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# Length and Energy of Quadratic Bézier Curves and Applications 

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#### Abstract

This paper derives expressions for the arc length and the bending energy of quadratic Bézier curves. The formulae are in terms of the control point coordinates. For fixed start and end points of the Bézier curve, the locus of the middle control point is analyzed for curves of fixed arc length or bending energy. In the case of arc length this locus is convex. For bending energy it is not. Given a line or a circle and fixed end points, the locus of the middle control point is determined for those curves that are tangent to the given line or circle. For line tangency, this locus is a parallel line. In the case of the circle, the locus can be classified into one of six major types. In some of these cases, the locus contains circular arcs. These results are then used to implement fast algorithms that construct quadratic Bézier curves tangent to a given line or circle, with given end points, that minimize bending energy or arc length.


Key words: quadratic Bézier curve; geometric constraint solving; arc length; bending energy; minimum arc length; minimum bending energy; optimization; GPU implementation.

## 1 Introduction

We consider the following set of problems:
Given end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ of a quadratic Bézier curve $\mathbf{q}(t)$, and a line or circle to which the curve should be tangent. Find a curve that has minimum arc length or minimum bending energy.

We address the problem of finding minimum length or minimum energy curves with tangency constraints by deriving, for these curves, analytical expressions for arc length and bending energy. We then solve the constraint problem by utilizing the high parallelism of the GPU, in the case of tangency to a circle. The tangency constraints are solved using a locus method. For example, given the points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ and the tangent line $T$, the locus of the middle control point $\mathbf{b}_{1}$ of all quadratic Bézier curves tangent to $T$ is a parallel to $T$ at a distance easily determined from the distances of $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ from $T$. We also prove that the solution of the minimum length tangency problem is unique.

## Applications

This paper is a continuation of work that seeks to use free-form curves in geometric constraint problems [1]. Geometric constraint solvers are a foundation of variational, parametric CAD; [2]. The geometric vocabulary of those CAD-centric constraint solvers is limited, however. Normally it consists of points, lines and circular arcs that are to be arranged according to imposed constraints of distance, angle, tangency, perpendicularity, and so on; e.g., [2].

The dominant approach to solving geometric constraint problems is algebraic, deducing small systems of equations whose solution delivers the coordinate assignment of the shape elements. This compact geometric vocabulary allows solvers to be fast. Moreover, the algebraic approach can find solutions regardless of initial conditions, a key advantage over most numerical iteration methods. Finally, algebraic solvers are well-suited to exploring alternative solutions, thus justifying that the approach dominates in practice.

When adding free-form curves to the traditional shape vocabulary, however, two major obstacles have to be addressed.
(1) In order to properly express constraints relating the free-form shape elements and the traditional ones, for instance circular arcs, tractable equation systems need to be found that are as simple as possible. This can be done by analyzing the possible geometric configurations.
(2) The degree of the algebraic equations that must be solved rises dramatically for many constraints. This degree burden complicates finding the few solutions that are meaningful to the applications, from among the many other solutions that, while mathematically correct, have no application significance. We overcome this barrier using a GPU-assisted approach; e.g., [1]. Prior art has often used variational approaches to solve such systems, but the numerical procedures to find solutions tend not to scale well.

It is the growing demand for parametric design, and the increasingly more sophisticated shapes needed, that lie at the core of developing an enlarged
shape vocabulary for constraint solvers. As stated in [3]: "Designing product shapes using geometric operations on free-form curves and surfaces is still a tedious task. ... This explains why constraint modeling in CAGD is an important challenge..." Indeed, designing a free-form shape for a family of products would be much simplified by constraint-based tools that allow varying the shape automatically by changing a few parameters. For example, in this context, a boundary curve may be required to have a particular clearance from a hole. This would be expressed as tangency of that curve to a concentric circle of an appropriate radius. These and similar constraint problems come up in parametric design with free-form curves.

Curves that minimize bending energy or have minimum arc length are sometimes considered to be fair in CAGD [4-9]. Finding such curves subject to geometric constraints offers the ability to find curves that are to have prescribed clearance from points, circles or straight-line borders. Design with deformable shape elements is a general area in which our work can make a contribution. Routing wires is one example where our work can be applied, in parametric CAD design, and where minimum length or minimum bending energy curves are natural; [10]. As pointed out in [10], it is difficult to combine energy constraints with geometric constraints. The design of leaf springs is another example. As analyzed in [11], parabolic shapes (i.e., quadratic Bézier curves) have advantages over linear springs. For both types of design, our constraint problems can be used to assess the operating volume and required clearances.

In motion planning, Moll and Kavraki [12] consider moving elastic objects through confined spaces. Roughly speaking, a flexible rectangular object is moved past a set of obstacles and is flexing along the path as required for the passage. Such motion problems arise in assembly design and analysis as well. The required flexing is determined numerically using a bending energy functional, a precise but expensive approach. In the first approximation one can consider how the cross-section must bend using our analysis of bending energy curves.

In [13], Sauvage et al. point out that in animation the boundary curve of characters must be deformed subject to preserving length. For efficiency reasons they restrict to piecewise linear boundaries and maintain constant length of segments. Using the results of this paper, such animations could use quadratic curve segments as well, for increased flexibility. Moreover, constraints of tangency could be imposed to confine the boundary, or parts of it, to a domain bounded by lines and circular arcs.

More generally, we believe that the combination of constraints on analytic and parametric shapes has the potential to contribute new and effective tools for direct manipulation as well as for variational design.

Prior work on finding Bézier curves with minimum arc length as constraint includes the following: [14] gives an analytic formula for the arc length of quadratic Bézier curves. This is a well-known result. Cubic Bézier curves, on the other hand, do not have a closed-form analytic arc length expression in general, and require approximation $[15,16]$. However, finding cubic Hermite interpolants subject to minimum arc length has been done using iteration; e.g., [17].

