Fluids for Computer Graphics

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Overview

- Some terms…
- Incompressible Navier-Stokes Equations
- Boundary conditions
- Lagrange vs. Euler
- Eulerian approaches
- Lagrangian approaches
- Shallow water
- Conclusions

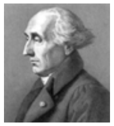
Some terms

- *Advect:*
  evolve some quantity forward in time using a velocity field. For example particles, mass, etc.

- *Convect:*
  transfer of heat by circulation of movement of fluid.

Some terms

- *Lagrangian:*
  methods that move fluid mass
  (for example by advecting particles)



- *Eulerian:*
  fluid quantities are defined on a grid that is fixed
  (the quantities can vary over time)

Equations

Fluids are governed by the incompressible Navier-Stokes Equations

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla \mathrm{p} = \vec{g} + \upsilon \nabla \cdot \nabla \vec{u} \qquad (1)$$

$$\nabla \cdot \vec{u} = 0 \qquad (2)$$

Equations

- $\vec{u} = (u, v, w)$ *velocity* of the fluid $[m/s]$
- ρ (rho) fluid *density* $[kg/m^3]$
  water ~1000
  air ~1.3
- $p = \frac{\vec{F}}{A}$ *pressure* [Pa]
  force per unit area that the fluid exerts

Equations

- $\vec{g} = (0, -9.81, 0)$ *accel. due gravity* $[m/s^{-2}]$
  (assuming:
  $z -$ points to you, $y -$ is up, $x -$ is right)
- υ (upsilon) *kinematics viscosity*
  how difficult it is to stir

Equations

The momentum equation ($\vec{F} = m\vec{a}$)

How the fluid accelerates due to the forces acting on it

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{acceleration}} + \underbrace{\vec{u} \cdot \nabla \vec{u}}_{\text{convection (internal movement)}} + \underbrace{\frac{1}{\rho} \nabla \mathrm{p}}_{\text{pressure}} = \overbrace{\vec{g}}^{\text{gravity and other external forces}} + \underbrace{\upsilon \nabla \cdot \nabla \vec{u}}_{\text{viscosity}}$$

drag

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Equations

The momentum equation ($\vec{F} = m\vec{a}$)

gravity

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} = -\frac{\partial p}{\partial y} + g_y + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial wu}{\partial x} + \frac{\partial wv}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

acceleration convection pressure drag

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Equations

- *Balance of momentum*.
  Internal + external forces = change in momentum.
- *Conservation of energy*.
  Kinetic + internal energy = const.

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Equations

Conservation of mass

Advecting mass through the velocity field cannot change total mass.

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

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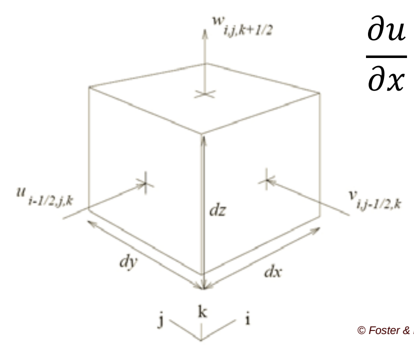
Equations

Conservation of mass



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

© Foster & Metaxas, 1996

Boundary Conditions

- Three types
  - Solid walls
  - Free surface
  - Other fluids

Boundary Conditions

- *Solid boundaries*
  - Normal component of fluid velocity = 0

$$\vec{u} \cdot \hat{n} = 0$$

- *Ideal fluids (slip boundary)*
  - The tangential component is unchanged.
- *Viscous fluids (no-slip boundary)*
  - The tangential component is set to zero.

Boundary Conditions

- *Free surface*
  Interface between the fluid
  and "nothing" (air)
  - Volume-of-fluid tracking (for Eulerian)
  - Mesh tracking (tracks evolving mesh)
  - Particle fluids (Lagrangian)
  - Level sets:
    advects a signed distance function

Solutions

- Lagrangian:
  - The world is a particle system
  - Each particle has an ID and properties (position, velocity, acceleration, etc.)
- Eulerian:
  - The point in space is fixes
  - Measure stuff as it flows past
- Analogy - temperature:
  - Lagrangian: a balloon floats with the wind
  - Eulerian: on the ground wind blows past

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Incompressibility

- Real fluids are compressible that is why we hear under water
- Not important for animation and expensive to calculate

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Eulerian Approach

Nick Foster and Dimitri Metaxas (1996) Realistic animation of liquids. *Graph. Models Image Process.* 58, 5 (September 1996), 471-483.

