On Spatial Constraint Solving Approaches*

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Abstract. Simultaneous spatial constraint problems can be approached algebraically, geometrically, or constructively. We examine how each approach performs, using several example problems, especially constraint problems involving lines. We also prove that there are at most 12 real tangents to four given spheres in $\mathbf{R}^3$.

1 Introduction

Spatial constraint solving involves decomposing the constraint schema into a collection of indecomposable subproblems, followed by a solution of those subproblems. Good algorithms for decomposing constraint problems have appeared recently, including [3, 6]. The best of those algorithms are completely general, adopting a generic degree-of-freedom reasoning approach that extends the older approach of searching for characteristic constraint patterns from a fixed repertoire such as [7].

In the spatial setting, even small irreducible problems give rise to nontrivial algebraic equation systems and yield a rich set of challenging problems. Restricting to points and planes, prior work has succeeded in elucidating and solving with satisfactory results the class of octahedral problems. An octahedral problem is an indecomposable constraint schema on six geometric entities, points and/or planes, with the constraint topology of an octahedron; see [1, 7, 10]. Such problems have up to 16 real solutions.

When lines are added as geometric primitives, even sequential problems become nontrivial, such as placing a single line at prescribed distances from four fixed points. In [1] line problems have been investigated and solved using several homotopy continuation techniques in conjunction with algebraic simplification. In particular, the problem 3p3L was analyzed and solved in which three lines and three points are pairwise constrained in the topology of the complete graph K_6. In this paper, we consider the problems 4p1L and 5p1L of placing four or five points and one line by spatial constraints. We also contrast them to the 6p octahedral problem. Our main purpose is to learn how successful the different approaches to solving these problems are.

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2 Three Ways to Solve Subproblems

Once a subproblem has been identified, it must be translated into a simultaneous system of nonlinear equations, usually expressed algebraically. The system is then solved. Due to application considerations, we are especially interested in solution strategies that can, in principle, identify all real solutions of such a system. Thus we exclude in particular the usual Newton iteration approach that, beginning with a particular initial configuration, numerically determines at most one solution of the system.

We are interested in three approaches to solving the algebraic equations that arise when evaluating a subproblem.

1. Simplify the equations using a systematic set of techniques that are appropriate for the problem. This is the approach taken in, e.g., [1].
2. Apply a pragmatic mixture of geometric reasoning that simplifies the equations, in conjunction with other algebraic manipulation. This approach has been taken in, e.g., [8, 7].
3. Adopt a procedural approach in which basic geometric reasoning results in a tractable, numerical procedure. This approach is familiar from, e.g., [5, 4].

In each case, the goal is to simplify the system so that it becomes tractable to evaluate all real solutions. Aside from the intrinsic repertoire of each of the three approaches, we note that the choice of a coordinate system in which to solve the system is of critical importance.

We will explore how each of these approaches performs by considering the constraint subproblem in which 5 points and one line are to be placed subject to constraints on them. In [1], it was argued that a good choice of the coordinate system seeks to place the lines in a fixed position, formulating the equations on the points and on lines that could not be placed. We have found this to be a good idea as well. However, in the sequential line placing problem, we will see that it is better to place the points.

In the following, we will consider three spatial irreducible constraint problems:

1. **The 6p Octahedral Problem:** Given six points in space and twelve prescribed distances between them in the topology of an octahedron, determine the six points relative to each other. This problem is related to the Stewart platform [8].
2. **The 4p1L Problem:** Given four known points, find a line that lies at prescribed distance from each of them. Equivalently, find the common tangents to four fixed spheres [1].
3. **The 5p1L Problem:** Given one line and five points, and thirteen constraints between them in the topology shown in Figure 2, determine their relative position.

We will see that the first problem yields to the systematic simplification approach. That it can be addressed with the second approach as well has been shown in [8], among others. An especially nice solution using the Cayley-Menger determinant was presented by Michelucci in [10].

The second problem is amenable to the algebraic approach as well, except that the coordinate system has to be chosen carefully. We will explain briefly two different choices and their consequences.

Finally, the third constraint problem has not yielded to the first two constraint solving approaches, and the only satisfactory approach we have found so far is the computational one.

3 The Spatial Constraint Problems

We explain each constraint problem we consider in turn, in increasing order of complexity.

