon's resultant, we obtain

$$-64(x + 1)(x^2 + y^2 + z^2 - 1)$$

Technically, a resultant is based on formulating a system of linear equations with symbolic coefficients. This is especially apparent in the derivation of the Sylvester resultant. Macaulay recognized that extraneous factors are technically related to dependent equations, and that they can be eliminated by division by a suitable minor [19]. Modern work on the multivariate resultant tries to find this minor algorithmically, that is, to recognize and eliminate extraneous factors; see, for example, [3] and [6]. In our approach, a linear system is formulated numerically; hence dependencies among the equations are easy to recognize. If the approximant is formulated with the exact degree of the implicit form, then our approach determines the implicit form without extraneous factors. If an approximant of higher degree is determined with our approach, then a reducible implicit form could be generated.

REFERENCES

1. BAJAJ, C. L. Local parameterization, implicitization and inversion of real algebraic curves. Tech. Rep. 89-863, Dept. of Computer Science, Purdue University, West Lafayette, Ind., 1989.
2. BAJAJ, C. L., AND IHM, I. Hermite interpolation of rational space curves using real algebraic surfaces. In *Proceedings of the 5th Annual Symposium on Computational Geometry* (Saarbrucken, West Germany, June 5–7). ACM, 1989, pp. 94–103.
3. BAJAJ, C. L., GARRITY, T., AND WARREN, J. On the applications of multi-equational resultants. Tech. Rep. 88-826, Dept. of Computer Science, Purdue University, West Lafayette, Ind., 1988.
4. BAJAJ, C. L., HOFFMANN, C. M., HOPCROFT, J. E., AND LYNCH, R. E. Tracing surface intersections. *Comput.-Aided Geom. Des. 5*, 1988, 285–307.

5. BUCHBERGER, B. Gröbner bases: An algorithmic method in polynomial ideal theory. In *Recent Results in Multidimensional Systems Theory*, N. K. Bose, Ed. Reidel, Amsterdam, 1985.
6. CANNY, J. Generalized characteristic polynomials. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, Rome, July 1988.
7. FAROUKI, R. T. The characterization of parametric surface sections. *Comput. Vision Gr. Image Process. 33* (1986), 209–236.
8. GARRITY, T., AND WARREN, J. On computing the intersection of a pair of algebraic surfaces. To be published.
9. HOFFMANN, C. M. Algebraic curves. In *Mathematical Aspects of Scientific Software*, J. Rice, Ed. IMA Vol. in Mathematics and Applications, Springer Verlag, New York, 1988, pp. 101–122.
10. HOFFMANN, C. M. A dimensionality paradigm for surface interrogations, Tech. Rep. TR-88-837, Dept. of Computer Science, Purdue University, West Lafayette, Ind., 1988. To appear in *Comput-Aided Geom. Des.*
11. HOFFMANN, C. M. *Geometric and Solid Modeling*, Chap. 6. Morgan Kaufmann, San Francisco, Calif., 1989.
12. HOUGHTON, E. G., EMNETT, R. F., FACTOR, J. D., AND SABHARWAL, C. L. Implementation of a divide-and-conquer method for intersection of parametric surfaces. *Comput.-Aided Geom. Des. 2* (1985), pp. 173–183.
13. KAHAN, W. A survey of error analysis. In *Proceedings of the International Federation for Information Processing Congress*, 2 (Ljubljana, Yugoslavia, August 1971), North-Holland, pp. 1214–1239.
14. KOPARKAR, P. A., AND MUDUR, S. P. Generation of continuous smooth curves resulting from operations on parametric surface patches. *Comput.-Aided Des. 18*, 4 (1986).
15. LANE, J. M., AND RIESENFELD, R. F. A theoretical development for the computer generation and display of piecewise polynomial surfaces. *IEEE Trans. Pattern Anal. Mach. Intel. PAMI-2*, 1, (1980).
16. LASSER, D. Intersection of parametric surfaces in the Bernstein-Bézier representation. *Comput.-Aided Des. 18*, 4 (1986).
17. LEVIN, J. Z. Mathematical models for determining the intersection of quadric surfaces. *Comput. Gr. Image Process. 11*, 73–87, (1979).
18. MACAULAY, F. S. Some formulae in elimination. *Proc. London Math. Soc. 35*, (1902), 3–27.
19. MACAULAY, F. S. *The Algebraic Theory of Modular Systems*, Cambridge University Press, New York, 1916.
20. DE MONTAUDOUIN, Y., TILLER, W., AND VOLD, H. Applications of power series in computational geometry. *Comput.-Aided Des. 18*, 10, (1986).
21. NETTO, E. *Algebra*, Vol. 1, p. 169 and Vol. 2, p. 79, Midland Press, Ann Arbor, Michigan, 1896.
22. PENG, Q. S. An algorithm for finding the intersection lines between two b-spline surfaces. *Comput.-Aided Des. 16*, 4 (1984).
23. PRAKASH, P. V., AND PATRIKALAKIS, N. M. Algebraic and rational polynomial surface intersections. To be published.
24. REQUICHA, A. A. G., AND VOELCKER, H. B. Solid modeling: Current status and research directions. *IEEE Comput. Gr. Appl.* (Oct. 1983).
25. SARRAGA, R. F. Algebraic methods for intersections of quadric surfaces in GMSOLID. *Comput. Vision, Gr. Image Process. 22* (1983).
26. SEDERBERG, T. W. Implicit and parametric curves and surfaces for computer aided geometric design. Ph.D. dissertation, Purdue University, Dept. of Mechanical Engineering, West Lafayette, Ind., 1984.
27. SEDERBERG, T. W. Improperly parametrized rational curves. *Comput.-Aided Geom. Des. 3* (1986).
28. WAGGENSPACK, JR., W. N. Parametric curve approximations for surface intersection, Ph.D. dissertation Purdue University, Dept. of Mechanical Engineering, West Lafayette, Ind., 1987.
29. WAGGENSPACK, JR., W. N., AND ANDERSON, D. C. Piecewise parametric approximations for algebraic curves. *Comput.-Aided Geom. Des. 6*, (1989), 33–53.

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