J. Math. Anal. Appl. 410 (2014) 257–266

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Circle approximation using LN Bézier curves of even degree and its application

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ARTICLE INFO

Article history:
Received 28 January 2013
Available online 14 August 2013
Submitted by H.R. Parks

Keywords:
Circle approximation
Offset approximation
LN Bézier curve
Hausdorff distance
Curvature continuous interpolation

ABSTRACT

We present an approximation method of circular arcs using linear-normal (LN) Bézier curves of even degree, four and higher. Our method achieves G^m continuity for endpoint interpolation of a circular arc by a LN Bézier curve of degree $2m$, for $m = 2, 3$. We also present the exact Hausdorff distance between the circular arc and the approximating LN Bézier curve. We show that the LN curve has an approximation order of $2m + 2$, for $m = 2, 3$. Our approximation method can be applied to offset approximation, so obtaining a rational Bézier curve as an offset approximant. We derive an algorithm for offset approximation based on the LN circle approximation and illustrate our method with some numerical examples.

1. Introduction

Offset approximation is an important task in CAGD. Over the last thirty years, many algorithms for offset approximation have been proposed. These algorithms fall, roughly speaking, into one of three types: One is the approximation of offset curves by Bézier or spline curves [18,17,20,29]. This approach achieves an approximation with the smallest number of curve segments for a prescribed tolerance. But this approach does not have a closed form error bound.

The second approach focuses on the use of Pythagorean hodographs, that is, on the use of polynomial curves having rational offsets. This elegant method was first proposed by Farouki and Sakkalis [12]. A vast corpus of this type of approximation methods, and of the properties of PH curves and Pythagorean normal (PN) vector surfaces, has been developed over the years; e.g., [7,13,11,24,28,30,32]. The approach also generalizes to Minkowski Pythagorean hodograph curves which have been elegantly used to compute the medial axis transform of a domain [8,21,23,26,33].

In the third approach, offsets are approximated based on circle approximations that use linear-normal Bézier curves. Lee et al. [25] proposed such a circle approximation using quadratic Bézier curves. Quadratic Bézier curves are the LN curves of least degree. Jüttler [19] introduced the concept of LN surfaces and developed an offset approximation using LN surfaces. Moreover, in [5,6,31] useful properties of LN surfaces have been proved. Recently, Ahn et al. [2] proved the invariance of the Hausdorff distance between two compatible curves under convolution if these curves have no cusp. This result provides an exact error analysis of offset approximations based on LN circle approximations.

The approximation method presented in this paper approximates circles using LN Bézier curves of even degree. Specifically, we present a G^m endpoint interpolation of circular arcs using LN Bézier curves of degree $2m$, for $m = 2, 3$. We also derive the exact Hausdorff distance of the approximant and show that it has an approximation order of $2m + 2$. We then apply our circle approximation method to offset approximation, obtaining a rational Bézier curve as an offset approximant

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.079

that has the same Hausdorff distance. We discuss the algorithm for our offset approximation method and illustrate it with some numerical examples.

Our manuscript is organized as follows. In Section 2, we derive the circle approximation using LN Bézier curves of even degree $2m$ for $m = 2, 3$. In Section 3, the algorithm for offset approximation based on this circle approximation is developed. We give some numerical examples in Section 4, and summarize our method in Section 5.

2. Circle approximation by LN Bézier curves of degree 2*m*

In this section we present the circle approximation using LN Bézier curves of even degree. For a given circular arc, the approximating LN Bézier curve is obtained, in a manner similar to previous work on approximation by Bézier curves [4,1,9,10,14,15,22,25].

Let $\mathbf{c}$ be a circular arc subtending the angle $2\alpha < \pi$,

$$\mathbf{c}(\theta) = \big(\cos(\theta), \sin(\theta)\big), \quad \theta \in [-\alpha, \alpha]. \tag{1}$$

The approximating Bézier curve $\mathbf{p}$ of degree $2m$, $m \geqslant 2$, is defined by

$$\mathbf{p}(t) = \sum_{i=0}^{2m} \mathbf{p}_i B_i^{2m}(t) \quad \text{for } t \in [0, 1]$$

where $\mathbf{p}_i$, $i = 0, 1, \ldots, 2m$ are the control points and $B_i^{2m}(t)$, $i = 0, 1, \ldots, 2m$ are the Bernstein polynomials of degree $2m$. The following are the necessary and sufficient conditions for the Bézier curve $\mathbf{p}(t)$ of degree $2m$ to be a LN curve:

(a) $\mathbf{p}'(t) = w(t)(X(t), Y(t))$ for some polynomial $w(t)$ of degree $2m - 2$ and linear functionals $X(t)$ and $Y(t)$;
(b) $\mathbf{p}'(t) \cdot ((1 - t)\mathbf{n}(0) + t\mathbf{n}(1)) = 0$ for all $t \in [0, 1]$, where $\mathbf{n}(t)$ is the rotation of $\mathbf{p}'(t)$ by a right angle (Eq. (18) of [31]), and $\cdot$ is scalar product.