- Space discretization
  - Uses fixed 3D regular grid (voxels) (two of them)
  - Each cell has pressure p in the middle
  - Each cell has its state:
    - FULL of fluid
    - SOLID material
    - EMPTY air
    - SURFACE boundary cell

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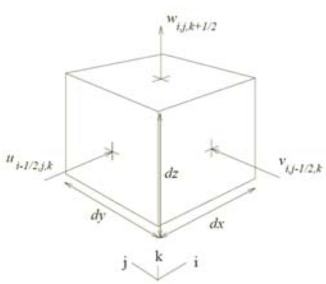
Eulerian Approach

- Each wall stores the velocity vector



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Eulerian Approach

- Discretization of the NS momentum equation gives new velocity $\tilde{u}$:

$$
\begin{aligned}
\tilde{u}_{i+1/2,j,k} = u_{i+1/2,j,k} + \delta t\{&(1/\delta x)[(u_{i,j,k})^2 - (u_{i+1,j,k})^2] \\
&+(1/\delta y)[(uv)_{i+1/2,j-1/2,k} - (uv)_{i+1/2,j+1/2,k}] \\
&+(1/\delta z)[(uw)_{i+1/2,j,k-1/2} - (uw)_{i+1/2,j,k+1/2}] + g_x \\
&+(1/\delta x)(p_{i,j,k} - p_{i+1,j,k}) + (\nu/\delta x^2)(u_{i+3/2,j,k} \\
&-2u_{i+1/2,j,k} + u_{i-1/2,j,k}) + (\nu/\delta y^2)(u_{i+1/2,j+1,k} \\
&-2u_{i+1/2,j,k} + u_{i+1/2,j-1,k}) + (\nu/\delta z^2)(u_{i+1/2,j,k+1} \\
&-2u_{i+1/2,j,k} + u_{i+1/2,j,k-1})\},
\end{aligned}
$$

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Eulerian Approach

- Time step and the
- Convergence condition

$$1 > max[u\frac{\delta t}{\delta x}, v\frac{\delta t}{\delta y}, w\frac{\delta t}{\delta z}]$$

- $\partial x, \partial y, \partial z$ are given, so we can only decrease the time step
- This causes the simulation to slow down over time.

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Eulerian Approach

- The divergence fluid ("missing mass")

$$\begin{aligned} D_{i,j,k} &= -((1/\delta x)(u_{i+1/2,j,k} - u_{i-1/2,j,k}) \\ &+ (1/\delta y)(v_{i,j+1/2,k} - v_{i,j-1/2,k}) \\ &+ (1/\delta z)(w_{i,j,k+1/2} - w_{i,j,k-1/2})). \end{aligned}$$

- Is a finite difference approx. of $\nabla \cdot \vec{u} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

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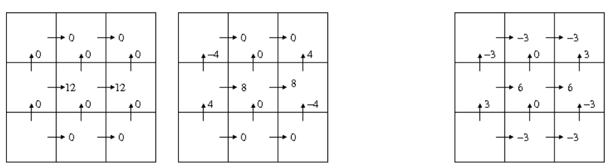
Eulerian Approach

$\vec{u}$



$\nabla \cdot \vec{u}$

init

0	0	0
−12	0	+12
0	0	0

iteration 1

−4	0	4
0	0	0
−4	0	4

iteration n

0	0	0
0	0	0
0	0	0

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Eulerian Approach

- Change of the mass causes change of the pressure

$$\partial p = \beta D$$

$$\beta = \beta_0 / 2\partial t(\frac{1}{\partial x^2} + \frac{1}{\partial y^2} + \frac{1}{\partial y^2})$$

$$\beta_0 \in [1,2]$$

Eulerian Approach

- The face vortices are then updated

$$\begin{aligned}
u_{i+1/2,j,k} &= u_{i+1/2,j,k} + (\delta t/\delta x)\delta p, \\
u_{i-1/2,j,k} &= u_{i-1/2,j,k} - (\delta t/\delta x)\delta p, \\
v_{i,j+1/2,k} &= v_{i,j+1/2,k} + (\delta t/\delta y)\delta p, \\
v_{i,j-1/2,k} &= v_{i,j-1/2,k} - (\delta t/\delta y)\delta p, \\
w_{i,j,k+1/2} &= w_{i,j,k+1/2} + (\delta t/\delta z)\delta p, \\
w_{i,j,k-1/2} &= w_{i,j,k-1/2} - (\delta t/\delta z)\delta p,
\end{aligned}$$

