COMPLIANT MOTION CONSTRAINTS*

CHRISTOPH M. HOFFMANN AND WEIQIANG YANG

Department of Computer Science
Purdue University
West Lafayette, IN 47907, USA

We examine a dual-quaternion formulation for expressing the relative rigid body motion between two objects when incidence constraints are to be observed. The incidences are between points, lines and planes, of the two parts. Both parametric and implicit representations are investigated. Several examples illustrate the techniques.

Keywords: Rigid body motion, compliant motion, quaternion, dual quaternion, relative motion, incidence constraint, virtual reality, kinematics, geometric constraint.

1. Introduction

Geometric constraints are used in two different contexts. In one application area we define a set of geometric primitives and constraints upon them, and then are asked to find an arrangement of the primitives such that the constraints are satisfied. Let us call this the *construction problem*. The construction problem arises for example when defining CAD models for discrete manufacturing.

In a second application area we are given a set of (usually composite) geometric objects as well as constraints upon their spatial relationship, with the objective of constraining the relative motion of the objects with respect to each other. Let us call this the *compliance problem*. The compliance problem arises in assembly modeling, kinematic simulation of machinery, and in virtual reality, to name a few uses.

In this paper we consider the compliance problem in 3-space and investigate basic techniques to solve it. There is a wealth of prior work,

*Work supported in part by NSF Grants EIA 02-16131, DMS/CCR 01-38098, CCR 99-02025, and by ARO contract 39136-MA.

and we give a few example references below. The area of kinematics to a large extent considers and solves compliance problems, for instance when investigating linkages, or, more generally, when designing and analyzing machinery [6,7]. Other relevant research is done in robotics [1], and in some areas of geometric constructions [5]. To some extent, the compliance problem overlaps with the construction problem, as seen in [4], where a system of equations is attacked by considering the residual compliant motion of geometric primitives when restricting to a subset of the given constraints.

Much of the research into compliance is dominated by seeking elegant mathematical formalisms that would simplify expressing and analyzing compliant motion. In addition to ad-hoc techniques that are highly successful in special cases such as four-bar linkages, three main formalisms have emerged: (4×4) transforms, screws, and dual quaternions. The three formalisms offer a general description of rigid body motion in 3-space. Note that screws are essentially dual quaternions, but the reduced coordinate set may introduce ambiguities in some cases. For this reason we do not consider them further.

2. Tools and Notation

In this section, we review some basics and notations on quaternions, their relations to rotations, and dual quaternions.

2.1. *Quaternions*

The field of quaternions has elements of the form $\mathbf{a} = a_0 + a_1 i + a_2 j + a_3 k$ where the coefficients a_r are real numbers and the units i, j, and k obey the equations:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j$$

Complex numbers are quaternions with $a_2 = a_3 = 0$. The *length* of a quaternion $\mathbf{a}$, is defined as $\|\mathbf{a}\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. Quaternions are due to Maxwell.

The *conjugate* of the quaternion $\mathbf{a} = a_0 + a_1 i + a_2 j + a_3 k$ is the quaternion $\bar{\mathbf{a}} = a_0 - a_1 i - a_2 j - a_3 k$. The *norm* of $\mathbf{a}$ is the square of the length of $\mathbf{a}$ and is equal to the quaternion product $\mathbf{a}\bar{\mathbf{a}}$. We define the *inner product* $(\mathbf{a} \cdot \mathbf{b})$ of two quaternions $\mathbf{a}$ and $\mathbf{b}$ as $(\mathbf{a} \cdot \mathbf{b}) = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$. Note that the norm of $\mathbf{a}$ is $(\mathbf{a} \cdot \mathbf{a}) = \mathbf{a}\bar{\mathbf{a}}$.

A quaternion with $a_0 = 0$ is called a *vector*. More generally, we can call a_0 the *real part* of the quaternion $\mathbf{a}$ and call $a_1 i + a_2 j + a_3 k$ the *vector part*. We denote quaternions with lower-case bold letters. Vectors in $\mathbf{R}^3$ are denoted by bold lower-case letters with an arrow; for example, $\vec{\mathbf{p}}$. The arrow is omitted when it is clear from the context that we speak of a vector. The vector part of a quaternion is denoted in the same way. Thus, if $\mathbf{a} = a_0 + a_1 i + a_2 j + a_3 k$, then $\vec{\mathbf{a}} = a_1 i + a_2 j + a_3 k$. We note that $\overline{\mathbf{a}} = a_0 - \vec{\mathbf{a}}$.

We will use the inner product, denoted by $\cdot$ and the cross product of vectors, denoted by $\times$, to express quaternion operations more succinctly. For example, the product of two quaternions $\mathbf{a} = a_0 + \vec{\mathbf{a}}$ and $\mathbf{b} = b_0 + \vec{\mathbf{b}}$ is the quaternion

$$\mathbf{ab} = a_0 b_0 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) + a_0 \vec{\mathbf{b}} + b_0 \vec{\mathbf{a}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

2.2. *Rotations*

With Cartesian point coordinates in 3-space, a rotation in 3-space about the origin can be represented by the orthogonal matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

where $RR^T = I$ and $\det(R) = 1$.