Prior work on minimum energy curves includes [18] which develops analytic formulae for the bending energy of Pythagorean Hodograph curves of degree 3 and 5. [19] devises tools to find minimum energy splines under prescribed end tangency conditions. The curves are approximated from first principles and are not Bézier curves. [20] considers the Hermite interpolation problem with minimal energy curves. The curves are integrated from piecewise polynomial curvature functions. We are not aware of a prior, closed-form expression for bending energy of quadratic Bézier curves; however, the derivation of such a formula is elementary and is only sketched for completeness.

Rahman [11] evaluates the energy integral numerically. Theetten [10] also evaluates the energy integral by numerical integration. This work considers space curves and geometric constraints that come from physical restraints. Thus, a cable might be restraint by a clamp, expressed as requiring the curve to interpolate a point. Sauvage [13] considers deformation-based editing of piecewise linear curves subject to length constraints, seeking to develop novel direct manipulation gestures. The piecewise linear skeleton is manipulated by moving the break points, and a numerical procedure generates a "wrinkled" curve that has the required fixed length.

Prior work on incorporating free-form curves and surfaces into constraint solvers is surveyed in [3]. That survey also discusses papers on subdivision and blobby modeling, in a constraint solving context.

In the remainder of this paper, we introduce notation and preliminaries in Section 2. Section 3 analyzes the locus of the middle control point of a quadratic Bézier curve when prescribing a fixed arc length. It is a convex curve. Section 3 also studies the expression for the bending energy. Here, the region for the middle control point, when prescribing a particular bending energy, is not convex, thus the corresponding constraint problem may have multiple solutions.

Section 4 solves the design problem when introducing tangency constraints, to a circle or a line. It also gives performance results when implementing the computations on the GPU in Section 5. Because of the preceding locus analysis, the algorithms for solving these constraint problems are very simple.

## 2 Definitions and Preliminaries

We consider quadratic Bézier curves $\mathbf{q}(t)=\sum_{i=0}^{2} \mathbf{b}_{i} B_{i}^{2}(t)$ where the $\mathbf{b}_{i}$ are control points and $B_{i}^{n}(t)$ are the familiar Bernstein-Bézier basis functions of degree $n$. We are interested in the arc length of those curves, given by the integral

$$
\begin{equation*}
L(\mathbf{q})=\int_{0}^{1} \sqrt{\mathbf{q}^{\prime}(t) \cdot \mathbf{q}^{\prime}(t)} d t \tag{1}
\end{equation*}
$$

where $\mathbf{q}^{\prime}(t)$ is the derivative of $\mathbf{q}(t)$ and $\cdot$ denotes the inner product. The integral has a closed form solution; see, e.g., [14]. Likewise, we consider those curves that have a given bending energy, given by

$$
\begin{equation*}
\mathcal{E}(\mathbf{q})=\frac{1}{2} \int_{\mathbf{q}} \kappa(s)^{2} d s \tag{2}
\end{equation*}
$$

where $\kappa(s)$ is the curvature at $s$. We will express some relations in terms of a quantity $\Lambda_{\alpha}$, defined as

$$
\Lambda_{\alpha}=\alpha\left|\Delta \mathbf{b}_{0}\right|^{2}+(1-\alpha)\left|\Delta \mathbf{b}_{1}\right|^{2}-\frac{1}{4}\left|\Delta \mathbf{b}_{2}\right|^{2}
$$

where $\Delta \mathbf{b}_{i}$ denotes the vector $\mathbf{b}_{i+1}-\mathbf{b}_{i}$, for $i=0,1$, and $\Delta \mathbf{b}_{2}=\mathbf{b}_{0}-\mathbf{b}_{2}$. Also, $|\cdot|$ denotes the length of a vector.

Throughout, we are interested in the locus of the middle control point $\mathbf{b}_{1}$. When considering tangency to a line, the tangency locus is a the curve on which the middle control point must lie for the curve to realize the prescribed tangency, a line parallel to the tangent. For a circle, the tangency locus is more complicated and is analyzed in Section 4. For a fixed arc length and fixed end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, the possible positions of $\mathbf{b}_{1}$ comprise an arc length locus. Likewise, for fixed energy we have an energy locus. We will solve the problems stated in the introduction by intersecting these loci using a highly parallel GPU implementation.

## 3 Length and Energy of Quadratic Bézier Curves

We give analytic expressions for arc length and bending energy of quadratic Bézier curves. Arc length is stated in terms of the length of the sides and the area of the triangle $\triangle\left(\mathbf{b}_{0}, \mathbf{b}_{1}, \mathbf{b}_{2}\right)$. We also prove that the level sets of the arc length, for fixed curve end points, are convex. That is, fixing the end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, the locus of $\mathbf{b}_{1}$ for quadratic Bézier curves of fixed length is a convex curve. We then give a bending energy formula. Here, the level sets for given bending energy are no longer convex.


Fig. 1. Orbits of the middle control points $\mathbf{b}_{1} \mathrm{~s}$ for curves with the same arc length 2.05 (red), 2.5(magenta), 3(green), 3.5(blud), respectively, when $\mathbf{b}_{0}=[-1,0]$ and $\mathbf{b}_{2}=[1,0]$.

### 3.1 Arc Length

Let $\mathbf{q}(t), t \in[0,1]$ be a quadratic Bézier curve with control points $\mathbf{b}_{0}, \mathbf{b}_{1}$ and $\mathbf{b}_{2}$. For such Bézier curves the arc length, defined by Equation (1), can be expressed as

$$
\begin{align*}
\mathcal{L}(\mathbf{q}) & =\frac{\Lambda_{1 / 4}\left|\Delta \mathbf{b}_{1}\right|+\Lambda_{3 / 4}\left|\Delta \mathbf{b}_{0}\right|}{2 \Lambda_{1 / 2}}  \tag{3}\\
& +\frac{\left|\Delta \mathbf{b}_{0} \times \Delta \mathbf{b}_{1}\right|^{2}}{8\left(\Lambda_{1 / 2}\right)^{3 / 2}}\left\{\ln \left(\Lambda_{1 / 4}+\left|\Delta \mathbf{b}_{1}\right| \sqrt{\Lambda_{1 / 2}}\right)-\ln \left(-\Lambda_{3 / 4}+\left|\Delta \mathbf{b}_{0}\right| \sqrt{\Lambda_{1 / 2}}\right)\right\}
\end{align*}
$$

See also [17].
When both end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ are given, we can determine the orbits of the control point $\mathbf{b}_{1}=\left[x_{1}, y_{1}\right]$ for which the Bézier curves have the same arc length. Figure 1 shows four orbits of $\mathbf{b}_{1} \mathrm{~s}$, for arc lengths $2.05,2.5,3$, and 3.5 , with $\mathbf{b}_{0}=[-1,0]$ and $\mathbf{b}_{2}=[1,0]$. We prove that the orbits are the boundary of convex sets.