Eulerian Approach

- Cell pressure is updated

$$\tilde{p} = p + \partial p$$

Eulerian Approach

Putting this all together:

1. Scene definition (material, sources, sinks)
2. Set initial pressure and velocity
3. In a loop
   I. Compute $\tilde{u}, \tilde{v}, \tilde{w}$ for all Full cells.
   II. Pressure iteration for all Full cells.

Eulerian Approach

- Where is the free level?
  - Use marching cubes
  - Use Marker Particles –
    particles that are advected with the velocity
    Markers and Cells (MAC)

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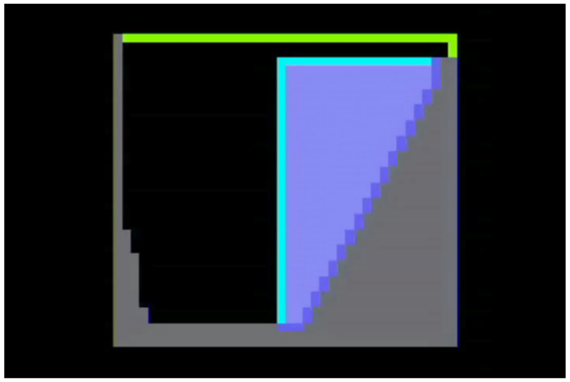
Eulerian Approach



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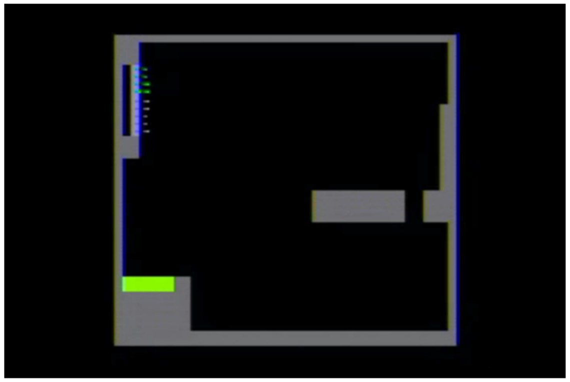
Eulerian Approach



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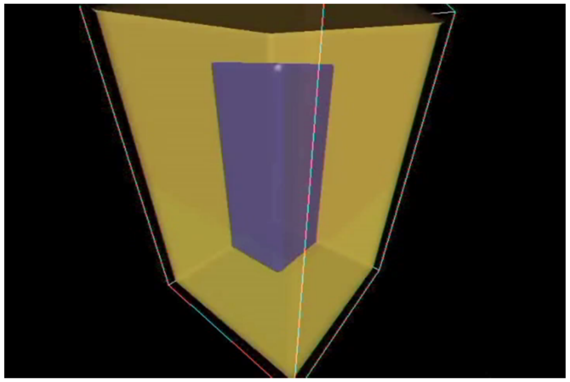
Eulerian Approach



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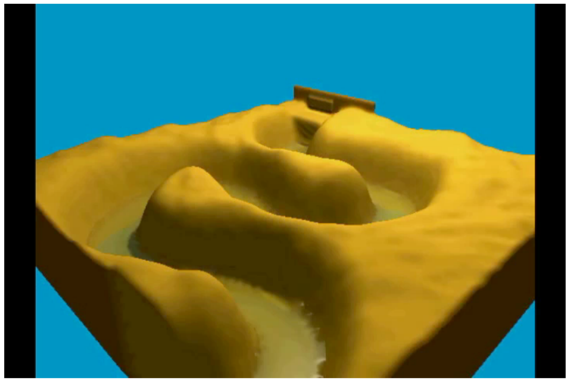
Eulerian Approach