To find the approximate LN Bézier curve $\mathbf{p}(t)$ for the circular arc $\mathbf{c}(\theta)$, we use condition (b) and set

$$\zeta(t) = \mathbf{p}'(t) \cdot \big((1 - t)\mathbf{n}(0) + t\mathbf{n}(1)\big). \tag{2}$$

Since the circular arc $\mathbf{c}(\theta)$, $\theta \in [-\alpha, \alpha]$, is symmetric with respect to the x-axis, we make the approximating curve $\mathbf{p}(t)$ symmetric by imposing $y_{2m-i} = y_i$, $i = 0, \ldots, m$. The approximating LN curve $\mathbf{p}(t)$ is a G^1 endpoint interpolation of the circular arc $\mathbf{c}(\theta)$ if and only if

$$\begin{aligned} \mathbf{p}_0 &= \mathbf{c}(-\alpha), & \mathbf{p}_1 &= \mathbf{p}_0 + u(\mathbf{m}_1 - \mathbf{p}_0), \\ \mathbf{p}_{2m} &= \mathbf{c}(\alpha), & \mathbf{p}_{2m-1} &= \mathbf{p}_{2m} + u(\mathbf{m}_1 - \mathbf{p}_{2m}) \end{aligned} \tag{3}$$

for some $u > 0$ where $\mathbf{m}_1 = (\sec\alpha, 0)$. Let $(x(t), y(t)) = \mathbf{p}(t)$ and

$$\psi(t) = x(t)^2 + y(t)^2 - 1.$$

It is well known [4,1,16] that for $k \geqslant 2$ the approximate curve $\mathbf{p}(t)$ is a G^k endpoint interpolation of circular arc $\mathbf{c}(\theta)$ if and only if Eq. (3) holds and

$$\left.\frac{d^i\psi(t)}{dt^i}\right|_{t=0} = 0, \quad i = 2, \ldots, k. \tag{4}$$

Since both curves $\mathbf{c}(\theta)$ and $\mathbf{p}(t)$ are symmetric with respect to x-axis, it is sufficient to find the maximum of the function

$$\phi(t) = \big|\left|\mathbf{p}(t)\right| - 1\big| = \left|\sqrt{x(t)^2 + y(t)^2} - 1\right|$$

in the interval $[0, 1/2]$.

2.1. Circle approximation by quartic LN Bézier curves

In this section we construct a quartic LN Bézier curve $\mathbf{p}$ which is a G^2 endpoint interpolation of the circular arc $\mathbf{c}(\theta)$, $\theta \in [-\alpha, \alpha]$. By Eq. (3) and the symmetry of two curves $\mathbf{p}$ and $\mathbf{c}$ with respect to the x-axis, the control points of $\mathbf{p}$ are

$$\begin{aligned} \mathbf{p}_0 &= (\cos\alpha, -\sin\alpha), \\ \mathbf{p}_1 &= (1 - u)\mathbf{p}_0 + u\mathbf{m}_1, \\ \mathbf{p}_2 &= (1 - v)\mathbf{m}_0 + v\mathbf{m}_1, \\ \mathbf{p}_3 &= (1 - u)\mathbf{p}_4 + u\mathbf{m}_1, \\ \mathbf{p}_4 &= (\cos\alpha, \sin\alpha), \end{aligned} \tag{5}$$

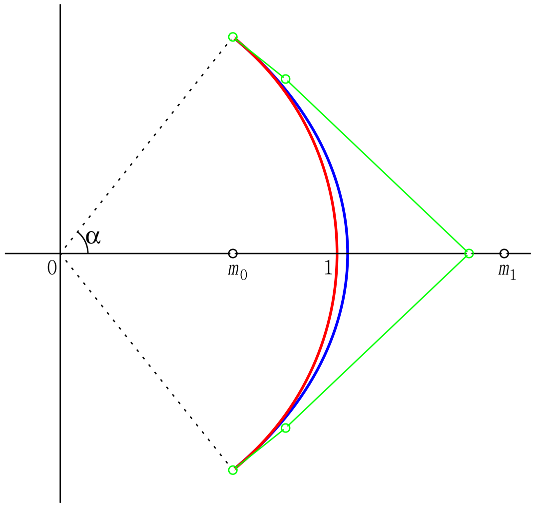


Fig. 1. Quartic Bézier curve (blue) $\mathbf{p}$ with its control polygon (green) $\mathbf{p}_i$, $i = 0, \ldots, 4$, which is a LN G^2 endpoint interpolation of the circular arc (red) $\mathbf{c}$ of angle 2α. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