Lemma 3.1 For fixed $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, the set $S_{L}=\left\{\mathbf{b}_{1}: \mathcal{L}(\mathbf{q}) \leq L\right\}$ is convex.

Proof. Let $\left[k_{1}, h_{1}\right],\left[k_{2}, h_{2}\right] \in S_{L}$ be two points in $S_{L}$. We will show that the convex combination $\left[k_{3}, h_{3}\right]=\lambda\left[k_{1}, h_{1}\right]+(1-\lambda)\left[k_{2}, h_{2}\right]$, with $\lambda \in(0,1)$, is also in $S_{L}$.

For $j=1,2,3$, let $\mathbf{q}^{j}(t)$ be the quadratic Bézier curve having the control points $\mathbf{b}_{0}, \mathbf{b}_{1}^{j}=\left[k_{j}, h_{j}\right], \mathbf{b}_{2}$, respectively, and let $\mathbf{b}_{0}^{j}=\mathbf{b}_{0}$ and $\mathbf{b}_{2}^{j}=\mathbf{b}_{2}$. By assumption, $\mathcal{L}\left(\mathbf{q}^{0}\right) \leq L$ and $\mathcal{L}\left(\mathbf{q}^{1}\right) \leq L$. Then

$$
\mathbf{q}^{3}(t)=\sum_{i=0}^{2} B_{i}^{2}(t) \mathbf{b}_{i}^{3}=\sum_{i=0}^{2} B_{i}^{2}(t)\left[\lambda \mathbf{b}_{i}^{1}+(1-\lambda) \mathbf{b}_{i}^{2}\right]=\lambda \mathbf{q}^{1}(t)+(1-\lambda) \mathbf{q}^{2}(t)
$$

For any partition $\left\{t_{0}=0, t_{1}, \cdots, t_{n-1}, t_{n}=1\right\}$, by the triangle inequality

$$
\left|\mathbf{q}^{3}\left(t_{k+1}\right)-\mathbf{q}^{3}\left(t_{k}\right)\right| \leq \lambda\left|\mathbf{q}^{1}\left(t_{k+1}\right)-\mathbf{q}^{1}\left(t_{k}\right)\right|+(1-\lambda)\left|\mathbf{q}^{2}\left(t_{k+1}\right)-\mathbf{q}^{2}\left(t_{k}\right)\right| .
$$

Since the secant is shorter than the arc subtended, we have

$$
\mathcal{L}\left(\mathbf{q}^{3}\right)=\sup \sum_{k=1}^{n}\left|\mathbf{q}^{3}\left(t_{k+1}\right)-\mathbf{q}^{3}\left(t_{k}\right)\right| \leq \lambda \mathcal{L}\left(\mathbf{q}^{1}\right)+(1-\lambda) \mathcal{L}\left(\mathbf{q}^{2}\right)
$$

where the sup is for all partitions of the whole interval $[0,1]$. Since $\mathcal{L}\left(\mathbf{q}^{1}\right)$ and $\mathcal{L}\left(\mathbf{q}^{2}\right)$ is less than or equal to $L$, so is $\mathcal{L}\left(\mathbf{q}^{3}\right)$. Thus the set $\left\{[k, h]: \mathbf{b}_{1}=\right.$ $[k, h], \mathcal{L}(\mathbf{q}) \leq L\}$ is convex.

It follows that the boundary $\left\{\mathbf{b}_{1}: \mathcal{L}(\mathbf{q})=L\right\}$ is a convex curve.

### 3.2 Bending Energy

We consider now the bending energy of the Bézier curve $\mathbf{q}(t)$ defined by Equation (2). By algebra and using integral formulae, the bending energy can be expressed as

$$
\mathcal{E}(\mathbf{q})=\frac{2}{3\left|\Delta \mathbf{b}_{0} \times \Delta \mathbf{b}_{1}\right|^{2}}(A+B)
$$

where

$$
\begin{equation*}
A=\frac{\Lambda_{1 / 4}\left(3 \Lambda_{1 / 2}\left|\Delta \mathbf{b}_{1}\right|^{2}-\Lambda_{1 / 4}^{2}\right)}{\left|\Delta \mathbf{b}_{1}\right|^{3}} \text { and } B=\frac{\Lambda_{3 / 4}\left(3 \Lambda_{1 / 2}\left|\Delta \mathbf{b}_{0}\right|^{2}-\Lambda_{3 / 4}^{2}\right)}{\left|\Delta \mathbf{b}_{0}\right|^{3}} \tag{4}
\end{equation*}
$$

For fixed $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, we can determine the locus of the control point $\mathbf{b}_{1}=$ [ $\left.x_{1}, y_{1}\right]$ for which the Bézier curves have the same energy. Figure 2 shows four contours, for energy $0.1,1,2$, and 3 , when $\mathbf{b}_{0}=[-1,0]$ and $\mathbf{b}_{2}=[1,0]$.