It is well-known that unit-length quaternions can represent rotations about the origin. Wittenburg [10] gives the following conversion formulae. For any unit-length quaternion $\mathbf{a}$, the entries of the rotation matrix are

$$\begin{aligned} r_{11} &= 2(a_0^2 + a_1^2) - 1, & r_{12} &= 2(a_1 a_2 + a_0 a_3), & r_{13} &= 2(a_1 a_3 - a_0 a_2), \\ r_{21} &= 2(a_1 a_2 - a_0 a_3), & r_{22} &= 2(a_0^2 + a_2^2) - 1, & r_{13} &= 2(a_2 a_3 + a_0 a_1), \\ r_{31} &= 2(a_1 a_3 + a_0 a_2), & r_{32} &= 2(a_2 a_3 - a_0 a_1), & r_{33} &= 2(a_0^2 + a_3^2) - 1, \end{aligned}$$

and for any rotation matrix with entries r_{pq} the quaternion coefficients are:

$$\begin{aligned} a_0^2 &= (r_{11} + r_{22} + r_{33} + 1)/4, \\ a_1^2 &= r_{11}/2 - u, \\ a_2^2 &= r_{22}/2 - u, \\ a_3^2 &= r_{33}/2 - u, \\ \text{where} \qquad u &= (r_{11} + r_{22} + r_{33} - 1)/4. \end{aligned}$$

Other, equivalent conversion formulae are given in, for example, [8].

There is a well-known geometric interpretation of the quaternion representation of such rotations. Let $\mathbf{v} = (v_1, v_2, v_3)$ be the unit-length direction vector of the axis of rotation, and let 2θ be the angle of rotation. With $c = \cos(\theta)$ and $s = \sin(\theta)$, the rotation is represented by the quaternion $c + sv_1 i + sv_2 j + sv_3 k$.

2.3. *Dual Numbers and Dual Quaternions*

A *dual number* is defined as $A = a + b\epsilon$, where a and b are from a field and $\epsilon^2 = 0$. Dual numbers form a Clifford algebra. If $A = a + b\epsilon$ is a dual number, $A_\epsilon = a - b\epsilon$ is its *conjugate*.

A *dual quaternion* is defined as $\mathbf{A} = \mathbf{a} + \mathbf{b}\epsilon$, where $\mathbf{a}$ and $\mathbf{b}$ are quaternions. Equivalently, a dual quaternion is a quaternion whose components are dual numbers (with real coefficients). Dual quaternions can represent points, lines and planes in 3-space, as well as general rigid body motions, as will be discussed in the next section.

As in [2], we define three different conjugations of a dual quaternion, according to whether the quaternion components are conjugated, the dual numbers are conjugated, or both. Let $\mathbf{A} = \mathbf{a} + \mathbf{b}\epsilon$, where $\mathbf{a}$ and $\mathbf{b}$ are the quaternions. We define

$$\overline{\mathbf{A}} = \overline{\mathbf{a}} + \overline{\mathbf{b}}\epsilon, \quad \mathbf{A}_\epsilon = \mathbf{a} - \mathbf{b}\epsilon, \quad \overline{\mathbf{A}}_\epsilon = \overline{\mathbf{a}} - \overline{\mathbf{b}}\epsilon.$$

3. Representations

We adopt the algebraic schema of [2] to represent points, lines and planes in 3-space, as well as rigid-body transformations on them.

3.1. *Points, Lines, and Planes*

In Cartesian coordinates, points are specified by their position vector (p_1, p_2, p_3) which we represent by the dual quaternion

$$\mathbf{P} = 1 + \mathbf{p}\epsilon$$

A plane has the equation $ap_1 + bp_2 + cp_3 + d = 0$, where we require that $a^2 + b^2 + c^2 = 1$. Such a plane is represented by the dual quaternion

$$\mathbf{E} = \mathbf{n} + d\epsilon.$$

The first quaternion is the plane normal vector $\mathbf{n} = ai + bj + ck$, and the second quaternion, which is real, is the constant of the implicit plane equation.

Using Plücker coordinates, lines in 3-space can be represented by two 3-vectors $\mathbf{t} = (t_1, t_2, t_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$, where $\mathbf{t}$ is the line direction vector, normed to unit length, and $\mathbf{m}$ is the moment vector $\mathbf{p} \times \mathbf{t}$ of some point $\mathbf{p}$ on the line. Clearly, the inner product of the moment vector and the direction vector is zero; that is, $(\mathbf{m} \cdot \mathbf{t}) = 0$. Identifying, as before, the vector (a, b, c) with the quaternion $ai + bj + ck$, we represent the line $(\mathbf{t}, \mathbf{m})$ as the dual quaternion

$$\mathbf{L} = \mathbf{t} + \mathbf{m}\epsilon.$$

The first quaternion is the unit-length direction vector, the second quaternion is the moment vector. For lines through the origin $\mathbf{m} = 0$.

3.2. *Rigid Body Motion*

The unit quaternion $\mathbf{q}$ was noted to represent a rotation about the origin. The dual quaternion $\mathbf{Q} = \mathbf{q}$ with the zero quaternion as the ϵ coordinate is chosen to represent the same rotation. Furthermore, we represent a translation by the vector $(2s_1, 2s_2, 2s_3)$ by the dual quaternion $\mathbf{S} = 1 + \mathbf{s}\epsilon$, where $\mathbf{s} = s_1 i + s_2 j + s_3 k$. A rigid body motion in 3-space can therefore be represented by a dual quaternion $\mathbf{T} = \mathbf{SQ}$ that is the product of the rotation quaternion $\mathbf{Q}$ and the translation quaternion $\mathbf{S}$, imitating the action of 4-by-4 transforms. The representation of rigid motions by dual quaternions is due to Study [9].