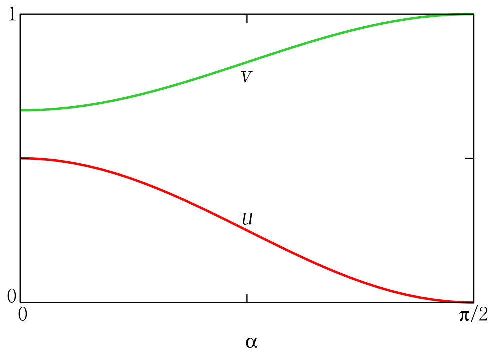


Fig. 2. u and v.

for some real numbers $u > 0$ and v, where $\mathbf{m}_0 = (\cos\alpha, 0)$, as shown in Fig. 1. By Eq. (2), we have

$$\zeta(t) = 4\sin^2\alpha(2u + 3v - 3)t(1 - t)(2t - 1).$$

Thus the quartic Bézier curve $\mathbf{p}(t)$ is LN if and only if

$$v = 1 - \frac{2}{3}u. \tag{6}$$

By series expansion of $\psi(t)$ near $t = 0$, we get

$$\psi(t) = 8u\tan^2\alpha\left(2u - \cos^2\alpha\right)t^2 + \mathcal{O}\left(t^3\right).$$

Note that $\psi(t)$ has zeros of order three at $t = 0$ and 1 if and only if

$$u = \frac{1}{2}\cos^2\alpha. \tag{7}$$

Thus by Eq. (4), we get a quartic LN curve that is a G^2 endpoint interpolation of circular arcs as follows (see Fig. 2).

Proposition 2.1. *The quartic Bézier curve $\mathbf{p}(t)$ with control points given in Eq.* (5) *is a G^2 LN endpoint interpolation of a circular arc if and only if*

$$u = \frac{1}{2}\cos^2\alpha \quad \textit{and} \quad v = 1 - \frac{1}{3}\cos^2\alpha. \tag{8}$$

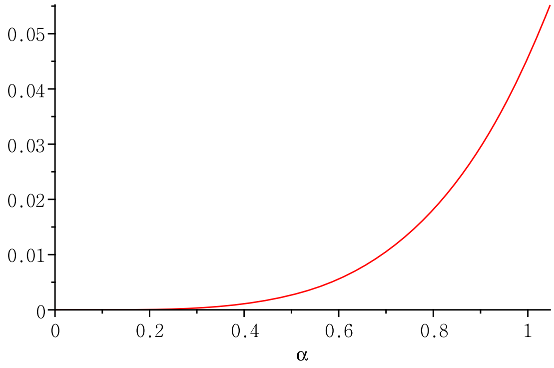


Fig. 3. $d_H(\mathbf{c},\mathbf{p}) = \varepsilon(\alpha) = \frac{(3+\cos\alpha)(1-\cos\alpha)^3}{8\cos\alpha}$.

Proof. The quartic Bézier curve $\mathbf{p}(t)$ with the control points of Fig. 1 is a G^2 LN endpoint interpolation of a circular arc if and only if both equations (6) and (7) are satisfied. These equations entail Eq. (8). □

It follows from Eq. (8) that $0 < u, v < 1$. All control points $\mathbf{p}_i$, $i = 0, \dots, 4$ are contained in the (closed) triangle $\triangle \mathbf{p}_0\mathbf{m}_1\mathbf{p}_4$, and so is the Bézier curve $\mathbf{p}(t)$. Thus the Hausdorff distance $d_H(\mathbf{c},\mathbf{p})$ can be obtained by

$$d_H(\mathbf{c},\mathbf{p}) = \max_{0\leqslant t\leqslant 1} \left|\phi(t)\right|. \tag{9}$$

Proposition 2.2. *Let* $\mathbf{p}$ *be the quartic LN* G^2 *endpoint interpolation of the circular arc* $\mathbf{c}$ *with control points as in Eq.* (5) *and satisfying Eq.* (8). *The Hausdorff distance between* $\mathbf{c}$ *and* $\mathbf{p}$ *is*

$$d_H(\mathbf{c},\mathbf{p}) = \frac{(3+\cos\alpha)(1-\cos\alpha)^3}{8\cos\alpha}$$

and its approximation order is six,

$$d_H(\mathbf{c},\mathbf{p}) = \frac{1}{16}\alpha^6 + \mathcal{O}(\alpha^8).$$

Proof. Since

$$\psi(t) = \frac{4\sin^6(\alpha)}{\cos^2(\alpha)}(1-t)^3t^3\left(2\cos(\alpha)^2 + 9t\sin^2(\alpha) - 9t^2\sin^2(\alpha)\right)$$

it has the maximum at $t = 1/2$. Thus we have

$$d_H(\mathbf{c},\mathbf{p}) = \phi(1/2) = \frac{(3+\cos\alpha)(1-\cos\alpha)^3}{8\cos\alpha}. \tag{10}$$

By series expansion, we obtain

$$d_H(\mathbf{c},\mathbf{p}) = \frac{1}{16}\alpha^6 + \mathcal{O}(\alpha^8),$$

whose approximation order is six. □

In Eq. (10) the Hausdorff distance $d_H(\mathbf{c},\mathbf{p})$ between the circular arc $\mathbf{c}$ of angle 2α and its quartic approximation $\mathbf{p}$ depends only on the angle α. We denote it by $\varepsilon(\alpha)$, as shown in Fig. 3.