We explore the level curves near the singularities. By symmetry, it suffices to explore the curves near the control point $\mathbf{b}_{0}$. In the following, we assume that $\mathbf{b}_{0}=[-1,0]$ and $\mathbf{b}_{2}=[1,0]$. Let $r$ be the distance $\left|\Delta \mathbf{b}_{0}\right|$ and let $\theta$ be the angle between the $x$-axis and the vector $\Delta \mathbf{b}_{0}$, in the positive orientation. Observing


Fig. 2. The level curve of energy function $\mathcal{E}(\mathbf{q})=E$ of the quadratic Bézier curve $\mathbf{q}(t)$ with respect to the middle control point $\mathbf{b}_{1}, E=.1$ (red), 1(magenta), 2(green), or 3 (blue), for fixed end points $\mathbf{b}_{0}=[-1,0]$ and $\mathbf{b}_{2}=[1,0]$.
that $\left|\Delta \mathbf{b}_{2}\right|=2$, we estimate $\mathcal{E}(\mathbf{q})$ for $r \ll 1$ (cf. Equation (4)). We have

$$
\begin{aligned}
\frac{2}{3\left|\Delta \mathbf{b}_{0} \times \Delta \mathbf{b}_{1}\right|^{2}} & =\frac{1}{6 r^{2} \sin ^{2} \theta} \\
\Lambda_{1 / 4} & \approx 2-3 r \cos \theta \\
\Lambda_{1 / 2} & \approx 1-2 r \cos \theta \\
\Lambda_{3 / 4} & \approx-r \cos \theta \\
A & \approx 2-6 r \cos \theta \\
B & \approx \cos ^{3} \theta-3 \cos \theta
\end{aligned}
$$

so that, neglecting the $O(r)$ term of the approximation of $A$, we obtain for sufficiently small $r$ :

$$
\begin{equation*}
\mathcal{E}(\mathbf{q}) \approx \frac{(\cos \theta+2) \tan ^{2}(\theta / 2)}{6 r^{2}} \tag{5}
\end{equation*}
$$

The estimate implies that the level curves of equal energy reach the singularities at $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ tangent to the $x$-axis. Moreover, fixing $\theta<2 \pi$ and for small $r$, the energy $\mathcal{E}(\mathbf{q})$ is proportional to $1 / r^{2}$. Figure 3 shows several level curves near the singularity at $\mathbf{b}_{0}$. As the energy threshold is increased, the level curves curl more tightly near the singularity. It follows that the curves $\mathcal{E}(\mathbf{q})=E$ are properly nested.


Fig. 3. Energy level curves near the singularity at $\mathbf{b}_{0}$ for $\mathrm{E}=1$ (red) to $\mathrm{E}=4$ (blue).

## 4 Tangency Locus

We fix the start and end points of the quadratic Bézier curve and construct the locus of the control point $\mathbf{b}_{1}$ of all curves that are tangent to a given geometric element $T$. This tangency locus is simple when $T$ is a line, but it is considerably more complex when $T$ is a circle.

### 4.1 Tangency to a Line

The tangency locus to a given line is particularly simple:
Proposition 4.1 Given a line $T$ and two points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ on the same side of $T$, let $\mathbf{q}(t)$ be a quadratic Bézier curve with control points $\mathbf{b}_{k}, k=0,1,2$. Then the following holds (Figure 4).
(a) $\mathbf{q}(t)$ is tangent to $T$ iff $\mathbf{b}_{1}$ is on the line $T^{\prime}$ parallel to $T$ at a distance that is the geometric mean of the distances of $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ to $T$, and on the opposite side of $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$. That is, $b^{2}=a c$.
(b) With $\mathbf{c}$ the point of tangency on $T, \mathbf{b}_{1}$ is the midpoint of the intersections $\mathbf{x}_{0}$ and $\mathbf{x}_{2}$ of $T^{\prime}$ with the lines through $\mathbf{c}$ and $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, respectively.
(c) Let $\mathbf{z}_{0}$ and $\mathbf{z}_{2}$ be the orthogonal projections of $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ onto $T$. If the normal $N$ through $\mathbf{c}$ contains the point $\mathbf{b}_{1}$, then $N$ and $T$ bisect the angles between the lines $\overline{\mathbf{b}_{0} \mathbf{c}}$ and $\overline{\mathbf{b}_{2} \mathbf{c}}$. Moreover, $\mathbf{c}$ divides the line segment $\overline{\mathbf{z}_{0} \mathbf{z}_{2}}$ in the ratio $a: c$.

Proof. The proof follows from the DeCasteljau algorithm by elementary reasoning.


Fig. 4. Quadratic Bézier control points and tangency to line $T$.

Given $\mathbf{b}_{0}, \mathbf{b}_{2}, \mathbf{c}$ and $T$, the missing control point $\mathbf{b}_{1}$ can be constructed using Part (b) of Proposition 4.1. If the control point is to lie on the normal through $\mathbf{c}$, then $\mathbf{c}$ can be found using Part (c).

### 4.2 Tangency to a Circle

The tangency locus for the circle is in general a curve with two real-valued branches. The branches correspond to solutions where the circle is tangent to the convex side of the Bézier curve, and where the circle is tangent on the concave side. In addition, there are several degenerate cases. We investigate these details because they allow us later to exclude solutions in which arc length would be maximized, instead of minimized. More than that, the analysis also reveals feasibility segments on the circle.

### 4.2.1 Locus Derivation

We derive the locus of the middle control point for tangency to a circle and explain the two branches it has.

Let a circle be given by $[x(\theta), y(\theta)]=\left[O_{x}+r \cos \theta, O_{y}+r \sin \theta\right]$, where $\mathbf{O}=$ $\left[O_{x}, O_{y}\right]$ is the center and $r$ is the radius of the circle, and $\mathbf{b}_{0}=\left[x_{0}, y_{0}\right]$ and $\mathbf{b}_{2}=\left[x_{2}, y_{2}\right]$ are the end points of the quadratic Bézier curve; Figure 5(a).