Screw Motion

Chasles' theorem [3] states that every rigid motion is equivalent to a screw motion. Here, the screw with axis $(\mathbf{t}, \mathbf{m})$, angle of rotation 2θ, and a displacement $2d$ is represented as the dual quaternion

$$\mathbf{M}_{\text{screw}} = \cos(\theta) + \sin(\theta)\,\mathbf{t} + (-d\sin(\theta) + \sin(\theta)\,\mathbf{m} + d\cos(\theta)\,\mathbf{t})\,\epsilon \quad (3.1)$$

Note that for $\theta = 0$, the motion $\mathbf{M}$ simplifies to $1 + d\,\mathbf{t}\epsilon$, a translation by $2d\,\mathbf{t}$, and for $d = 0$ and $\mathbf{m} = 0$ it simplifies to $\cos(\theta) + \sin(\theta)\,\mathbf{t}$, a rotation about the origin.

Other Motion Representation

The general rigid body motion can be expressed as

$$\mathbf{M} = \mathbf{q} + \mathbf{u}\epsilon, \quad \|\mathbf{q}\| = 1, \quad (\mathbf{q} \cdot \mathbf{u}) = 0. \quad (3.2)$$

We prove that this is true.

The conditions of Equation (3.2) are clearly satisfied by the screw motions of Equation 3.1. Thus, all rigid motions can be represented.

Conversely, assume that the above conditions are satisfied by the dual quaternion $\mathbf{M} = \mathbf{q} + \mathbf{u}\epsilon$. Writing $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$, we may define θ, t_1, t_2, t_3 by setting

$$q_0 = \cos(\theta), \quad \text{and } q_r = \sin(\theta) t_r, \qquad r = 1, 2, 3.$$

Let $\mathbf{t} = t_1 i + t_2 j + t_3 k$. Since $\|\mathbf{q}\| = 1$, we have $\|\mathbf{t}\| = 1$ in general.

If $|q_0| = 1$, then $q_1 = q_2 = q_3 = 0$ and $u_0 = 0$. In that case $\mathbf{M}$ is a translation.

If $u_0 = 0$, the vector $\vec{\mathbf{u}}$ must be perpendicular to the vector $\vec{\mathbf{q}}$. We may assume $|q_0| \neq 1$ and define the vector $\mathbf{m} = \vec{\mathbf{u}} / \sin(\theta)$. We now see that $\mathbf{M}$ represents a pure rotation about an axis with direction $\mathbf{t}$ and moment $\mathbf{m}$.

Otherwise, with $|q_0| \neq 1$ and $u_0 \neq 0$, we have $\sin(\theta) \neq 0$, and we can define the nonzero quantity d from $u_0 = -d \sin(\theta)$. Define $m_0 = 0$ and $m_r \sin(\theta) = u_r - d t_r \cos(\theta)$, $r = 1, 2, 3$. Then

$$\begin{aligned} 0 &= q_0 u_0 + q_1 u_1 + q_2 u_2 + q_3 u_3 \\ &= -d \sin(\theta)\cos(\theta) + \textstyle\sum_{r=1}^{3} \sin(\theta) t_r (m_r \sin(\theta) + d t_r \cos(\theta)) \\ &= -d \sin(\theta)\cos(\theta) + \textstyle\sum_{r=1}^{3} (d \sin(\theta)\cos(\theta) t_r^2 + t_r m_r \sin^2(\theta)) \\ &= \sin^2(\theta)(\mathbf{t} \cdot \mathbf{m}) \end{aligned}$$

Therefore the vector $\mathbf{m} = m_1 i + m_2 j + m_3 k$ is perpendicular to the vector $\mathbf{t}$, which means that $\mathbf{M}$ is a screw motion with axis $(\mathbf{t}, \mathbf{m})$, rotation angle 2θ, and displacement $2d$.

3.3. *Motion of Points, Lines and Planes*

Let $\mathbf{P}$ be a dual quaternion representing a point or a plane. Then the dual quaternion $\mathbf{P}'$ that represents the result of a rigid body motion $\mathbf{M}$, applied to the point or plane represented by $\mathbf{P}$, is calculated as

$$\mathbf{P}' = \mathbf{M}\mathbf{P}\overline{\mathbf{M}}_\epsilon. \tag{3.3}$$

Similarly, the line represented by the dual quaternion $\mathbf{L}$ is transformed into the line represented by $\mathbf{L}'$, where

$$\mathbf{L}' = \mathbf{M}\mathbf{L}\overline{\mathbf{M}} \tag{3.4}$$

An algebraic computation verifies this definition; see also [2].

Summarizing, dual quaternions allow us to represent points, lines and planes in 3-space uniformly, and express rigid body transformations of them uniformly as well.

4. Constrained Motion

We investigate what relative motion is possible when requiring a single incidence constraint of a point, line or plane on another point, line or plane. First, we formulate incidence conditions in terms of dual quaternions. Then we investigate relative motion assuming that the incidences are currently satisfied.

4.1. *Incidence*

Six elementary incidence conditions arise when requiring points, lines and planes to be incident to each other. Among features of equal type, incidence is trivial, as it requires equal coordinates. Note, however, that for planes and lines incidence with opposite orientation should be accounted for.