2.2. Circle approximation by a degree six LN Bézier curve

In this section we construct the degree six LN Bézier curve $\mathbf{p}$ that is a G^2 endpoint interpolation of the circular arc $\mathbf{c}(\theta)$, $\theta \in [-\alpha,\alpha]$ (see Fig. 4). By Eq. (3) and the symmetry of the two curves $\mathbf{p}$ and $\mathbf{c}$ w.r.t. the x-axis, the control points of $\mathbf{p}$ are

$$\mathbf{p}_0 = \left(\cos(\alpha), -\sin(\alpha)\right),$$
$$\mathbf{p}_1 = (1-u)\mathbf{p}_0 + u\mathbf{m}_1,$$
$$\mathbf{p}_2 = \mathbf{m}_0 + w_1(\mathbf{m}_1 - \mathbf{m}_0) + w_2(\mathbf{p}_0 - \mathbf{m}_0),$$

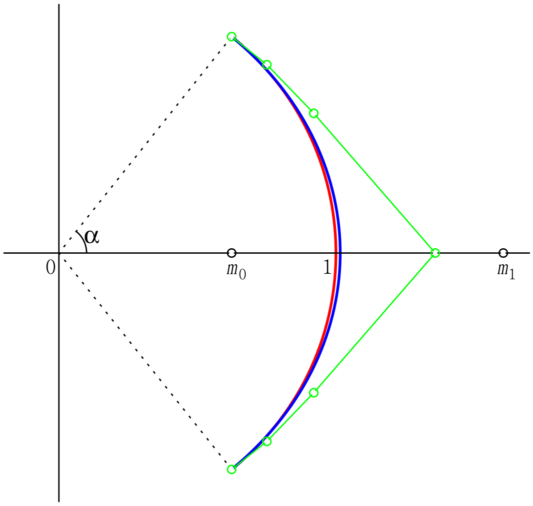


Fig. 4. Degree six Bézier curve $\mathbf{p}$ (blue) with its control polygon $\mathbf{p}_i$, $i = 0, \ldots, 6$ (green). Here, $\mathbf{p}$ is a LN G^3 endpoint interpolation of the unit circle arc $\mathbf{c}$ (red) subtending the angle 2α. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\begin{aligned}
\mathbf{p}_3 &= \mathbf{m}_0 + v(\mathbf{m}_1 - \mathbf{m}_0),\\
\mathbf{p}_4 &= \mathbf{m}_0 + w_1(\mathbf{m}_1 - \mathbf{m}_0) + w_2(\mathbf{p}_6 - \mathbf{m}_0),\\
\mathbf{p}_5 &= (1-u)\mathbf{p}_6 + u\mathbf{m}_1,\\
\mathbf{p}_6 &= \big(\cos(\alpha), \sin(\alpha)\big),
\end{aligned} \tag{11}$$

for real numbers $u > 0$, v, w_1, and w_2. By Eq. (2), $\zeta(t)$ is a polynomial of degree six and is given by

$$\zeta(t) = \sin^2\alpha\big(2u + 5(w_1 + w_2 - 1)\big)\big(B_1^6(t) - B_5^6(t)\big) - 2\sin^2\alpha(2u - 2v + w_1 + 3w_2 - 1)\big(B_2^6(t) - B_4^6(t)\big).$$

Thus $\zeta(t) \equiv 0$ if and only if

$$w_1 = \frac{2}{5}u - v + 1, \qquad w_2 = -\frac{4}{5}u + v. \tag{12}$$

By series expansion of $\psi(t)$ near $t = 0$, we have

$$\psi(t) = 12u\tan^2\alpha\big(3u - \cos^2\alpha\big)t^2 + 4\tan^2\alpha\big(\cos^2\alpha\big(10v + 18u^2 + 12u - 10\big) - 45uv - 72u^2 + 45u\big)t^3 + \mathcal{O}\big(t^4\big)$$

and so $\psi(t)$ has zeros of order four at $t = 0$ if and only if

$$u = \frac{1}{3}\cos^2\alpha \quad \text{and} \quad v = 1 - \frac{4}{5}\cos(\alpha)^2 + \frac{2}{5}\cos(\alpha)^4, \tag{13}$$

which implies the existence of a G^3 LN endpoint interpolation of the circular arc $\mathbf{c}$ as follows.