3.1 The 6p Octahedral Problem

We are given six points and twelve distance constraints, as indicated in Figure 1. The position of the points relative to each other, or with respect to a global coordinate system, is not known. As noted in [7], this problem has several instances

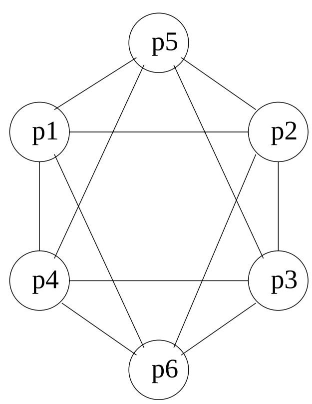

Fig. 1. The 6p Octahedral Problem: Graph vertices represent points, graph edges distance constraints

when replacing some of the points with planes and considering angle constraints between planes. In every case, the problem cannot be further decomposed and requires solving a simultaneous system of nonlinear equations. A solution is a coordinate assignment to the points that satisfies all twelve distance constraints. As we will explain, this problem yields to both the algebraic and to the reasoning approach.

3.2 4p1L – Common Tangent to Four Spheres

We are given four points in fixed location in a global coordinate system. We are asked to find a line in space that lies at prescribed distance from each of the four points. Equivalently, we are given four fixed spheres in 3-space, not necessarily of the same radius, and are asked to find a line that is tangent to each sphere.

This problem is a sequential construction problem. A line has four independent coordinates, so four conditions such as the required distances determine its position. Suppose that we have a constraint problem in which each geometric element can be placed in a global coordinate system one-by-one in some order. If we admit as possible elements points, lines and planes, then this subproblem arises naturally. Note that geometric constraint problems that can be solved by such a sequential procedure are among the simplest problems.

We will discuss an algebraic approach to solving this problem that relies on a good choice of the coordinate system. Geometric reasoning approaches appear to fail to lead to more simplification.

3.3 The Problem 5p1L

Consider a configuration of five points and one line in 3-space that is constrained as shown in Figure 2. All constraints are distances. The subgraph of the five

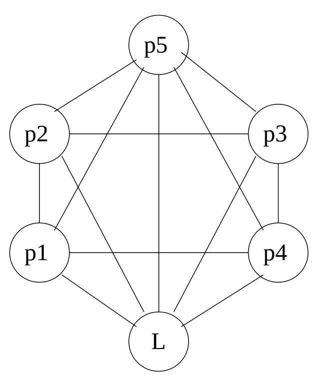

Fig. 2. The 5p1L Problem: Graph vertices represent points and a line, graph edges distances.

points has the topology of a square pyramid and is therefore not rigid. The point p_5 is the apex of the pyramid. In all, the configuration requires 19 generalized coordinates subject to 13 constraints, and is therefore generically a rigid body in 3-space.

4 Solving Strategies

4.1 Algebraic Approach

In the algebraic approach, we choose a coordinate system and formulate a set of algebraic equations based on that choice. The equations are then simplified and brought into a form that gives greater insight into the number of distinct solutions and is sufficiently simple that root finding or reliable numerical techniques can be applied to solve the system. Ideally, the approach follows a systematic generic framework for manipulating the equations.

The 6p – Octahedral Problem The octahedral problem 6p has an elegant solution discovered by Michelucci [10] that is based on the Cayley-Menger determinant. Recall that the determinant relates the squared distances between five points in space. Consider the two unknown diagonal distances $d_{13} = d(p_1, p_3)$ and $d_{24} = d(p_2, p_4)$. Choosing the five points $\{p_1, p_2, p_3, p_4, p_5\}$, a quadratic relationship between d_{13}^2 and d_{24}^2 is obtained from the determinant. A similar relationship is obtained from the set $\{p_1, p_2, p_3, p_4, p_6\}$. Thus, we obtain two quartic equations in two unknowns, a system of total degree 16.

Michelucci's solution is independent of a coordinate system choice, a strong point, but it does not follow a systematic procedure. A systematic framework was developed by Durand in [1, 2]. Choosing to place one point at the origin, one point on the x-axis, and one point in the positive quadrant of the xy-plane, the initial system consists of nine quadratic equations in nine unknowns. This system is then simplified by the following steps:

1. Gaussian elimination.
2. Solving univariate equations.
3. Parameterization of variables in bilinear and biquadratic equations.

The resulting system for 6p are three quartic equations in three variables, a system of total degree 64. By applying techniques from homotopy continuation, the final system required evaluating only 16 roots, of which, in the examples studied, 8 were real and 8 were complex.

4p1L – Tangent to Four Spheres The problem would appear to be classical, but we did not find much helpful literature on it. A systematic algebraic treatment of the problem was given by Durand in [1]. Durand found an equation system of degree 64 (the BKK bound) and experimentally determined that 40 of the 64 paths led to infinity. Thus, only 24 paths had to be explored. We improve this result now.