Let $\mathbf{P}$ be a point, $\mathbf{E}$ a plane, and $\mathbf{L}$ a line. We require that the plane normal and the line direction vectors have unit length. The following are the incidence conditions between features of different type.

$$\begin{aligned} \mathbf{EP} + \mathbf{PE} &= 0 && \text{point on plane} \\ \mathbf{LP} - \mathbf{P}\overline{\mathbf{L}}_\epsilon &= 0 && \text{point on line} \\ \mathbf{LP} + \mathbf{P}\overline{\mathbf{L}}_\epsilon &= 0 && \text{line on plane} \end{aligned}$$

See [2] for a proof.

4.2. *Parametric Relative Motion*

It is not difficult to express parametrically the relative motion that obeys a single elementary incidence constraint. In particular, if the elements are of the same type, we are asking for motion expressions that leave a point, a line or a plane invariant. However, a parametric representation in the presence of multiple incidence constraints between different features of two rigid bodies is not so easy. We will show that it can be done based on the parametric representation, in a number of cases. A related problem commonly investigated in robotics is to synthesize the motion of a kinematic chain, such as an articulated robotic arm. Such work typically assumes fixed common lower-pair connections between the links such as a revolute or a prismatic joint.

If we express the relative motion of a single incidence constraint parametrically, then we can combine the equations into a single system and obtain a combined parameterization using elimination computations. It is advantageous to keep the equation system as simple as possible, and this would argue for performing every algebraic simplification possible as a preprocessing step before undertaking the actual evaluation. We begin with expressing the elementary constraints. As before, we denote points, lines and planes with dual quaternions $\mathbf{P}$, $\mathbf{L}$ and $\mathbf{E}$, respectively.

4.2.1. *Incidences of Equal Type*

These are incidences of point on point, line on line, and plane on plane. To express the relative motions, we ask which rigid body transformations fix a point, a line or a plane.

For the point represented by the dual quaternion $\mathbf{P} = 1 + \epsilon\mathbf{p}$ we obtain

$$\mathbf{M_P} = \mathbf{q} + \epsilon\vec{\mathbf{p}} \times \vec{\mathbf{q}}, \quad \|\mathbf{q}\| = 1. \tag{4.1}$$

Note that we require that $\mathbf{q}$ has length 1, and that $\mathbf{p}$ is a vector quaternion, that is, $p_0 = 0$.

We can derive $\mathbf{M_P}$ by conjugating a general rotation about the origin by the translation of the fixed point to the origin. Let $\mathbf{T}$ be the translation from the point $\mathbf{P}$ to the origin, represented as a dual quaternion, and let its inverse be $\mathbf{T}'$. Then $\mathbf{M_P} = \mathbf{T}'\mathbf{Q}\mathbf{T}$. The representation has four parameters which reduce to three independent ones because of the unit-length requirement on $\mathbf{q}$.

A different parameterization derivation is possible by considering a screw motion that has a zero displacement along the axis. With $\mathbf{t}$ an arbitrary unit length vector, we then obtain the equivalent form

$$\begin{aligned} &\mathbf{M_P} = \cos(\theta) + \mathbf{t}\sin(\theta) + \mathbf{m}\sin(\theta), \\ &\|\mathbf{t}\| = 1, \qquad \mathbf{m} = \mathbf{p} \times \mathbf{t}. \end{aligned}$$

Again, there are 4 parameters reducing to three independent ones because of the unit-length requirement. The resulting parameterization is identical to (4.1).

Next, we consider the motion that leaves the plane $\mathbf{E} = \mathbf{n} + d\epsilon$ invariant. Here, $\mathbf{n}$ is the unit-length normal vector of the plane. The motion that leaves the plane invariant can be considered as a rotation about an axis through the origin in the direction $\vec{\mathbf{n}}$ plus a translation by a vector $\mathbf{t}$ in the plane, which therefore satisfies $(\mathbf{t} \cdot \mathbf{n}) = 0$. We obtain

$$\begin{gathered} \mathbf{M_E} = \cos(\theta) + \sin(\theta)\mathbf{n} + \epsilon(\cos(\theta)\mathbf{t} + \sin(\theta)(\mathbf{t} \times \mathbf{n})), \\ (\mathbf{t} \cdot \mathbf{n}) = 0. \end{gathered} \tag{4.2}$$

The four parameters reduce to three independent ones by the (linear) equation $(\mathbf{t} \cdot \mathbf{n}) = 0$.

Finally, the motion that leaves the line $\mathbf{L}$ with direction $\mathbf{t}$ and moment $\mathbf{m}$ invariant is given by

$$\begin{gathered}\mathbf{M_L} = \cos(\theta) + \sin(\theta)\mathbf{t} + \epsilon(-d\sin(\theta) + \sin(\theta)\mathbf{m} + d\cos(\theta)\mathbf{t}), \\ (\mathbf{t}\cdot\mathbf{m}) = 0.\end{gathered} \tag{4.3}$$

It represents the screw with axis $\mathbf{L}$, displacement $2d$, and angle of rotation 2θ. The two parameters d and θ are independent. Thus, this relative motion parameterization is irredundant.

4.2.2. *Incidences of Different Types*

Unequal type incidence constraints may be obtained by combining motions that include fixing one of the features, to account for symmetries, followed by displacing it within the geometry of the other feature.