Let $\theta_{j}, j=1, \cdots, 4$ be the signed angles of the four tangent points on the circle of the lines(skyblue and orange) passing $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$, as shown in Figure 5(a). By Proposition 4.1, the quadratic Bézier curve is tangent to the circle at the point $\mathbf{c}(\theta)=[x(\theta), y(\theta)]$ if and only if the middle control point $\mathbf{b}_{1}(\theta)=\left[x_{1}(\theta), y_{1}(\theta)\right]$


Fig. 5. (a) The quadratic Bézier curve with end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ has the contact point $\mathbf{c}(\theta)$ on the circle, $\theta_{1}<\theta<\theta_{2}$ (closer arc, blue) and $\theta_{3}<\theta<\theta_{4}$ (farther arc, red). (b) On the closer branch, the center $\mathbf{O}$ and the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ are on opposite sides of the common tangent(green line). (c) $\mathbf{c}^{\prime}(\theta)$ and $\mathbf{b}_{1}^{\prime}(\theta)$ are parallel and have the same direction for the closer branch of the tangency locus(blue curve). (d) The tangency locus of $\mathbf{b}_{1}$ consists of two branches, closer branch(blue curve) and farther branch(red curve).
satisfies

$$
\begin{align*}
& {\left[x_{1}(\theta), y_{1}(\theta)\right]=} \\
& \quad[x(\theta), y(\theta)]+\frac{1}{2}\left(\sqrt{\frac{d_{2}}{d_{0}}}\left[x(\theta)-x_{0}, y(\theta)-y_{0}\right]+\sqrt{\frac{d_{0}}{d_{2}}}\left[x(\theta)-x_{2}, y(\theta)-y_{2}\right]\right) \tag{6}
\end{align*}
$$

$\theta_{1}<\theta<\theta_{2}$ or $\theta_{3}<\theta<\theta_{4}$, where $d_{i}$ is the distance from $\mathbf{b}_{i}$ to the common
tangent(green line) of circle and Bézier curve,

$$
d_{i}=\frac{1}{r}\left|\left(x-x_{i}\right)\left(x-O_{x}\right)+\left(y-y_{i}\right)\left(y-O_{y}\right)\right|=\frac{\left|\left(\mathbf{c}-\mathbf{b}_{i}\right) \cdot(\mathbf{c}-\mathbf{O})\right|}{r} .
$$

Let $\sigma=\left(\mathbf{c}-\mathbf{b}_{0}\right) \cdot(\mathbf{c}-\mathbf{O}) /\left|\left(\mathbf{c}-\mathbf{b}_{0}\right) \cdot(\mathbf{c}-\mathbf{O})\right|$. It is obvious that

$$
\sigma=\left(\mathbf{c}-\mathbf{b}_{2}\right) \cdot(\mathbf{c}-\mathbf{O}) /\left|\left(\mathbf{c}-\mathbf{b}_{2}\right) \cdot(\mathbf{c}-\mathbf{O})\right|
$$

since the two points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ are on the same side of the common tangent. If $\mathbf{b}_{0}$ and $\mathbf{O}$ lie on the same side of the common tangent, then $\sigma=1$ and if they are on the opposite side, then $\sigma=-1$. Thus we obtain

$$
\begin{equation*}
d_{i}=\frac{\sigma}{r}\left\{\left(\mathbf{c}-\mathbf{b}_{i}\right) \cdot(\mathbf{c}-\mathbf{O})\right\} . \tag{7}
\end{equation*}
$$

Proposition 4.2 The tangent of the tangency locus at $\left[x_{1}(\theta), y_{1}(\theta)\right]$ is parallel to the tangent of the circle at $[x(\theta), y(\theta)]$, for each $\theta \in\left(\theta_{0}, \theta_{1}\right) \cup\left(\theta_{3}, \theta_{4}\right)$. Furthermore they have the same tangent direction if the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ and the center $\mathbf{O}$ lie on opposite sides of the common tangent of circle and Bézier curve.

Proof. The derivative of $\left[x_{1}(\theta), y_{1}(\theta)\right]$ in Equation (6) is

$$
\begin{align*}
{\left[x_{1}^{\prime}(\theta), y_{1}^{\prime}(\theta)\right] } & =\left[x^{\prime}(\theta), y^{\prime}(\theta)\right]\left(1+\frac{1}{2}\left(\sqrt{\frac{d_{2}}{d_{0}}}+\sqrt{\frac{d_{0}}{d_{2}}}\right)\right)  \tag{8}\\
& +\frac{1}{2} \frac{d_{0} d_{2}^{\prime}-d_{0}^{\prime} d_{2}}{d_{0}^{3 / 2} d_{2}^{3 / 2}}\left\{d_{2}\left(\mathbf{c}(\theta)-\mathbf{b}_{0}\right)-d_{0}\left(\mathbf{c}(\theta)-\mathbf{b}_{2}\right)\right\}
\end{align*}
$$

From Equation (7) we get

$$
\begin{equation*}
d_{2}\left(\mathbf{c}(\theta)-\mathbf{b}_{0}\right)-d_{0}\left(\mathbf{c}(\theta)-\mathbf{b}_{2}\right)=\frac{\sigma}{r}\left(\left(\mathbf{c}-\mathbf{b}_{0}\right) \times\left(\mathbf{c}-\mathbf{b}_{2}\right)\right)\left[y-O_{y},-\left(x-O_{x}\right)\right] \tag{9}
\end{equation*}
$$

Since the vector $\left[y-O_{y},-\left(x-O_{x}\right)\right]$ is parallel to $\left[x^{\prime}(\theta), y^{\prime}(\theta)\right]$, we have

$$
\left[x_{1}^{\prime}(\theta), y_{1}^{\prime}(\theta)\right] \|\left[x^{\prime}(\theta), y^{\prime}(\theta)\right] .
$$