Proposition 2.3. *The sixth-degree Bézier curve $\mathbf{p}(t)$ with control points in Eq. (11) is G^3 LN endpoints interpolation of the circular arc $\mathbf{c}$ if and only if*

$$\begin{aligned}
u &= \frac{1}{3}\cos^2\alpha, & v &= 1 - \frac{4}{5}\cos^2\alpha + \frac{2}{5}\cos^4\alpha,\\
w_1 &= \frac{14}{15}\cos^2\alpha - \frac{2}{5}\cos^4\alpha, & w_2 &= 1 - \frac{16}{15}\cos^2\alpha + \frac{2}{5}\cos^4\alpha.
\end{aligned} \tag{14}$$

Proof. The degree six Bézier curve $\mathbf{p}(t)$ with control points in Eq. (11) is a G^3 LN endpoint interpolation of a circular arc if and only if both equations (12) and (13) are satisfied. These equations yield Eq. (14). □

By Eqs. (13)–(14), for all $\alpha \in (0, \pi/2)$, $0 < u, v, w_1, w_2 < 1$, and also

$$0 < w_1 + w_2 = 1 - \frac{2}{15}\cos^2\alpha < 1$$

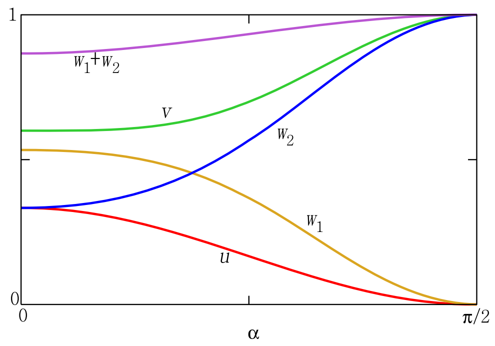


Fig. 5. u, v, w_1 and w_2.

as shown in Fig. 5. Thus all control points $\mathbf{p}_i$, $i = 0, \ldots, 6$ are contained in the (closed) triangle $\triangle \mathbf{p}_0\mathbf{m}_1\mathbf{p}_6$, and so is the approximation curve $\mathbf{p}(t)$. Hence we can have the Hausdorff distance $d_H(\mathbf{c}, \mathbf{p})$ easily, using Eq. (9) as articulated in the following proposition.

Proposition 2.4. *The Hausdorff distance is*

$$d_H(\mathbf{c}, \mathbf{p}) = \frac{(1-\cos\alpha)^4(5+4\cos\alpha+\cos^2\alpha)}{16\cos\alpha}$$

and the approximation order is eight,

$$d_H(\mathbf{c}, \mathbf{p}) = \frac{5}{128}\alpha^8 + \mathcal{O}\big(\alpha^{10}\big). \tag{15}$$

Proof. The error function $\psi(t)$ has zeros of order four at both endpoints

$$\psi(t) = \frac{4}{3}\sin^6\alpha\tan^2\alpha t^4(1-t)^4\psi_1(t)$$

where

$$\psi_1(t) = 15\cos^2\alpha\big(B_0^4(t) + B_4^4(t)\big) + 3\cos^2\alpha\big(11 - 6\cos^2\alpha\big)\big(B_1^4(t) + B_3^4(t)\big) + \big(50 + 61\cos^2\alpha + 26\cos^4\alpha\big)B_2^4(t).$$

Since $\psi_1(t)$ is positive on $[0, 1]$, increasing on $[0, 1/2)$ and decreasing on $(1/2, 1]$, it has the maximum at $t = 1/2$. Thus $\psi(t)$ is also the maximum at $t = 1/2$, and so

$$\max_{0\leqslant t\leqslant 1}\big|\psi(t)\big| = \psi(1/2) = \frac{1}{256}\sin^6\alpha\tan^2\alpha\big(\cos^4\alpha - 6\cos^2\alpha + 25\big).$$

Hence

$$d_H(\mathbf{c}, \mathbf{p}) = \phi(1/2) = \varepsilon(\alpha) = \frac{(1-\cos\alpha)^4(5+4\cos\alpha+\cos^2\alpha)}{16\cos\alpha}$$

and by its series expansion, Eq. (15) follows. □

3. An application

The circle approximation algorithm with LN curves can be used to approximate curve offsets; [2,3,5,6,25,27,31]. In particular, Ahn et al. [3] showed that the Hausdorff distance between two planar compatible curves is invariant under the convolution with a third compatible curve when their convolutions have no cusps. Using this result, we can give an explicit error analysis of offset approximations that are based on our circle approximation. We sketch the offset approximation first.