The relative motion subject to requiring that the point $\mathbf{P}$ stay in the plane $\mathbf{E}$ can be obtained by composing the relative motion that fixes $\mathbf{P}$ with a subsequent translation in the plane. With $2\mathbf{s}$ the vector of translation, we obtain

$$\begin{aligned}\mathbf{M_{PE}} &= \mathbf{TM_P} \\ &= \mathbf{q} + (-(\vec{\mathbf{s}}\cdot\vec{\mathbf{q}}) + q_0\vec{\mathbf{s}} + \vec{\mathbf{p}}\times\vec{\mathbf{q}} + \vec{\mathbf{s}}\times\vec{\mathbf{q}})\epsilon \\ &= \mathbf{q} + (\mathbf{sq} + \vec{\mathbf{p}}\times\vec{\mathbf{q}})\epsilon, \\ \|\mathbf{q}\| &= 1, \qquad (\vec{\mathbf{s}}\cdot\vec{\mathbf{n}}) = 0.\end{aligned} \tag{4.4}$$

The condition that the translation vector be perpendicular to the plane normal implies two independent parameters in the choice of $\mathbf{s}$, bringing the total degrees of freedom of the motion to 5.

Applying the same procedure we obtain the following for keeping the point $\mathbf{P}$ on the line $\mathbf{L}$. The translation must be along the line direction, thus we obtain

$$\mathbf{M_{PL}} = \mathbf{TM_P} = \mathbf{q} + (\mathbf{tq} + \mathbf{t}\times\vec{\mathbf{q}})\epsilon. \tag{4.5}$$

Finally, consider keeping a line $\mathbf{L} = \mathbf{t} + \mathbf{m}\epsilon$ incident to a plane $\mathbf{E} = \mathbf{n} + d\epsilon$. Geometrically, the motion can be considered a screw motion with axis $\mathbf{L}$ followed by a translation of the line in the plane which can be restricted to a displacement perpendicular to the line. Since $\mathbf{t}$ is perpendicular to the plane normal, the subsequent translation is in the direction $\mathbf{t}\times\mathbf{n}$.

We obtain the following representation:

$$\begin{aligned}\mathbf{M_{LE}} &= \cos(\theta) + \sin(\theta)\mathbf{t} + \epsilon(-d_1\sin(\theta) + \sin(\theta)\mathbf{m}) + \\ &\quad \epsilon(d_1\cos(\theta)\mathbf{t} + d_2\cos(\theta)\mathbf{u} + d_2\sin(\theta)\mathbf{n}), \\ \mathbf{u} &= \mathbf{t}\times\mathbf{n}, \quad \|\mathbf{t}\| = 1, \quad (\mathbf{m}\cdot\mathbf{t}) = 0.\end{aligned} \tag{4.6}$$

Here 2θ is the angle of rotation, $2d_1$ the displacement in the $\mathbf{t}$ direction, and d_2 the displacement in the perpendicular $\mathbf{u} = \mathbf{t}\times\mathbf{n}$ direction.

5. Combining Constraints

5.1. *Parametric Approach*

Consider now moving a part A relative to another part B where there are multiple incidence constraints between features of the two parts. The parametric representations of the relative motion can be used when combining several incidence constraints as follows. Let $F_1, ..., F_r$ be the parametric forms of the residual motion taken separately for each incidence constraints.

By equating the rigid body motions of the F_i, we obtain a system $E_1, ..., E_s$ of implicit equations in the parameters. We solve this system for a set of independent parameters. This is an elimination computation and therefore potentially expensive. Then we can evaluate relative motion by evaluating the dependent parameters as necessary and substituting into F_1, thus obtaining an admissible relative motion.

Example 5.1. Consider a fixed part A with two plane features, $\mathbf{E}_1 = j$ and $\mathbf{E}_2 = i$, namely, the planes $x = 0$ and $y = 0$. On a moving part B we fix the points $\mathbf{P}_1 = 1 + i\epsilon$ and $\mathbf{P}_2 = 1 + j\epsilon$, that is, the points $(1, 0, 0)$ and $(0, 1, 0)$, respectively. Evidently $\mathbf{P}_1$ is on $\mathbf{E}_1$ and $\mathbf{P}_2$ is on $\mathbf{E}_2$. We use for the translation in $\mathbf{E}_1$, the vector $\mathbf{s} = (s, 0, t)$ and for the translation in $\mathbf{E}_2$, the vector $\mathbf{s}' = (0, s', t')$. Then the parametric forms for the relative motion, considering the incidence constraints separately, are, in detail,

$$\begin{aligned}F_1 : \mathbf{M}_1 &= \mathbf{q} + \epsilon\big(-sq_1 - tq_3 + (sq_0 - tq_2)i \\ &\qquad + (tq_1 - sq_3 - q_3)j + (sq_2 + tq_0 + q_2)k\big), \\ F_2 : \mathbf{M}_2 &= \mathbf{q}' + \epsilon\big(-s'q_2' - t'q_3' + (s'q_3' - t'q_2' + q_3')i \\ &\qquad + (s'q_0' + t'q_1')j + (t'q_0' - s'q_1' - q_1')k\big).\end{aligned}$$

We equate the parameters $\mathbf{q}$ and $\mathbf{q}'$, and determine the relationships between the other parameters by equating the components of the ϵ quaternion.

Accounting for $\|\mathbf{q}\| = 1$, we obtain

$$\begin{aligned} s &= q_0q_3 - q_1q_2 - q_2^2 - q_3^2, \\ s' &= -q_0q_3 - q_1^2 - q_1q_2 - q_3^2, \\ t' &= t + q_0q_1 + q_0q_2 - q_1q_3 + q_2q_3. \end{aligned}$$

Thus, we have four independent parameters. Three of them specify $\mathbf{q}$ and this determines s and s' as well. The fourth parameter is t which, in conjunction with $\mathbf{q}$, determines t'.