Placing three points at the origin, on the x-axis, and in the xy-plane, our initial equation system consists of six quadratic equations in six unknowns, (1–6). The unknowns are the point (x, y, z) nearest to the origin on the sought line, and the unit length tangent (u, v, w) of the line. Assume that r_i is the distance of point i from the line, and that the point coordinates are (a_i, b_i, c_i). Then the initial equation system is

$$
\begin{aligned}
x^2 + y^2 + z^2 - r_1^2 &= 0 \qquad (1)\\
(a_2 - x)^2 + y^2 + z^2 - (a_2 u)^2 - r_2^2 &= 0 \qquad (2)\\
(a_3 - x)^2 + (b_3 - y)^2 + z^2 - (a_3 u + b_3 v)^2 - r_3^2 &= 0 \qquad (3)\\
(a_4 - x)^2 + (b_4 - y)^2 + (c_4 - z)^2 - (a_4 u + b_4 v + c_4 w)^2 - r_4^2 &= 0 \qquad (4)\\
xu + yv + zw &= 0 \qquad (5)\\
u^2 + v^2 + w^2 - 1 &= 0 \qquad (6)
\end{aligned}
$$

We use equation (1) to eliminate the terms x^2, y^2 and z^2 from equations (2–4). Then those equations can be solved symbolically, yielding a solution that expresses the variables x, y and z as a quadratic expression in u, v and w. This eliminates x, y and z from equations (5) and (6) and factors out a subsystem of three equations in u, v, w of degree 2, 3 and 4, respectively. Thus, a degree reduction to 24 has been accomplished.

We note that for each solution (x, y, z, u, v, w) of the system $(x, y, z, -u, -v, -w)$ is also a solution.[1] Geometrically, this says that the orientation of the lines is immaterial, which one expects. Therefore, the 24 solutions of the system, counted by Bezout's theorem, reduce to 12 geometric solutions. That this is the smallest number possible follows from the result by Theobald et al. [9]. They prove there are up to 12 distinct real tangents when all radii are equal, that is, when $r_1 = r_2 = r_3 = r_4 = r$.

It would seem that one could place the unknown line on the x-axis and seek equations to place the four points as a rigid structure subject to the distance constraints. Doing so yields equations with a high degree of symmetry and structure, but we have not found an attractive simplification of those equations.

5pL1 We can choose a coordinate system in which the line L is on the x-axis and the point p_5 on the z-axis as shown in Figure 3. We denote the distance between L and the point p_i with r_i, $i = 1, \ldots, 5$. The distance between and point p_5 and p_i, $i = 1, \ldots, 4$, is denoted d_i, and the distance between points p_i and p_j with d_{ij}. This choice leads to a system consisting of 12 equations in 12 unknowns:

$$
\begin{array}{ll}
y_i^2 + z_i^2 = r_i^2 & i = 1, \ldots, 4 \\
x_i^2 + y_i^2 + (z_i - r_5)^2 = d_i^2 & i = 1, \ldots, 4 \\
(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = d_{ij}^2 & ij = 12, 23, 34, 41
\end{array}
\qquad (7)
$$

Naive counting of the number of possible solutions would yield 4096. Using the multi-homogeneous Bezout theorem of [11], a tighter bound of 512 finite solutions is obtained. That bound does not make it practical to explore all solutions.

[1] This is clearly true for the original system. Moreover, the expressions substituted for x, y and z also exhibit the sign symmetry; hence the claim is true for the resulting system of degree 24.

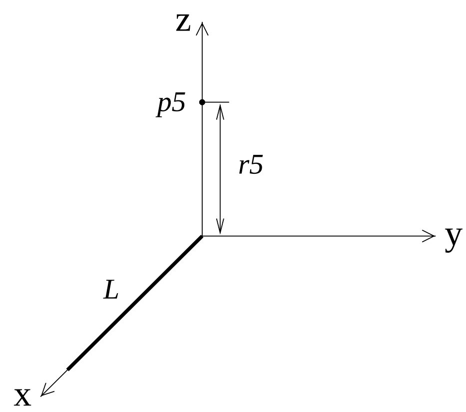

Fig. 3. Coordinate Assignment for the 5p1L Problem

Moreover, the system of equations resisted meaningful simplification, both ad-hoc manipulations as well as the systematic simplification steps developed before.

We could choose to place the coordinate system such that three points are put into a special position, say one point at the origin, one on the x-axis, and one in the xy-plane, but doing so did not lead to better equations.