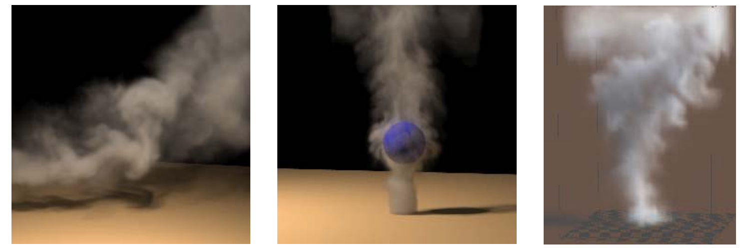


Stability

- Slowing down because of stability
- Some iterations for divergence needed
- Addressed by
  Jos Stam. 1999. Stable fluids.
  In *Proceedings of the 26th annual conference on Computer graphics and interactive techniques* (SIGGRAPH '99).
- Demo

Follow up works

- Ronald Fedkiw, Jos Stam, and Henrik Wann Jensen. 2001. Visual simulation of smoke. In *Proceedings of the 28th annual conference on Computer graphics and interactive techniques*(SIGGRAPH '01).



Follow up works

- Douglas Enright, Stephen Marschner, and Ronald Fedkiw. 2002. Animation and rendering of complex water surfaces. *ACM Trans. Graph.* 21, 3 (July 2002), 736-744.

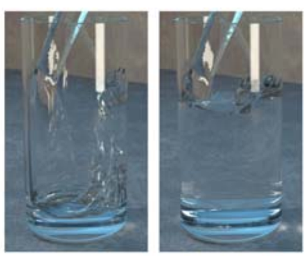
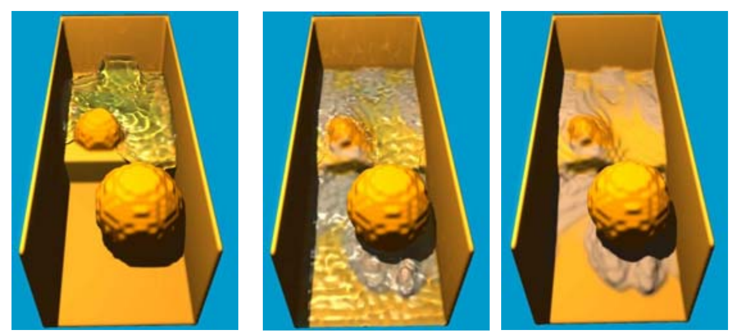


Figure 1: **Water being poured into a glass (55x120x55 grid cells).**

Follow up works

- Erosion simulation



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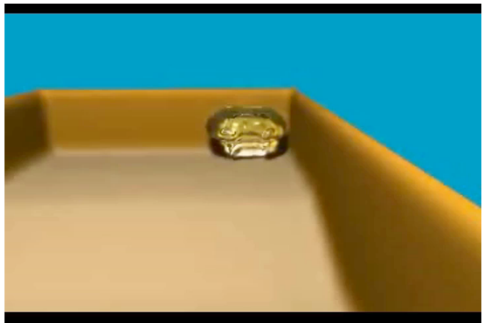
Follow up works

- Erosion



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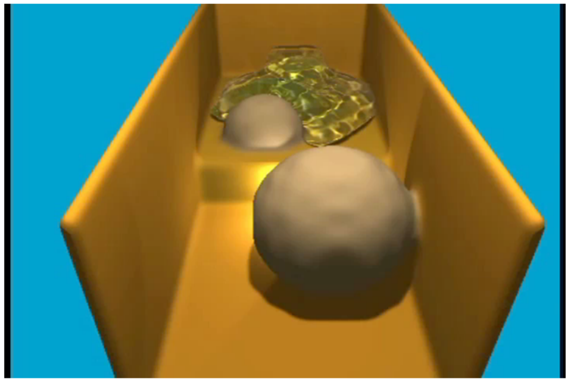
More…



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Lagrangian approaches

- *Lagrangian:*
  methods that move fluid mass
  (for example by advecting particles)

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Smoothed Particle Hydrodynamics (SPH)

- R.A. Gingold and J.J. Monaghan, (1977) Smoothed particle hydrodynamics: theory and application to non-spherical stars, Mon. Not. R. Astron. Soc., Vol 181, pp. 375–89.
- M. Desbrun, and M-P. Cani. (1996). Smoothed Particles: a new paradigm for animating highly deformable bodies. In Proceedings of Eurographics Workshop on Computer Animation and Simulation