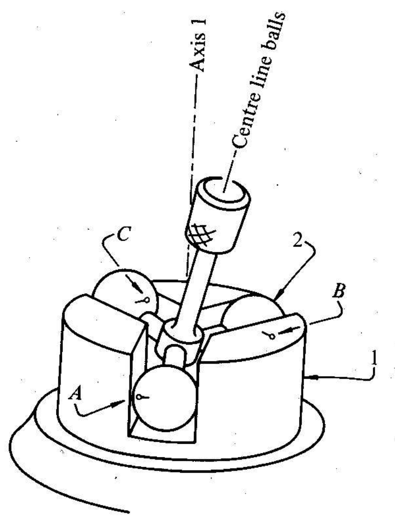

Figure 1. Configuration of Example 2; drawing reproduced from [6]

Example 5.2. Consider the joint constructed by fitting a tripod of balls into three slots whose center planes intersect in a common line. See Figure 1 above. Here we have three point/plane incidence constraints. The features of the fixed part are the three planes

$$\begin{aligned} \mathbf{E}_1 &= j, \\ \mathbf{E}_2 &= -\frac{\sqrt{3}}{2}i - \frac{1}{2}j, \\ \mathbf{E}_3 &= \frac{\sqrt{3}}{2}i - \frac{1}{2}j. \end{aligned}$$

The respective parametric tangential motions are

$$\begin{aligned}\mathbf{s}_1 &= (s, 0, t),\\ \mathbf{s}_2 &= (-s', \sqrt{3}s', t'),\\ \mathbf{s}_3 &= (-s'', -\sqrt{3}s'', t'').\end{aligned}$$

The features of the moving part are

$$\begin{aligned}\mathbf{P}_1 &= 1 + \epsilon i,\\ \mathbf{P}_2 &= 1 + \epsilon(-\frac{1}{2}i + \frac{\sqrt{3}}{2}j),\\ \mathbf{P}_3 &= 1 + \epsilon(-\frac{1}{2}i - \frac{\sqrt{3}}{2}j).\end{aligned}$$

The three transformations obtained are

$$\begin{aligned}F_1 : \mathbf{M}_1 = \mathbf{q} + \epsilon(&-sq_1 - tq_3\\ &+ (sq_0 - tq_2)i + (tq_1 - sq_3 - q_3)j + (sq_2 + tq_0 + q_2)k),\\ F_2 : \mathbf{M}_2 = \mathbf{q} + \epsilon(&s'(q_1 - \sqrt{3}q_2) - t'q_3 +\\ &+ (s'(-q_0 + \sqrt{3}q_3) - t'q_2 + \sqrt{3}q_3/2)i\\ &+ (s'(\sqrt{3}q_0 + q_3/2) + t'q_1 + q_3/2)j\\ &+ (s'(-\sqrt{3}q_1 - q_2) + t'q_0 - q_2/2 - \sqrt{3}q_1/2)k),\\ F_3 : \mathbf{M}_3 = \mathbf{q} + \epsilon(&s''(q_1 + \sqrt{3}q_2) - t''q_3 +\\ &+ (s''(-q_0 - \sqrt{3}q_3) - t''q_2 - \sqrt{3}q_3/2)i\\ &+ (s''(-\sqrt{3}q_0 + q_3/2) + t''q_1 + q_3/2)j\\ &+ (s''(\sqrt{3}q_1 - q_2) + t''q_0 + \sqrt{3}q_1/2 - q_2/2)k).\end{aligned}$$

We equate the coordinates of the three dual quaternion expressions. Note that this results in a linear system of equations in the parameters s, s', s'', t, t', t''. The equations yield the following

$$\begin{aligned}2s &= q_1^2 - 3q_2^2 - 2q_3^2,\\ 2s' &= -q_1^2 - \sqrt{3}q_1q_2 - q_3^2,\\ 2s'' &= -q_1^2 + \sqrt{3}q_1q_2 - q_3^2,\\ 2(t - t') &= -3q_0q_2 + 3q_1q_3 - \sqrt{3}(q_0q_1 + q_2q_3),\\ 2(t - t'') &= -3q_0q_2 + 3q_1q_3 + \sqrt{3}(q_0q_1 + q_2q_3),\\ q_0q_3 &= 0.\end{aligned}$$

It follows from these six equations that the relative motion has three degrees of freedom.

5.2. *Implicit Approach*

In the implicit approach we express the relative motion as relations on the parameters of the general rigid-body motion. This will allow us to conjoin the parameter relations without having to resort to algebraic elimination.

Implicit Incidence Constraints, Equal Types

We translate the parametric formulations of the incidence constraints into implicit form. We will work with the generic rigid body transformation expression $\mathbf{M}$ of Equation (3.2).

We derived the parametric form of Equation (4.1) for keeping a point $\mathbf{p}$ invariant. Accordingly, the implicit conditions on $\mathbf{M}$ are

$$\mathbf{M_p} = \mathbf{q} + \mathbf{u}\epsilon, \quad \mathbf{p} \times \vec{\mathbf{q}} = \mathbf{u}, \quad \|\mathbf{q}\| = 1. \tag{5.3}$$

These conditions imply in particular that $u_0 = 0$. When $\mathbf{p} = 0$ the point is at the origin and the condition on $\mathbf{u}$ simplifies to $u_0 = u_1 = u_2 = u_3 = 0$.