Assume we are given a spline curve or Bézier curve whose offset curve is to be approximated. Without loss of generality, we may assume that the given curve is a Bézier curve since spline curves can be subdivided into Bézier segments. For the planar Bézier curve $\mathbf{b}(s)$, $s \in [0, 1]$ of degree d, its offset is $\mathbf{b}_r(s) = \mathbf{b}(s) + rN(s)$, where the unit normal vector $N(s)$ is oriented by rotating the unit tangent vector $T(s)$ of $\mathbf{b}(s)$ counter-clockwise by the angle $\pi/2$. The graph of $N(s)$, $s \in [0, 1]$, is a circular arc. If $\mathbf{b}(s)$ has no inflection point and the angle of the circular arc $N(s)$, $s \in [0, 1]$, is less than π, then $N(s)$ can be approximated by a LN Bézier curve $\mathbf{p}(t)$ of degree $2m$ and the offset curve $\mathbf{b}_r(s)$ can be approximation by $\mathbf{b} * \mathbf{p}(t)$, where the binary operator $*$ denotes the convolution of two compatible curves. Here $\mathbf{b} * \mathbf{p}(t)$ is a rational Bézier curve of degree $(2m+1)d - 2m$ (see Table 1).

Table 1
Offset approximation based on the circle approximation by LN Bézier curves of even degree for a given spline curve **b** of degree d.

	Quadratic [25]	Quadratic biarc [2]	Quartic LN	Sixth-degree LN
G^k endpoint interpolation	G^1	G^2	G^2	G^3
Approximation order	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^6)$	$\mathcal{O}(\alpha^8)$
Degree of rational curve $\mathbf{b} * r\mathbf{p}$	$3d-2$	$3d-2$	$5d-4$	$7d-6$

Before approximating $N(s)$, which is compatible with the circular arc $\mathbf{c}(\theta)$ approximated by the LN Bézier curve $\mathbf{p}(t)$, we will need to subdivide $\mathbf{b}(s)$. Now $\mathbf{b}(s)$ has no inflection points if and only if $N(s)$ is a circular arc without cusps. Thus $\mathbf{b}(s)$ should be subdivided at the inflection points. Also, the sufficient condition that the convolution $\mathbf{b} * r\mathbf{c}$ has no singular point in the domain interior is that there is no point on the curve $\mathbf{b}(s)$ with the signed curvature $\kappa(s) = -1/r$, where r is the offset distance. Thus, we should also subdivide at the points where $\kappa(s) = -1/r$. Subject to these subdivisions, the invariance property of the Hausdorff distance, $d_H(\mathbf{b} * r\mathbf{c}, \mathbf{b} * r\mathbf{p}) = r d_H(\mathbf{c}, \mathbf{p})$, can be applied. Let I be the number of these subdivisions, and let $s_i \in (0, 1)$, $i = 1, \ldots, I$ be the subdivision points. Set $s_0 = 0$ and $s_{I+1} = 1$.

Now, for each segment, let α_i be the half angle of the circular arc $N(s)$, $u \in [s_i, s_{i+1}]$. We find the smallest positive integer J_i satisfying

$$\varepsilon\left(\frac{\alpha_i}{J_i}\right) < TOL \tag{16}$$

where TOL is the prescribed tolerance. Then each segment $b(s)$, $s \in [s_i, s_{i+1}]$, is subdivided into J_i smaller segments. In all, the curve is subdivided into

$$K = \sum_{i=0}^{I} J_i$$

segments. The algorithm yields the segments $(\mathbf{b} * r\mathbf{p})_{i,j}$ of the offset approximation, for $i = 0, \ldots, I$ and $j = 1, \ldots, J_i$. Note that the approximating method of this paper can be also be applied to NURBS curves using their subdivision into rational Bézier segments.

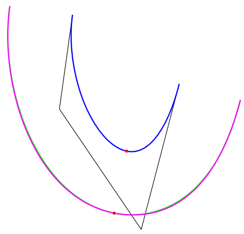
ALGORITHM
input : Bézier curve $\mathbf{b}(s)$, offset distance r, tolerance TOL.
find $s_i \in (0, 1)$, $i = 1, \ldots, I$ satisfying $\kappa(s_i) = 0$ or $-1/r$.
set $s_0 = 0$, $s_{I+1} = 1$.
for i from 0 to I do
subdivide $\mathbf{b}(s)$ into the segments $[s_i, s_{i+1}]$.
let α_i be the half angle of the circular arc $N(s)$, $s \in [s_i, s_{i+1}]$.
let J_i be the smallest positive integer satisfying $\varepsilon(\frac{\alpha_i}{J_i}) < TOL$.
find $s_{i,j}$, $j = 0, \ldots, J_i$ such that