SPH

- Fluid is divided into discrete particles
- Particles advect mass
- The values of different properties in the space where no particles are present are calculated by using *smoothing functions*

$$\phi = \sum_j m_j \frac{\phi_j}{\rho_j} W(\boldsymbol{x} - \boldsymbol{x}_j)$$

SPH

$$\phi = \sum_j m_j \frac{\phi_j}{\rho_j} W(\boldsymbol{x} - \boldsymbol{x}_j, h)$$

ϕ – the physical value at location $\boldsymbol{x}$

m_j- mass of the j –th particle

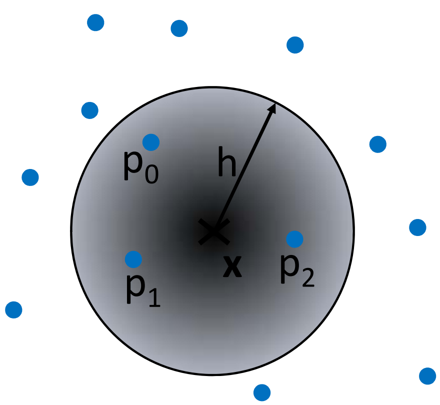
ϕ_j – the physical value of the j –th particle

$\boldsymbol{x}_j$ – location of the j –th particle

W – the smoothing kernel

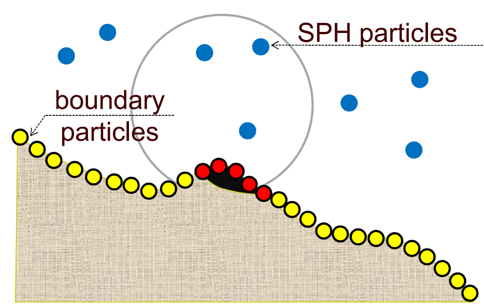
h – radius of influence

SPH

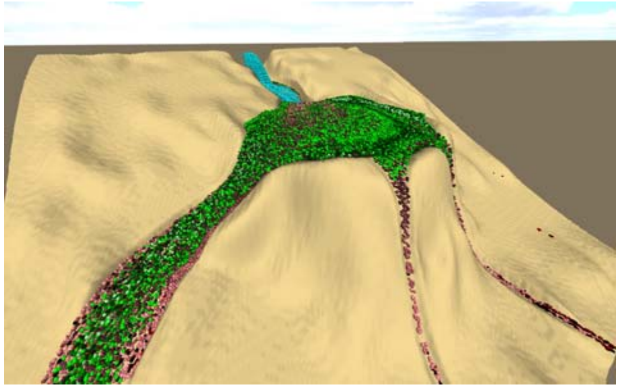


Boundary

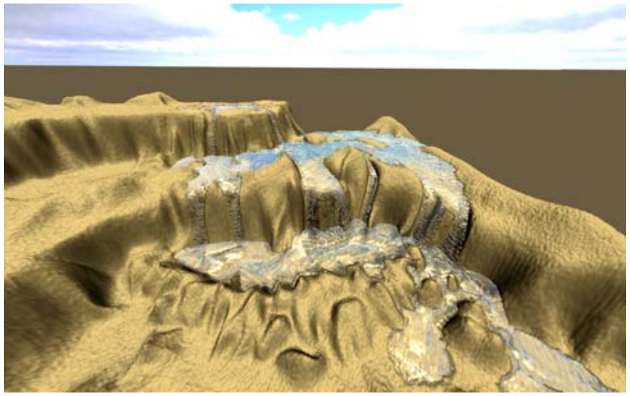
- Can be represented as a triangular mesh
- or as boundary particles (slip, no-slip)