To fix the plane $\mathbf{E}$, we derived the parametric form of Equation (4.2). It implies the condition $u_0 = 0$. The direction of the rotation axis implied by $\mathbf{q}$ has to be normal to the plane, hence we require $(\mathbf{n} \cdot \vec{\mathbf{q}}) = \sqrt{1 - q_0^2}$. Since both $\mathbf{t}$ and $\mathbf{t} \times \mathbf{n}$ in Equation (4.2) are perpendicular to the plane normal, we obtain the following conditions:

$$\mathbf{M_e} = \mathbf{q} + \mathbf{u}\epsilon, \quad u_0 = 0, \quad (\mathbf{n} \cdot \vec{\mathbf{q}}) = \sqrt{1 - q_0^2}, \quad (\vec{\mathbf{n}} \cdot \mathbf{u}) = 0. \tag{5.4}$$

The second condition degenerates when the motion is a pure translation since, in that case, the right-hand side vanishes. However, in that case the condition $\|\mathbf{q}\| = 1$ forces $q_1 = q_2 = q_3 = 0$, so a pure translation within the plane is implied by the formulation.

Now consider a line $\mathbf{L} = \mathbf{t} + \epsilon\mathbf{m}$ with direction $\mathbf{t}$ and moment vector $\mathbf{m}$. The line is invariant under $\mathbf{M} = \mathbf{q} + \epsilon\mathbf{u}$ if the transformed line $\mathbf{L}' = \mathbf{M}\mathbf{L}\overline{\mathbf{M}}$ has the same tangent and moment vectors. This implies the following relations, in which $\mathbf{m}$ and $\mathbf{t}$ are known quaternions:

$$\mathbf{M_l} = \mathbf{q} + \mathbf{u}\epsilon, \quad \mathbf{t} = \mathbf{q}\mathbf{t}\overline{\mathbf{q}}, \quad \mathbf{m} = \mathbf{u}\mathbf{t}\overline{\mathbf{q}} + \mathbf{q}\mathbf{m}\overline{\mathbf{q}} + \mathbf{q}\mathbf{t}\overline{\mathbf{u}}.$$

Implicit Incidence Constraints, Different Types

Consider now keeping a point $\mathbf{P}$ on a plane $\mathbf{E}$, for which we derived the parametric form of Equation (4.4). We derive the implicit condition on $\mathbf{M}$ by requiring that the transformed point $\mathbf{P}'$ is again in the plane $\mathbf{E}$. Let $\mathbf{P} = 1 + \epsilon\mathbf{p}$. We obtain

$$\begin{aligned}\mathbf{P}' &= (\mathbf{q} + \epsilon\mathbf{u})(1 + \epsilon\mathbf{p})(\overline{\mathbf{q}} - \epsilon\overline{\mathbf{u}}) \\ &= 1 + \epsilon(\mathbf{u}\overline{\mathbf{q}} - \mathbf{q}\overline{\mathbf{u}} + \mathbf{q}\mathbf{p}\overline{\mathbf{q}}) \\ &= 1 + \epsilon(-2(\mathbf{u} \times \vec{\mathbf{q}}) - 2u_0\vec{\mathbf{q}} + 2q_0\vec{\mathbf{u}} + \mathbf{q}\mathbf{p}\overline{\mathbf{q}}) \\ &= 1 + \epsilon\mathbf{p}'.\end{aligned}$$

An algebraic computation verifies that the real component of $\mathbf{p}'$ is zero, that is, $\mathbf{p}'$ is the position vector of the transformed point. Assuming the original point is in the plane with unit normal vector $\mathbf{n}$, we obtain the condition $(\mathbf{n} \cdot \mathbf{p}) = (\mathbf{n} \cdot \mathbf{p}')$, or equivalently:

$$(\mathbf{n} \cdot \mathbf{p}) = -2(\mathbf{n} \cdot (\mathbf{u} \times \vec{\mathbf{q}}) - 2u_0(\mathbf{n} \cdot \vec{\mathbf{q}}) + 2q_0(\mathbf{n} \cdot \vec{\mathbf{u}}) + (\mathbf{n} \cdot (\mathbf{q}\mathbf{p}\overline{\mathbf{q}})) \qquad (5.5)$$

Example 5.6. Consider the plane $\mathbf{E}_1 = j$ and the point $\mathbf{P}_1 = 1 + \epsilon i$ in the plane. Any motion $\mathbf{M}$ that keeps this point in the plane must satisfy according to Equation (5.5)

$$\begin{aligned}q_0q_3 + q_1q_2 - u_0q_2 + u_1q_3 + u_2q_0 - u_3q_1 &= 0 \\ \|\mathbf{q}\| &= 1 \\ (\mathbf{q} \cdot \mathbf{u}) &= 0\end{aligned}$$

Example 5.7. We consider the points and planes of Example 1. The conditions from the point incidences are then

$$\begin{aligned}q_0q_3 + q_1q_2 - u_0q_2 + u_1q_3 + u_2q_0 - u_3q_1 &= 0 \\ -q_0q_3 + q_1q_2 - u_0q_1 + u_1q_0 - u_2q_3 + u_3q_2 &= 0 \\ \|\mathbf{q}\| &= 1 \\ (\mathbf{q} \cdot \mathbf{u}) &= 0\end{aligned}$$

Note that a translation in the z-direction, $1 + \epsilon dk$, satisfies these conditions.