4.2 Geometric Reasoning to Assist Simplification

In this approach we try to introduce auxiliary geometric structures, such as the curves described by a particular point when restricting to a subset of the constraints, especially if this can lead to a reasonable parameterization. Often, one can then introduce the additional constraints and derive a simpler equation system.

6p – Octahedron Geometric reasoning was used in [7] to yield a system of equations that, in conjunction with resultant techniques, succeeded in deriving a univariate polynomial of degree 16. It improves on the systematic approach by a factor of 4 and matches the Cayley-Menger solution.

4p1L – Sphere Tangents Presently, we do not have a good solution that exploits the geometry of the configuration. We believe that it should be possible to find one of total degree to 24 or less.

5p1L Placing the coordinate system as before, with the line on the x-axis and the point p_5 on the z-axis, we could proceed by parameterizing the locus of the point p_1 as function of the z-coordinate Z. From the distance constraints r_1 and

d_1 we obtain for the point p_1:

$$p_1 = \begin{cases} x_1(t) = \pm\sqrt{d_1^2 - r_1^2 - r_5^2 + 2r_5 t} \\ y_1(t) = \pm\sqrt{r_1^2 - t^2} \\ z_1(t) = t \end{cases} \tag{8}$$

We can then construct the remaining points whose coordinates are now a function of the parameter t, using the distance constraints for r_2, d_2, and d_{12} for p_2, the distance constraints r_4, d_4 and d_{41} for p_4. Finally, point p_3 is constructed using r_3, d_3 and d_{23}. This leaves the distance constraint d_{34} to be used to determine the parameter t. The equations so derived have the following form:

$$\left\{ \begin{array}{r} -4d_2^2x_1(t)x_2 + 8x_1(t)y_1(t)x_2y_2 - 8r_5x_1(t)z_1(t)x_2 \\ -8r_5y_1(t)z_1(t)y_2 + 4x_1(t)^2x_2^2 + 4y_1(t)^2y_2^2 + 8r_5^2x_1(t)x_2 \\ -4d_1^2x_1(t)x_2 + 4d_{12}^2x_1(t)x_2 - 4d_2^2y_1(t)y_2 + 8r_5^2y_1(t)y_2 - 4d_1^2y_1(t)y_2 \\ +4d_{12}^2y_1(t)y_2 + 4d_2^2r_5z_1(t) - 4d_{12}^2r_5z_1(t) - 8r_5z_1(t)y_2^2 + 4r_5^2z_1(t)^2 \\ -8r_5^3z_1(t) + 4z_1(t)^2y_2^2 + 4r_5^2y_2^2 - 4r_2^2z_1(t)^2 + 8r_5r_2^2z_1(t) = D_1 \\ -z_1(t)x_2^2 + r_5x_2^2 - 2r_5x_1(t)x_2 - 2r_5y_1(t)y_2 + 2r_5^2z_1(t) \\ +d_2^2z_1(t) - r_5^2z_1(t) - r_2^2z_1(t) = D_2 \\ -4d_4^2x_1(t)x_4 + 8x_1(t)y_1(t)x_4y_4 - 8r_5x_1(t)z_1(t)x_4 - 8r_5y_1(t)z_1(t)y_4 \\ +4x_1(t)^2x_4^2 + 4y_1(t)^2y_4^2 + 8r_5^2x_1(t)x_4 - 4d_1^2x_1(t)x_4 + 4d_{41}^2x_1(t)x_4 \\ -4d_4^2y_1(t)y_4 + 8r_5^2y_1(t)y_4 - 4d_1^2y_1(t)y_4 + 4d_{41}^2y_1(t)y_4 + 4d_4^2r_5z_1(t) \\ -4d_{41}^2r_5z_1(t) - 8r_5z_1(t)y_4^2 + 4r_5^2z_1(t)^2 - 8r_5^3z_1(t) \\ +4z_1(t)^2y_4^2 + 4r_5^2y_4^2 - 4r_4^2z_1(t)^2 + 8r_5r_4^2z_1(t) = D_3 \\ -z_1(t)x_4^2 + r_5x_4^2 - 2r_5x_1(t)x_4 - 2r_5y_1(t)y_4 + 2r_5^2z_1(t) + d_4^2z_1(t) \\ -r_5^2z_1(t) - r_4^2z_1(t) = D_4 \\ -12z_1(t)x_2x_3 + 12r_5x_2x_3 - 4r_5x_1(t)x_2 - 4r_5y_1(t)y_2 + 4r_5^2z_1(t) \\ +4r_5z_1(t)z_3 - 4r_5^2z_3 + 2d_2^2z_1(t) + 2d_3^2z_1(t) \\ +2d_{23}^2z_1(t) - 4r_5^2z_1(t) = D_5 \\ 4z_1(t)x_3x_4 - 4r_5x_3x_4 + 4z_1(t)y_3y_4 - 4r_5y_3y_4 - 4x_1(t)x_4z_3 \\ -4y_1(t)y_4z_3 + 2d_4^2z_3 + 2d_1^2z_3 - 2d_{41}^2z_3 + 4r_5x_1(t)x_4 + 4r_5y_1(t)y_4 \\ -4r_5z_1(t) - 2d_3^2z_1(t) - 2d_4^2z_1(t) + 2d_{34}^2z_1(t) + 4r_5^2z_1(t) = D_6 \\ y_3^2 + z_3^2 = D_7 \\ x_3^2 - 2r_5z_3 = D_8 \end{array} \right. \tag{9}$$