If the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ and the center $\mathbf{O}$ lie on opposite sides of the common tangent, then $\sigma=-1$ and Equation (9) yields

$$
\begin{equation*}
d_{2}\left(\mathbf{c}(\theta)-\mathbf{b}_{0}\right)-d_{0}\left(\mathbf{c}(\theta)-\mathbf{b}_{2}\right)=\frac{1}{r}\left(\left(\mathbf{c}-\mathbf{b}_{0}\right) \times\left(\mathbf{c}-\mathbf{b}_{2}\right)\right) \mathbf{c}^{\prime}(\theta), \tag{10}
\end{equation*}
$$

which is the same direction as $\left[x^{\prime}(\theta), y^{\prime}(\theta)\right]$. Since $\mathbf{c}^{\prime}(\theta) \cdot(\mathbf{c}(\theta)-\mathbf{O})=0$, we
have $d_{i}^{\prime}(\theta)=\frac{-1}{r} \mathbf{c}^{\prime}(\theta) \cdot\left(\mathbf{c}-\mathbf{b}_{i}\right)$ for $i=0,2$, and
$d_{0} d_{2}^{\prime}-d_{0}^{\prime} d_{2}=\frac{1}{r^{2}}\left(\left(\mathbf{c}-\mathbf{b}_{0}\right) \cdot(\mathbf{c}-\mathbf{O})\right)\left(\mathbf{c}^{\prime} \cdot\left(\mathbf{c}-\mathbf{b}_{2}\right)\right)-\frac{1}{r^{2}}\left(\left(\mathbf{c}-\mathbf{b}_{2}\right) \cdot(\mathbf{c}-\mathbf{O})\right)\left(\mathbf{c}^{\prime} \cdot\left(\mathbf{c}-\mathbf{b}_{0}\right)\right)$.
Let $\alpha_{i}$ be the angle between the two vectors $\mathbf{c}^{\prime}$ and $\mathbf{b}_{i}-\mathbf{c}$, for $i=0,2$, as shown in Figure 5(b). Then
$d_{0} d_{2}^{\prime}-d_{0}^{\prime} d_{2}=\frac{1}{r^{2}}\left|\mathbf{c}-\mathbf{b}_{0}\right|\left|\mathbf{c}-\mathbf{O} \| \mathbf{c}^{\prime}\right|\left|\mathbf{c}-\mathbf{b}_{2}\right|\left(-\cos \left(\frac{\pi}{2}+\alpha_{0}\right) \cos \alpha_{2}+\cos \left(\frac{\pi}{2}+\alpha_{2}\right) \cos \alpha\right)$.
Since $0<\alpha_{2}<\alpha_{0}<\pi$, we have

$$
\begin{equation*}
d_{0} d_{2}^{\prime}-d_{0}^{\prime} d_{2}=\frac{1}{r^{2}}\left|\mathbf{c}-\mathbf{b}_{0}\right|\left|\mathbf{c}-\mathbf{O} \| \mathbf{c}^{\prime}\right|\left|\mathbf{c}-\mathbf{b}_{2}\right| \cdot \sin \left(\alpha_{0}-\alpha_{2}\right)>0 \tag{11}
\end{equation*}
$$

By Equations (8)-(11), $\left[x_{1}(\theta), y_{1}(\theta)\right]$ has the same tangent direction to $[x(\theta), y(\theta)]$.

REmARK 4.3 We note three observations:
(a) As shown in Figure 5, the contact points $\mathbf{c}$ of the circle and the quadratic Bézier curve lie on two arcs, a closer arc(blue curve) and a farther arc(red curve), from the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$. Likewise, the middle control points $\mathbf{b}_{1}$ form two branches of the tangency locus.
(b) By Proposition 4.2, the Gauss maps of the tangency locus and of the contact point arcs are equal by components: ${ }^{1}$

$$
\begin{aligned}
& \mathcal{N}\left(\left\{\mathbf{c}(\theta): \theta_{1}<\theta<\theta_{2}\right\}\right)=\mathcal{N}\left(\left\{\mathbf{b}_{1}(\theta): \theta_{1}<\theta<\theta_{2}\right\}\right) \\
& \mathcal{N}\left(\left\{\mathbf{c}(\theta): \theta_{3}<\theta<\theta_{4}\right\}\right)=\mathcal{N}\left(\left\{\mathbf{b}_{1}(\theta): \theta_{3}<\theta<\theta_{4}\right\}\right) .
\end{aligned}
$$

Even if the curve $[x(\theta), y(\theta)]$ is not a circle, the above two equations are true whenever the curve is $C^{1}$-continuous.
(c) By Equations (6) and (7), the tangency locus of $\mathbf{b}_{1}$ has the four assymptotic lines(skyblue and orange).

Corollary 4.4 The closer branch of tangency locus of $\mathbf{b}_{1}$ is nonsingular.

Proof. By Proposition 4.2, $\mathbf{b}_{1}^{\prime}(\theta)$ is parallel and has the same direction to $\mathbf{c}^{\prime}(\theta)$, i.e., $\mathbf{b}_{1}^{\prime}(\theta)=k \cdot \mathbf{c}^{\prime}(\theta)$ for some positive real number $k>0$. Since $\mathbf{c}^{\prime}(\theta)$ is a nonzero vector for all $\theta \in\left(\theta_{1}, \theta_{2}\right)$, so is $\mathbf{b}_{1}^{\prime}(\theta)$, and the closer branch(blue curve) of tangency locus of $\mathbf{b}_{1}$ is nonsingular, as illustrated in Figure 5 .
${ }^{1}$ For more about the Gauss map, see [21-23].

### 4.2.2 Minimum Length Branch

Since the equal-length locus is a convex curve, we can exclude one of the tangency locus branches for minimizing arc length.

Proposition 4.5 If any point on the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ is neither inside nor on the circle $C$, then the minimum length quadratic Bézier curve has the middle control point $\mathbf{b}_{1}$ on the closer branch $\left\{\mathbf{b}_{1}(\theta): \theta_{1}<\theta<\theta_{2}\right\}$, not on the farther branch $\left\{\mathbf{b}_{1}(\theta): \theta_{3}<\theta<\theta_{4}\right\}$.

Proof. If any point on the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ is neither inside nor on the circle $C$, then the closer branch $\left\{\mathbf{b}_{1}(\theta): \theta_{1}<\theta<\theta_{2}\right\}$ is not empty. By the definition of $\mathbf{b}_{1}$ in Equation (6), the closer branch and the farther branch each are connected sets in the plane. By Remark 4.3(c), the closer branch has two asymptotic lines, so it separates the plane into two regions. One region contains the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$, and the other region contains the far branch.