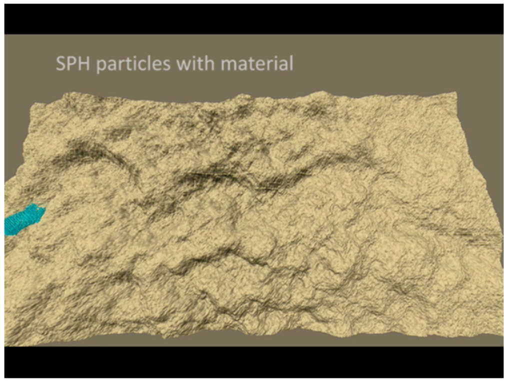
SPH



SPH



SPH



SPH

- Varying viscosity

http://www.youtube.com/watch?feature=player_embedded&v=6bdIHFTfTdU

http://www.youtube.com/watch?feature=player_embedded&v=Kt4oKhXngBQ

http://www.youtube.com/watch?feature=player_detailpage&v=UYIPg8TEMmU

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SPH

- Pros
  - No problems with losing material
  - Implicitly saves data (particles are where the fluid is)
  - Splashes are easy…
- Cons
  - Large number of particles is necessary
  - Not as good for GPU as Eulerian (but still pretty good)

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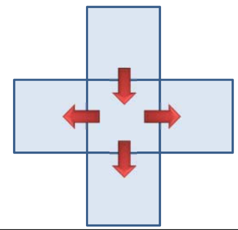
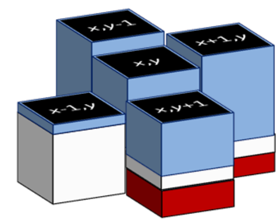
Shallow Water Simulation

- The fluid is a simple 2-D grid with layers
- Neighbor cells are connected by pipes
- Cannot simulate splashes and overhangs
- Good enough for near-still water with boats





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Shallow Water Simulation

- Algorithm:
  1) Get acceleration from unequal levels
  2) Calculate flow between cells
  3) Change levels of water
  4) Go to 1)

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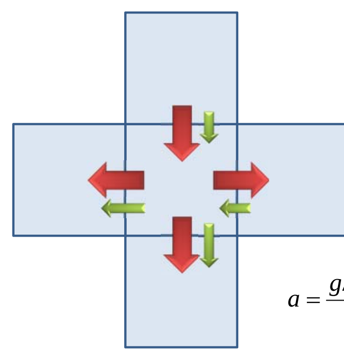
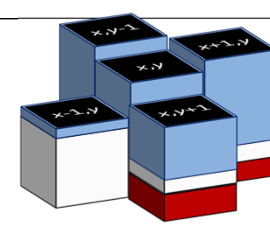
Shallow Water Simulation

- Acceleration

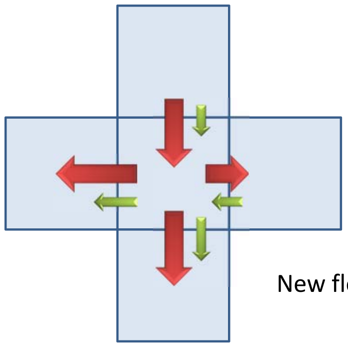


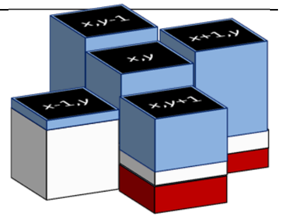


$$a = \frac{g\Delta h}{l}$$ ← Distance between cells

Shallow Water Simulation

- New flow



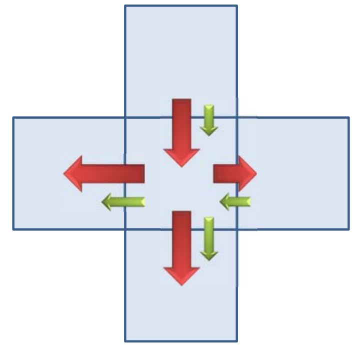


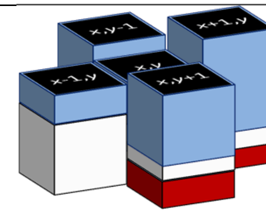
$$f^{t+\Delta t} = f^{t} + \Delta t C a$$

New flow Old flow Cross-sectional area of the pipe

Shallow Water Simulation

- Update levels





Shallow Water Simulation

Shallow Water Simulation

Conclusions

- Fluid simulation is a complex topic
- Fluid simulation for CG uses simplifications that are aimed at
  - Speed
  - Visual quality
- Still an open problem
- lot of work to do…

Reading

- Robert Bridson
  Fluid Simulation for
  Computer Graphics

- Siggraph proceedings