where $D_1, D_2, D_3, D_4, D_5, D_6$ are constants. The system is unattractive.

4.3 Construction by Computation

The closed-form algebraic expressions for the point coordinates of the 5p1L problem that were obtained by the geometric reasoning described before, do not seem to be simple enough to lead to further massive algebraic simplification. However, they are very easy to evaluate computationally, and can be used to define numerically a curve in a 2D coordinate space defined by the parameter and the distance d_{34}. When the curve is intersected with the nominal distance line, the real solutions are obtained. As illustrated in Figure 4, p_{10} is on line L and $\overline{p_{10}p_1} \perp L$, the angle between $\overline{p_{10}p_1}$ and the xy-plane is θ. We use θ as parameter to calculate point p_1:

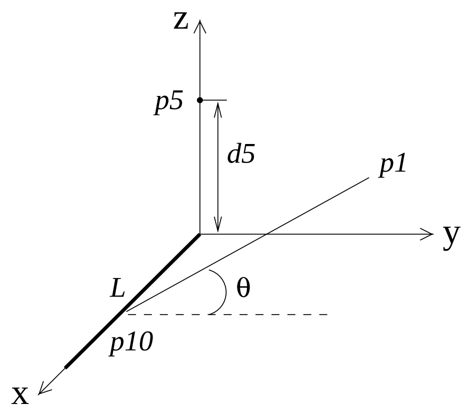

Fig. 4. Parameterization with θ

$$p_1 = \begin{cases} x_1(\theta) = \pm\sqrt{d_1^2 - r_1^2 - r_5^2 + 2r_1 r_5 \sin(\theta)} \\ y_1(\theta) = r_1 \cos(\theta) \\ z_1(\theta) = r_1 \sin(\theta) \end{cases} \tag{10}$$

For practical purposes, the approach is satisfactory, since it gives a systematic, and sufficiently simple, procedure to find all real solutions. Moreover, the solutions so found can be further refined with other numerical processes, since they provide good starting points. From a theoretical perspective, the draw-back of the procedural approach is its inability to produce, with certainty, a bound on the number of solutions. Here are the details for our 5p1L problem, and several example solutions.

p_2 can be solved using the constraints $dist(L, p_2) = r_2$, $dist(p_5, p_2) = d_2$ and $dist(p_1, p_2) = d_{12}$. As illustrated in Figure 5, the point $p_s = (x_s, y_s, z_s)$ in

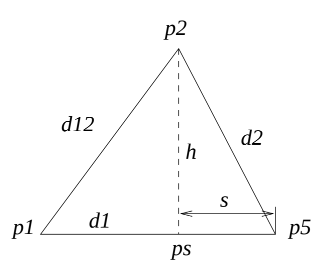

Fig. 5. Triangle $p_1p_2p_5$

triangle $\triangle p_1p_2p_5$ is on the line $\overline{p_1p_5}$ and $\overline{p_2p_s} \perp \overline{p_1p_5}$. So, we have

$$s = |p_sp_5| = \frac{(d_1^2 + d_2^2 - d_{12}^2)}{2d_1}$$

$$h = |p_sp_2| = \sqrt{d_2^2 - s^2}$$

We obtain

$$p_s = p_5 + \frac{s}{d_1}(p_1 - p_5)$$

Consider the vector $\boldsymbol{w} = \frac{p_1-p_5}{|p_1-p_5|} = \frac{p_1-p_5}{d_1}$. We define a plane Π through the point p_s perpendicular to $\boldsymbol{w}$. Since $dist(L, p_2) = r_2$, the point p_2 is on the cylinder $\Sigma : y^2 + z^2 = r_2^2$ whose axis is the line L and whose radius is r_2. Let p_c be the interaction point of line L and plane Π, then $p_c = (x_c, y_c, z_c)$ where

$$\begin{aligned} x_c &= x_s + \tfrac{w_y}{w_x} y_s + \tfrac{w_z}{w_x} z_s \\ y_c &= 0 \\ z_c &= 0 \end{aligned}$$