Assume that the minimum length is attained at $\mathbf{b}_{1}^{f}$ on the far branch. Let $\mathbf{m}$ be the midpoint of $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$. The line segment $\mathbf{b}_{1}^{f} \mathbf{m}$ intersects the closer branch, say, $\mathbf{b}_{1}^{c}$. By convexity, Lemma 3.1, there exists some $\lambda \in(0,1)$ such that

$$
\mathcal{L}\left(\mathbf{b}_{1}^{c}\right)=\lambda \mathcal{L}\left(\mathbf{b}_{1}^{f}\right)+(1-\lambda) \mathcal{L}(\mathbf{m})<\mathcal{L}\left(\mathbf{b}_{1}^{f}\right)
$$

which is a contradiction. Hence the minimum length is obtained on the closer branch $\left\{\mathbf{b}_{1}(\theta): \theta_{1}<\theta<\theta_{2}\right\}$ and not on the far branch $\left\{\mathbf{b}_{1}(\theta): \theta_{3}<\theta<\theta_{4}\right\}$.

### 4.2.3 Special Cases

We have a number of special cases that arise when the line segment $\overline{\mathbf{b}_{0} \mathbf{b}_{2}}$ intersects the circle $C$ or lies inside it. These can be classified into six different cases, as shown in Figure 6.

Proposition 4.6 The tangency locus of $\mathbf{b}_{1}$ has the following properties.
(a) The tangency locus of $\mathbf{b}_{1}$ is bounded if and only if both end-points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ lie inside the circle.
(b) If one end-point is on $C$ and the other is inside $C$, then the tangency locus of $\mathbf{b}_{1}$ consists of one straight line and a bounded curve.
(c) If two control points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ lie on the circle, then the tangency locus consists of two straight lines and two circular arcs that are centered at the points $\mathbf{O} \pm \frac{r}{|\mathbf{m}-\mathbf{O}|}(\mathbf{m}-\mathbf{O})$, on the circle perimeter, with the radius $r \mp|\mathbf{m}-\mathbf{O}|$.


Fig. 6. Tangency locus for a circle. There are six special cases: (a) both end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ lie outside the (gray) circle $C$, (b) one end point is outside and the other is inside, (c) one is outside and the other lies on $C$, (d) both are inside $C$, (e) one is on $C$ and the other is inside, or ( f ) both lie on the circle $C$. Green line connects $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$. In cases (c), (e) and (f) the tangency locus includes, as a component, the circle tangent of an end point on the perimeter.

Proof. (a) Assume that at least one of the two points $\mathbf{b}_{0}$ and $\mathbf{b}_{1}$ lies outside the circle $C$, say $\mathbf{b}_{0}$. Let $\mathbf{c}\left(\theta_{1}\right)$ be the tangent point from $\mathbf{b}_{0}$ to the circle, and $\left\{\mathbf{b}_{1}(\theta): \theta \in\left(\theta_{1}, \theta_{2}\right)\right\}$ a part of the tangency locus of $\mathbf{b}_{1}$. As $\theta$ approaches $\theta_{1}$ from the right, $d_{0}(\theta)$ converges to zero, and $d_{2}(\theta)$ and $\mathbf{c}(\theta)-\mathbf{b}_{0}$ converge to nonzero limits, so $\left|\mathbf{b}_{1}(\theta)\right|$ diverges to $\infty$ by Equation (6). Thus the tangency locus is unbounded and the line $\mathbf{b}_{0} \mathbf{c}\left(\theta_{1}\right)$ is an asymptote, as shown in Figures 5 and 6(a)-(c).

Let at least one of two end-points lie on the circle $C$, say $\mathbf{b}_{0}$. Let $\ell_{0}$ be the circle tangent at $\mathbf{b}_{0}$. If $\mathbf{b}_{1}$ lies on the line $\ell_{0}$, then the quadratic Bézier curve $\mathbf{q}$ having the control points $\mathbf{b}_{i}, i=0,1,2$ is tangent to the circle $C$ at $\mathbf{q}(0)=\mathbf{b}_{0}$. Thus the straight line $\ell_{0}$ is a part of the tangency locus of $\mathbf{b}_{1}$ and so the tangency locus is unbounded, as shown in Figure 6(c), (e) and (f).

Let both end-points be inside $C$. Then $\left\{\mathbf{b}_{1}(\theta): \theta \in[0,2 \pi]\right\}$ is the tangency locus of $\mathbf{b}_{1}$. Since for $i=0,2, d_{i}(\theta)$ is non-zero continuous and $\mathbf{c}(\theta)-\mathbf{b}_{i}$ is continuous, so is $\mathbf{b}_{1}(\theta)$ by Equation (6). Thus the tangency locus is a compact set in the plane, which means that it is closed and bounded[24], as shown in

Figure 6(d).
(b) Let one of $\mathbf{b}_{1}$ or $\mathbf{b}_{2}$ lie on $C$ and the other inside $C$. Without loss of generality, we may assume that $\mathbf{b}_{0}$ is on $C$ and $\mathbf{b}_{2}$ inside $C$. Let $\ell_{0}$ be the tangent line of the circle at the point $\mathbf{b}_{0}$. For some $\theta_{0} \in[0,2 \pi), \mathbf{b}_{0}=\mathbf{c}\left(\theta_{0}\right)$, and the tangency locus of $\mathbf{b}_{1}$ consists of $\left\{\mathbf{b}_{1}(\theta): \theta \in\left(\theta_{0}, \theta_{0}+2 \pi\right)\right\}$ and $\ell_{0}$. Since $d_{0}(\theta)=\left|\mathbf{c}-\mathbf{b}_{0}\right|^{2} / 2 r$, both limits