$$N\big(\mathbf{b}(s_{i,j})\big) = \frac{j}{J_i} \cdot N\big(\mathbf{b}(s_i)\big) + \left(1 - \frac{j}{J_i}\right) \cdot N\big(\mathbf{b}(s_{i+1})\big).$$

for j from 1 to J_i do
find the LN approximation $\mathbf{p}(t)$ which is compatible with $\mathbf{b}(s)$, $u \in [s_{i,j-1}, s_{i,j}]$.
find $t = t(s)$ satisfying $\mathbf{b}'(s) \parallel \mathbf{p}'(t)$, $u \in [s_{i,j-1}, s_{i,j}]$.
calculate $(\mathbf{b} * r\mathbf{p})_{i,j}(s) = \mathbf{b}(s) + r\mathbf{p}(t(s))$, $u \in [s_{i,j-1}, s_{i,j}]$.
end for
end for
set $K = \sum_{i=0}^{I} J_i$.
output : I, J_i, K, $(\mathbf{b} * r\mathbf{p})_{i,j}(s)$, $s_{i,j}$

4. Examples

In this section we consider three examples. Two examples were proposed by Lee et al. [25]. The first example is a cubic Bézier curve (blue) with control polygon (gray) and offset distance $r = 1$, as shown in Fig. 6. The angle of the unit normal vector $N(s)$, $s \in [0, 1]$, is larger than π. Hence we need to subdivide until the error ε is less than the tolerance TOL. Fig. 6 shows the offset approximation based on a circle approximation using quartic LN Bézier curves with tolerance $TOL = 0.1$. The offset approximation is achieved by two segments (green), i.e. $m = 2$. The subdivision point is marked by a small red circle.

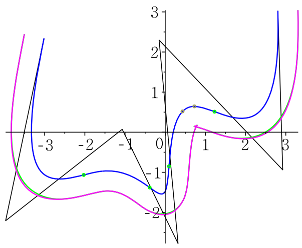


(a)

TOL	quartic LN	sixth-degree LN
10^{-1}	23	31
10^{-2}	34	31
10^{-3}	45	46
10^{-4}	56	61
10^{-5}	78	76

(b)

Fig. 6. (a) Cubic Bézier curve (blue), control polygon (gray), offset curve (green) with offset distance $r = 1$, and an approximation (magenta) based on circle approximation using quartic LN curve. One subdivision (small red circle) is needed with $TOL = 0.1$. (b) The number of control points of approximating rational spline curves based on circle approximation using quartic or degree six LN curves. This achieves an error within TOL. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a)

TOL	quartic LN	sixth-degree LN
10^{-1}	111	151
10^{-2}	133	181
10^{-3}	144	196
10^{-4}	177	196
10^{-5}	243	241

(b)

Fig. 7. (a) Cubic spline (blue) with three knots, control polygon (gray), offset curve (green) with offset distance 0.5, and approximation (magenta) based on circle approximation using quartic LN curves. There are four inflection points (small green circle), two points (small gray circle) satisfying $\kappa(t) = 1/r$ and no other subdivision point for $TOL = 0.1$. (b) The numbers of control points of the approximating rational spline curve, based on circle approximation using quartic or sixth-degree LN curves, which achieves the error within TOL. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The second example is a cubic spline with control polygon (gray line segments) and with uniform knot vectors, and offset distance $r = 0.5$, as shown in Fig. 7. This spline consists of four Bézier segments, has four inflection points (small green circle), and has two points (small gray circle) at which the offset has cusps. For the tolerance $TOL = 0.1$, the offset approximation, based on circle approximation using quartic LN Bézier curves, is achieved by ten segments.

The third example is the offset of the outline of font "S" whose height is about one hundred units. The offset distance is five units. As shown in Fig. 8(a), the outline (blue curve) consists of an upper-side cubic spline, a lower-side cubic spline, and six line segments. The upper and lower-side cubic splines consist of seven and six segments, respectively, that are G^2 continuous at the junction points (small violet circle); see Fig. 8(b). Their control polygons are plotted with gray lines in Fig. 8(a) and their curvatures are shown in Fig. 9. There are four inflection points on the outline curve. One of them is a junction point on the upper-side cubic spline, so subdivision at that point is not needed. The other three inflection points (small green circles) require subdivision. There are four points (small gray circles) at which the offset has a cusp. In all, nine subdivision points are needed on each cubic spline. Using quartic LN circle approximation, the rational offset approximation is obtained and the Hausdorff distance to the true offset is 0.0165 units. The offset curve (magenta) is constructed by twenty rational Bézier curves of degree eleven, eight circular arcs, and six line segments, as shown in Fig. 8(b).