Using the vectors

$$\begin{aligned} \boldsymbol{v} &= \tfrac{\boldsymbol{w}\times\boldsymbol{L}}{|\boldsymbol{w}\times\boldsymbol{L}|} \\ \boldsymbol{u} &= \boldsymbol{v} \times \boldsymbol{w} \end{aligned}$$

we set up a local coordinate system: $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$, where

$$\begin{aligned} o' &= p_c \\ \boldsymbol{x}' &= \boldsymbol{u} \\ \boldsymbol{y}' &= \boldsymbol{v} \\ \boldsymbol{z}' &= \boldsymbol{w} \end{aligned}$$

The matrix transform from the global coordinate system $(o, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to the local system $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ is

$$M = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Figure 6 illustrates the local coordinate system $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ situated in the global system $(o, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$.

From the construction process we know that the point p_2 lies on a circle in the plane Π with radius h. Let $p_s' = Mp_s$, in the local coordinate system $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$. Then the equation of the circle is

$$(x' - x_s')^2 + (y' - y_s')^2 = h^2 \tag{12}$$

Now the vector along line L is $\boldsymbol{L} : (1, 0, 0)$, the angle between $\boldsymbol{L}$ and $\boldsymbol{w}$ is β, and the intersection of plane Π and cylinder Σ is an ellipse on plane Π. In the local system $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$, the ellipse equation is

$$\frac{x'^2}{r_2^2 \csc^2(\beta)} + \frac{y'^2}{r_2^2} = 1 \tag{13}$$

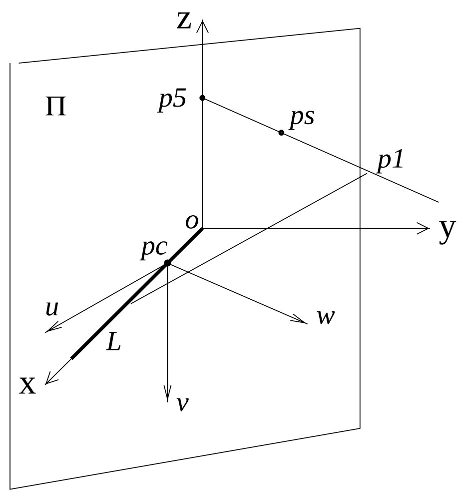

Fig. 6. Local Coordinate System $(o', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$

Solving equations (12) and (13) simultaneously, we get p_2' and from it, in turn, p_2:

$$p_2 = (x_2(\theta), y_2(\theta), z_2(\theta))$$

Note that there are up to 4 real solutions for p_2.

Similarly, we compute p_4 from its constraints with the line L and the points p_1 and p_5. Finally, we compute p_3 from the constraints with line L and points p_2 and p_5

$$p_3 = (x_3(\theta), y_3(\theta), z_3(\theta))$$
$$p_4 = (x_4(\theta), y_4(\theta), z_4(\theta))$$

$d_{34}(\theta)$ is a complicated curve in a coordinate space defined by the parameter θ and the distance $dist(p_3(\theta), p_4(\theta))$. The curve would be hard to express symbolically. However, we can trace it numerically.

Given a step length $d\theta$, we calculate $d_{34}(\theta)$ for every step $\theta = \theta + d\theta$, and so obtain the curve $C_\theta : d_{34}(\theta) - \theta$ numerically. Let the absolute error of $d_{34}(\theta)$ and the nominal distance line d_{34} be

$$\rho(\theta) = |d_{34}(\theta) - d_{34}|$$

Obviously, the smaller $\rho(\theta)$ is, the nearer θ is to a real solution of the 5pL1 problem. Call a point $(\theta, d_{34}(\theta))$ a *coarse solution* if θ satisfies

$$d_{34}(\theta) < \delta$$

for a chosen tolerance δ. The coarse solution set S_δ is then

$$S_\delta = \{q_\theta = (\theta, d_{34}(\theta)) | \rho(\theta) < \delta, q_\theta \in C_\theta\}$$

δ is the threshold of the coarse solutions, and the size of $|S_\delta|$ diminishes with δ. The coarse solutions can be further refined with Newton-Raphson iteration since they provide good starting points.