$$
\lim _{\theta \rightarrow \theta_{0}^{ \pm}} \sqrt{\frac{d_{2}(\theta)}{d_{0}(\theta)}}\left(\mathbf{c}-\mathbf{b}_{0}\right)=\sqrt{2 r d_{2}\left(\theta_{0}\right)} \cdot\left( \pm T_{0}\right)
$$

exist, where $T_{0}$ is the unit tangent vector of $\mathbf{c}(\theta)$ at $\theta=\theta_{0}$. Thus $\mathbf{b}_{1}(\theta)$ can be extended continuously at both end points $\theta=\theta_{0}$ and $\theta_{0}+2 \pi$, which is the compactification[24]. Hence the extended set $\left\{\mathbf{b}_{1}(\theta): \theta \in\left[\theta_{0}, \theta_{0}+2 \pi\right]\right\}$ is compact, and so the tangency locus consists $\ell_{0}$ and a bounded curve, as shown in Figure 6(e).
(c) Let both end points $\mathbf{b}_{0}$ and $\mathbf{b}_{2}$ lie on the circle. For $i=0,2$, let $\ell_{i}$ be the tangent line of the circle at the point $\mathbf{b}_{i}$. For some $\theta_{0}$ and $\theta_{2}$ with $\theta_{0}<\theta_{2}<$ $\theta_{0}+2 \pi, \mathbf{c}\left(\theta_{0}\right)=\mathbf{b}_{0}$ and $\mathbf{c}\left(\theta_{2}\right)=\mathbf{b}_{2}$. The tangency locus of $\mathbf{b}_{1}$ consists $\ell_{0}, \ell_{2}$ and $\left\{\mathbf{b}_{1}(\theta): \theta \in\left(\theta_{0}, \theta_{2}\right),\left(\theta_{2}, \theta_{0}+2 \pi\right)\right\}$. For $\theta \in\left(\theta_{0}, \theta_{2}\right), d_{i}(\theta)=\left|\mathbf{c}-\mathbf{b}_{i}\right|^{2} / 2 r$, $i=0,2$ and

$$
\left|\mathbf{c}-\mathbf{b}_{0}\right|=2 r \sin \frac{\theta-\theta_{0}}{2}, \quad\left|\mathbf{c}-\mathbf{b}_{2}\right|=2 r \sin \frac{\theta_{2}-\theta}{2} .
$$

Thus by Equation (6) and trigonometry, we have
$\mathbf{b}_{1}(\theta)=r\left(1-\cos \frac{\theta_{2}-\theta_{0}}{2}\right)[\cos \theta, \sin \theta]+\left[O_{x}+r \cos \frac{\theta_{0}+\theta_{2}}{2}, O_{y}+r \sin \frac{\theta_{0}+\theta_{2}}{2}\right]$.
Analogously, for $\theta \in\left(\theta_{2}, \theta_{0}+2 \pi\right)$,
$\mathbf{b}_{1}(\theta)=r\left(1+\cos \frac{\theta_{2}-\theta_{0}}{2}\right)[\cos \theta, \sin \theta]+\left[O_{x}-r \cos \frac{\theta_{0}+\theta_{2}}{2}, O_{y}-r \sin \frac{\theta_{0}+\theta_{2}}{2}\right]$.
Hence the tangency locus consists of two strait lines and two circular arcs centered at $\mathbf{O} \pm \frac{r}{|\mathbf{m}-\mathbf{O}|}(\mathbf{m}-\mathbf{O})$ with the radius $r \mp|\mathbf{m}-\mathbf{O}|$, as shown in Figure 6(f).

## 5 Implementation and Results

We have implemented the construction of quadratic Bézier curves subject to tangency and length or energy minimization constraints. The basic algorithm amounts to sampling candidate contact points, along the stipulated tangent or
tangency circle, and evaluate the resulting arc length or bending energy. This very fast computation uncovers local minima that can be refined iteratively or by oversampling subregions. The sampling can be implemented in the GPU. However, based on the derived simplicity of the task and the simplicity of the domain, we elected to keep this computation in the CPU. Our implementation includes the option of computing the $\mathbf{b}_{1}$ tangency locus, analyzed before, as well as the $\mathbf{b}_{1}$ locus of the energy and arc length level sets. The latter curves are computed in the GPU using continuation.

By Propositions 4.1, the $\mathbf{b}_{1}$ tangency locus is a parallel to the tangent line. By Lemma 3.1, moreover, the minimum-length solution is unique. We noticed empirically that the minimum length solution is achieved near a tangency at which the curve normal contains the $\mathbf{b}_{1}$ control point. This is a suitable starting point for iteratively determining the solution. For tangency to a circle we cannot expect a unique minimum solution. For instance, when the end points lie symmetrically with respect to the circle center there will be two global arc length minima.

We measured the program performance on a desktop PC outfitted with a desktop PC running Windows Vista (32bit) with the following configuration: Intel Xeon X5460 CPU at 3.16 GHz , 4GB main memory, and an nVidia GeForce GTX 285 graphics card driving a display with $2560 \times 1600$ pixels. The program was run in release mode. Performance is impacted by whether level set loci are computed, and so we measured performance with and without this computation. All performance numbers are in frames per second (fps) and are obtained by moving the start or endpoint with the mouse. Thus, 300 fps means that the computation to update the display takes only 3.3 msec . The slowest computation, minimum energy tangency to a circle with level set evaluation included achieves a very respectable $275 \mathrm{fps}(3.6 \mathrm{msec})$ - despite the high algebraic complexity.

|  | no level sets | level sets |
| :--- | :---: | :---: |
| Minimum length line tangent | 2000 | 340 |
| Minimum length circle tangent | 1400 | 300 |
| Minimum energy line tangent | 2000 | 300 |
| Minimum energy circle tangent | 1290 | 275 |

Figure 7 shows a representative screen shot.


Fig. 7. Left: minimum length quadratic Bézier curve (black) tangent to a circle; length locus red, tangency locus brown. Right: minimum energy quadratic Bézier curve (black) tangent to a circle; energy locus red, tangency locus brown.

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