Example 5.8. We consider the planes and points of Example 2. The incidence conditions, after some simplification, define the equations

$$\begin{aligned}\|\mathbf{q}\| &= 1\\ (\mathbf{q}\cdot\mathbf{u}) &= 0\\ 2q_0q_3 - q_1q_2 &= 0\\ q_0q_3 + q_1q_2 + u_2q_0 - u_0q_2 + u_1q_3 - u_3q_1 &= 0\\ 2\sqrt{3}(u_0q_1 - u_1q_0 + u_2q_3 - u_3q_2) + \sqrt{3}(q_1^2 - q_2^2) + 2(q_0q_3 + q_1q_2) &= 0\end{aligned}$$

6. Discussion

The uniformity of the representation and the algebraic nature of the representation are the main attractions when using dual quaternions. Points, lines and planes are simple to represent as dual quaternions, and so are rigid-body motions. Moreover, as we have seen, there is considerable geometric intuition in this representation schema.

Another advantage of dual quaternions, from a computational perspective, is that they describe a general rigid-body motion with only eight parameters, whereas a 4×4 matrix representation would require twelve. Thus, the system of equations describing a particular contact configuration is smaller. A screw representation would lower this to six parameters, but the resulting equations may fail in particular instances and do not differ, in essence, from the dual quaternion representation.

There are some drawbacks to using dual quaternions as well. In the implicit form of the constraint encoding, for instance, the conditions can become fairly complex. An example is the implicit representation of motions that keep a line invariant. Here, the parametric form does better. Moreover, the implicit form we derived has some redundancies. Consider again all conditions on $\mathbf{M_l}$:

$$\mathbf{t} = \mathbf{q}\mathbf{t}\overline{\mathbf{q}} \tag{6.1}$$

$$\mathbf{m} = \mathbf{u}\mathbf{t}\overline{\mathbf{q}} + \mathbf{q}\mathbf{m}\overline{\mathbf{q}} + \mathbf{q}\mathbf{t}\overline{\mathbf{u}} \tag{6.2}$$

$$\|\mathbf{q}\| = 1 \tag{6.3}$$

$$(\mathbf{q}\cdot\mathbf{u}) = 0 \tag{6.4}$$

Conditions (6.1) and (6.2) each yields three scalar equations, giving eight equations in eight variables total. Therefore, there must be two redundant equations. With a pure translation ($|q_0| = 1$), Condition (6.1) is trivial. With a pure rotation ($u_0 = 0$ and $\mathbf{u} = \sin(\theta)\mathbf{m}$), on the other hand, Condition (6.1) is not trivial. Thus, redundancy depends on the parameter values.

Parametric motion representations often lead to motion descriptions in which the system of parameter relations is linear, a computational plus. However, nonlinear relations may ensue, for example on the **q** coordinates of the transformation. Here, symbolic algebraic computations may require reasoning that is not entirely automated in, for instance, *Maple*.

Another drawback is that the representation of the relative motion may not include certain special cases. For instance, given a plane **E**, we may choose a line $(\mathbf{t}, \mathbf{m})$ that lies in the plane and use it as the axis for a rigid-body motion that has a rotation angle of $180°$. Those motions also preserve the plane, albeit with a reversal of the plane orientation. Thus, the geometry and the algebra diverge in this case.

In contrast to the parametric expression of relative motion, the implicit formulation does not require intermediate positions to satisfy the constraints. For example, the point **P** is required to be on the plane **E** only at the start and at the end of the motion. As the motion progresses, it may very well leave the plane **E** at the other times. This is true in particular of the special case of a rotation by $180°$ about an axis in the plane **E**.

From a computational perspective, dual quaternions do not reduce the number of arithmetic operations that must be done to compute the image of a feature under a given transformation. Using a 4×4 matrix representation, transforming a point requires 12 multiplications and 9 additions. The dual quaternion representation, on the other hand, requires more.

References

1. S. Ahlers and J. M. McCarthy. The Clifford algebra of double quaternions and the optimization of TS robot design. In E. Bayro and G. Sobczyk, editors, *Applications of Clifford Algebras in Computer Science and Engineering.* Birkhäuser Verlag, 2000.
2. W. Blaschke. *Kinematik und Quaternionen.* VEB Verlag der Wissenschaften, Berlin, Germany, 1960.
3. M. Chasles. Note sur les propriétés générales du système de deux corps... *Bull. des Sciences Mathématiques, Astronomiques, Physiques et Chimiques*, 14:321–326, 1830.
4. X.-S. Gao, C. Hoffmann, and W. Yang. Solving spatial basic geometric constraint configurations with locus intersection. In *Proc. 7th ACM Symp. Solid Modeling and Applic.*, pages 95–104. ACM Press, 2002.
5. X. S. Gao, C. C. Zhu, and Y. Huang. Building dynamic mathematical models with Geometry Expert. In W. C. Yang, editor, *3rd Asian Techn. Conf. in Mathematics*, pages 216–224, New York, 1998. Springer.

6. J. Phillips. *Freedom in Machinery, Vol I: Introducing Screw Theory*. Cambridge University Press, Cambridge, UK, 1984.
7. J. Phillips. *Freedom in Machinery, Vol II: Screw Theory Exemplified*. Cambridge University Press, Cambridge, UK, 1990.
8. H. Pottmann and J. Wallner. *Computational Line Geometry*. Springer, Heidelberg, Germany, 2001.
9. E. Study. Die Geometrie der Dynamen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 8:204–216, 1899.
10. J. Wittenburg. *Dynamics of Systems of Rigid Bodies*. B. G. Teubner, Stuttgart, Germany, 1977.