r_1	5.12863551744133
r_2	3.4797204504532
r_3	5.12009033478805
r_4	4.48866237372967
r_5	0.854823450422681
d_1	5.40391247291482
d_2	4.92751853999451
d_3	6.556901760918
d_4	5.04776146732994
d_{12}	2.49916074098941
d_{23}	9.55687124240852
d_{34}	9.15
d_{41}	7.1858882412183

Table 1. An Constraint Set of the 5p1L Problem

4.4 An Example

Table 1 gives an example of constraint set of 5pL1 problem, by defining $d\theta = 1.0^o$, Figure 7 gives the discrete curve. In our example, if $\delta = 0.1$, $|S_c| = 108$, if $\delta = 0.2$, $|S_c| = 224$. When $\delta = 0.1$ we can get 20 refined real solutions; when $\delta = 0.2$ we can get 24 refined real solutions; when $\delta > 0.2$ we have more than 224 coarse solutions but the refined real solution number is still 24. Therefore, the maximum real solution number of the example is 24. The circles on the nominal distance line in Figure 7 represent the real solutions, Table 2 gives all the 24 real solutions of this example.

The computation was carried out using a tolerance-driven search for potential solutions followed by a Newton iteration refining the initial values. On a PC with a 500MHz Pentium 3 the initial search took 100 milliseconds with a tolerance of 0.2, and the subsequent refinement took an additional 233 ms. This contrasts favorably with the computation times obtained by Durand on a Sun SPARC 20 using homotopy continuation where 24 paths were evaluated in approximately 30 sec. The homotopy evaluation on the slower machine was a completely general implementation, while our computation of the solution was specifically designed for this particular problem. It would be interesting to test this problem on general multi-variate interval Newton solvers.

5 Further Discussion

The Construction by Computation approach can be used more generally. Let $F(X) = 0$ be a system of n nonlinear equations $F = \{f_1, \ldots, f_n\}$ with n unknowns $X = \{x_1, \ldots, x_n\}$. To find all real solutions of $F(X) = 0$, we can choose a real parameter set $T = \{t_1, \ldots, t_k\}_{k<n}$ such that X can be solved as

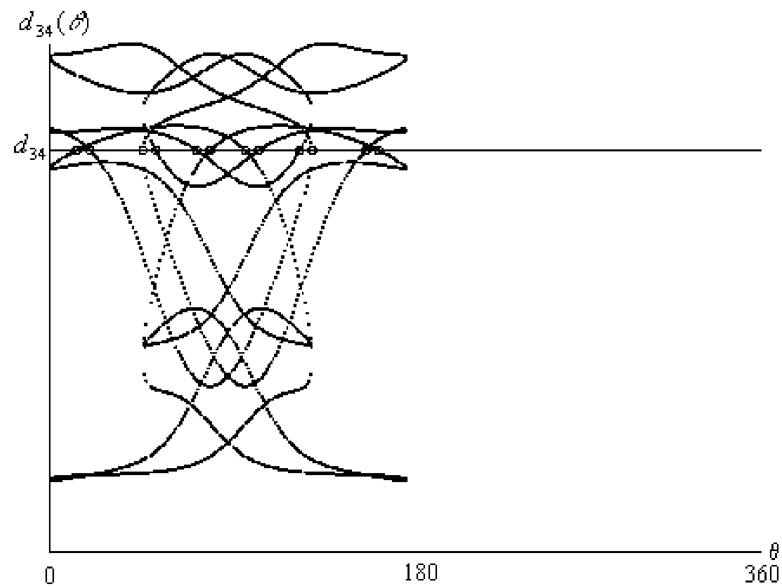

Fig. 7. $d_{34}(\theta) - \theta$ Curve

$X(T) = \{x_1(T), \ldots, x_n(T)\}$ by using $n-k$ equations

$$F_{n-k} = \{f^i | f^i \in F, 1 \leq i \leq n-k\} \subset F$$

Let

$$F_k = F - F_{n-k} = \{f^j | f^j \notin F_{n-k}, f^j \in F, 1 \leq j \leq k\} \subset F$$

and define

$$\rho(T) = \max_{\forall f^j \in F_k} (|f^j|)$$

Let domain of T be $D_T = [t_{1min}, t_{1max}] \times \cdots \times [t_{kmin}, t_{kmax}]$, and for every $t_i \in T$ define a step size dt_i such that we can calculate $\rho(T)$ on D_T numerically for every $[t_1 = t_1 + dt_1] \times \cdots \times [t_k = t_k + dt_k]$. Obviously, $T \times \rho(T) \subset \Re^{k+1}$ is a hypersurface. Given a small positive real number δ, we can get the Coarse Solution Set

$$S_c = \{q_T = (T, \rho(T)) | \rho(T) < \delta, q_T \in T \times \rho(T)\}$$

For every $q_T \in S_c$ we can get an starting point X^0. Using Newton-Raphson iteration, we may refine the starting point to a real solution of $F(X) = 0$. After calculating all $q_T \in S_c$ we can get the real solution set S_r. If the step sizes $dt_i, i = 1, \ldots, k$, are small enough and δ is large enough, we can find all real solutions of $F(X) = 0$.