5. Conclusion and future work

In this paper we presented a method of circle approximation using LN Bézier curves of even degree $2m$, $m = 2, 3$, that is, a G^m endpoint interpolation of circular arcs that has the approximation order $2m + 2$. Our approximation method can be applied to offset approximation which yields a rational Bézier approximation. Using the fact that the Hausdorff distance is invariant under convolution, an error analysis for offset approximation based on our circle approximation using LN Bézier curves can be obtained. We provided some numerical examples to illustrate our approximation method.

In Section 2, we presented the circle approximation by LN Bézier curve of degree $2m$, only for $m = 2, 3$. The same result for $m = 4$ can be obtained by the same method in Section 2 as shown in Fig. 10. But we could not find the generalized circle approximation by LN Bézier curve of all even degree $2m$ which has the approximation order $2m + 2$. The problem of seeking the generalized circle approximation is one of our future works. We also plan to find a sphere approximation using curvature continuous LN Bézier surfaces which can be applied to offset surface approximation.

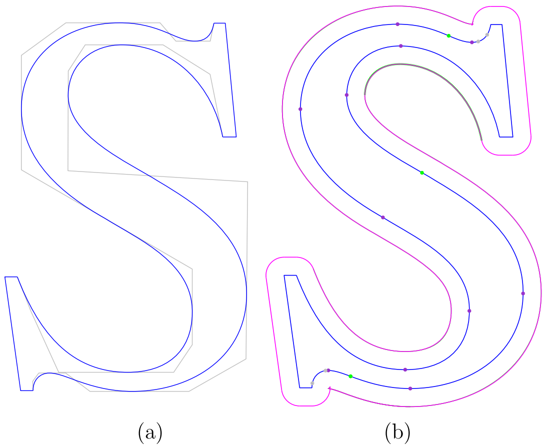


Fig. 8. (a) The outline of the font character "S" consists of thirteen cubic Bézier curves (blue) with control polygon (gray) and six straight lines (blue). (b) The exterior offset approximation based on circle approximation using quartic LN: Consecutive Bézier curves are G^2 continuous at each junction point (small violet circle). There are three inflection points (small green circle) that are not junction points and four points (small gray circle) satisfying $\kappa(t) = -1/r$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

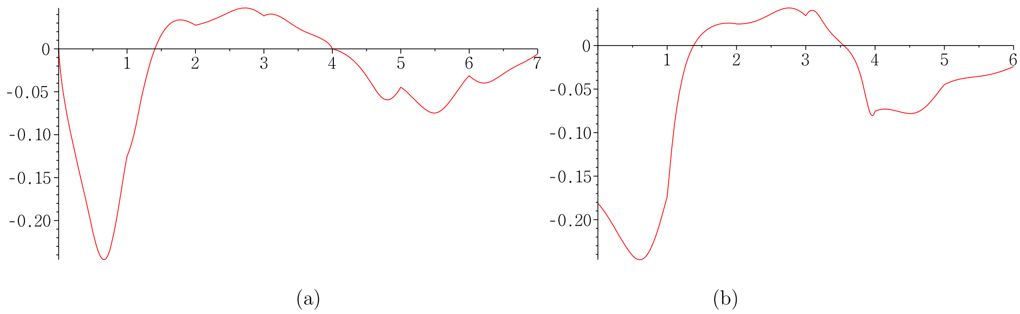


Fig. 9. Curvature plots of (a) upper side of the G^2 cubic spline consisting of seven segments, and (b) lower side of the G^2 cubic spline of six segments.

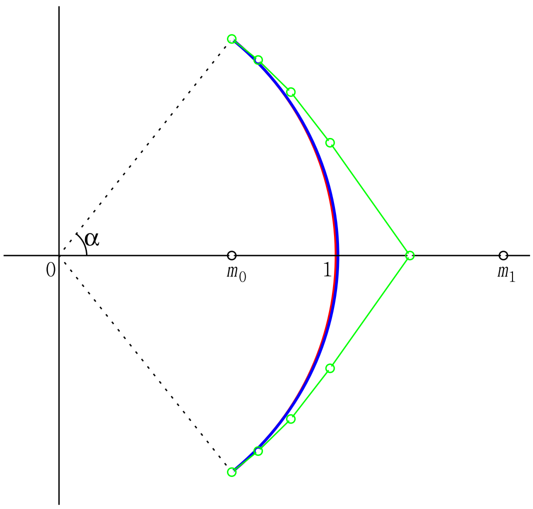


Fig. 10. Eighth-degree Bézier curve (blue) $\mathbf{p}$ with its control polygon (green) $\mathbf{p}_i$, $i = 0, \ldots, 8$, which is a LN G^4 endpoint interpolation of the circular arc (red) $\mathbf{c}$ of angle 2α. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0010147). Partial support by the National Science Foundation under grant CNS-0938999 is also gratefully acknowledged. The authors are grateful to the associate editor and an anonymous referee for their valuable comments and constructive suggestions.

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