	p_1	p_2	p_3	p_4
1	(2.06,4.98,1.21)	(3.30,3.46,-0.34)	(-4.87,2.43,4.51)	(3.45, -1.36, 4.28)
2	(-2.06,4.98,1.21)	(-3.30,3.46,-0.34)	(4.87, 2.43, 4.51)	(-3.45, -1.36,4.28)
3	(2.28,4.81,1.77)	(3.85,2.88,1.96)	(-2.79,1.73,-4.82)	(-3.46,1.30,4.30)
4	(-2.28, 4.81,1.77)	(-3.85,2.88,1.96)	(2.79,1.73,-4.82)	(3.46,1.30,4.30)
5	(2.93,3.48,3.77)	(3.72,3.18,1.42)	(-4.95,1.29,4.95)	(-0.37,3.65,-2.61)
6	(-2.93,3.48,3.77)	(-3.72,3.18,1.42)	(4.95,1.29,4.95)	(0.37,3.65,-2.61)
7	(3.04,3.04,4.13)	(4.14,0.94,3.35)	(-4.93,-1.65,4.85)	(-1.58,4.32,-1.23)
8	(-3.04,3.04,4.13)	(-4.14,0.94,3.35)	(4.93,-1.65,4.85)	(1.58,4.32,-1.23)
9	(3.26,1.39,4.94)	(4.01,2.20,2.70)	(-3.22,3.90,-3.32)	(-3.47,-1.07,4.36)
10	(-3.26,1.39,4.94)	(-4.01,2.20,2.70)	(3.22,3.90,-3.32)	(3.47,-1.07,4.36)
11	(3.29,0.79,5.07)	(4.15,-0.84,3.38)	(-4.92,1.82,4.79)	(1.72,4.39,-0.95)
12	(-3.29,0.79,5.07)	(-4.15,-0.84,3.38)	(4.92,1.82,4.79)	(-1.72,4.39,-0.95)
13	(3.29,-0.79,5.07)	(4.15,0.84,3.38)	(4.92,-1.82,4.79)	(1.72,-4.39,-0.95)
14	(-3.29,-0.79,5.07)	(-4.15,0.84,3.38)	(4.92,-1.82,4.79)	(-1.72,-4.39,-0.95)
15	(3.26,-1.39,4.94)	(4.01,-2.20,2.70)	(-3.22,-3.90,-3.32)	(-3.47,1.07,4.36)
16	(-3.26,-1.39,4.94)	(-4.01,-2.20,2.70)	(3.22,-3.90,-3.32)	(3.47,1.07,4.36)
17	(3.04,-3.04,4.13)	(4.14, -0.94,3.35)	(-4.93,1.65,4.85)	(-1.58,-4.32,-1.23)
18	(-3.04, -3.04,4.13)	(-4.14,-0.94,3.35)	(4.93,1.65,4.85)	(1.58,-4.32,-1.23)
19	(2.93,-3.48,3.77)	(3.72,-3.18,1.42)	(-4.95,-1.29,4.95)	(-0.37,-3.65,-2.61)
20	(-2.93,-3.48,3.77)	(-3.72,-3.18,1.42)	(4.95,-1.29,4.95)	(0.37,-3.65,-2.61)
21	(2.06,-4.98,1.21)	(3.30,-3.46,-0.34)	(-4.87,-2.43,4.51)	(3.45,1.36,4.28)
22	(-2.06,-4.98,1.21)	(-3.30,-3.46,-0.34)	(4.87,-2.43,4.51)	(-3.45,1.36,4.28)
23	(2.28,-4.81,1.77)	(3.85,-2.88,1.96)	(-2.79,-1.73,-4.82)	(-3.46,-1.30,4.30)
24	(-2.28,-4.81,1.77)	(-3.85,-2.88,1.96)	(2.79,-1.73,-4.82)	(3.46,-1.30,4.30)

Table 2. Real Solution Set of the Example

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