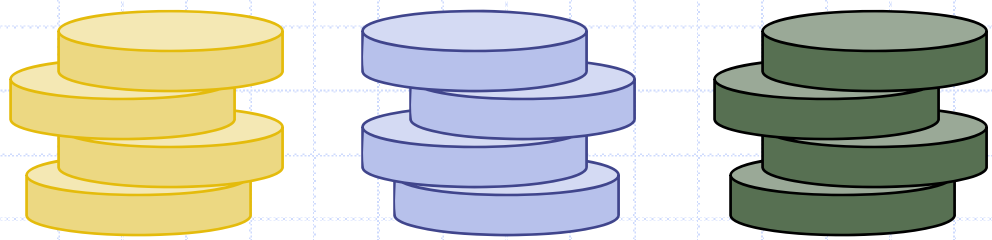
Stacks



Outline

- The Stack ADT
- Applications of Stacks
- Array-based implementation
- Growable array-based stack

Abstract Data Types (ADTs)

- An abstract data type (ADT) is an abstraction of a data structure
- An ADT specifies:
  - Data stored
  - Operations on the data
  - Error conditions associated with operations
- Example: ADT modeling a simple stock trading system
  - The data stored are buy/sell orders
  - The operations supported are
    - order buy(stock, shares, price)
    - order sell(stock, shares, price)
    - void cancel(order)
  - Error conditions:
    - Buy/sell a nonexistent stock
    - Cancel a nonexistent order

The Stack ADT

- The Stack ADT stores arbitrary objects
- Insertions and deletions follow the last-in first-out scheme
- Think of a spring-loaded plate dispenser
- Main stack operations:
  - push(object o): inserts element o
  - pop(): removes and returns the last inserted element
- Auxiliary stack operations:
  - top(): returns a reference to the last inserted element without removing it
  - size(): returns the number of elements stored
  - isEmpty(): returns a Boolean value indicating whether no elements are stored

Exceptions

- Attempting the execution of an operation of ADT may sometimes cause an error condition, called an exception
- Exceptions are said to be “thrown” by an operation that cannot be executed
- In the Stack ADT, operations pop and top cannot be performed if the stack is empty
- Attempting the execution of pop or top on an empty stack throws an EmptyStackException

Applications of Stacks

- Direct applications
  - Page-visited history in a Web browser
  - Undo sequence in a text editor
  - Saving local variables when one function calls another, and this one calls another, and so on.
- Indirect applications
  - Auxiliary data structure for algorithms
  - Component of other data structures

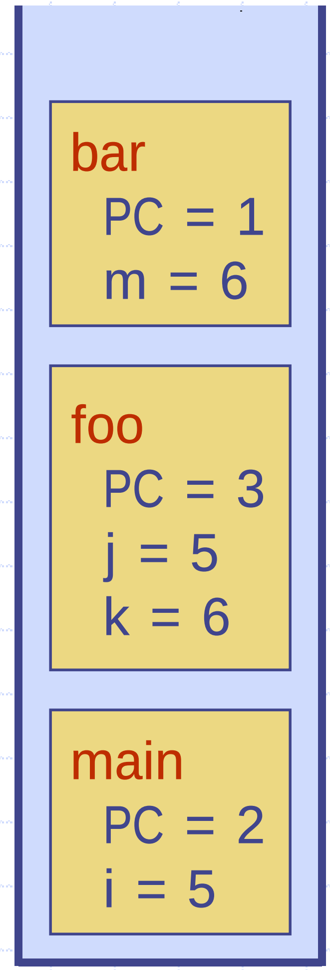
C++ Run-time Stack

- The C++ run-time system keeps track of the chain of active functions with a stack
- When a function is called, the run-time system pushes on the stack a frame containing
  - Local variables and return value
  - Program counter, keeping track of the statement being executed
- When a function returns, its frame is popped from the stack and control is passed to the method on top of the stack

```
main() {
  int i = 5;
  foo(i);
  }

foo(int j) {
  int k;
  k = j+1;
  bar(k);
  }

bar(int m) {
  …
  }
```

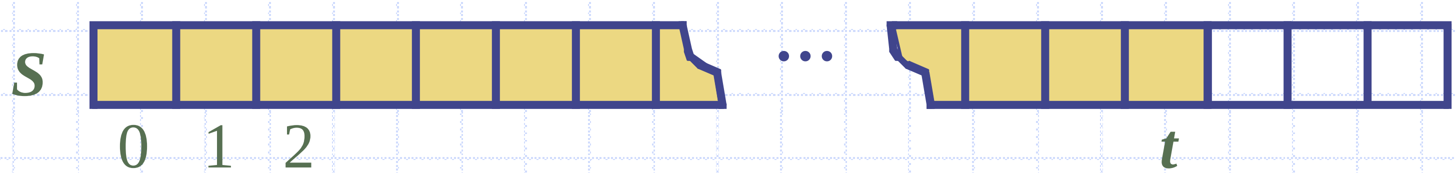


Array-based Stack

- A simple way of implementing the Stack ADT uses an array
- We add elements from left to right
- A variable keeps track of the index of the top element

Algorithm *size*()
  return $t + 1$

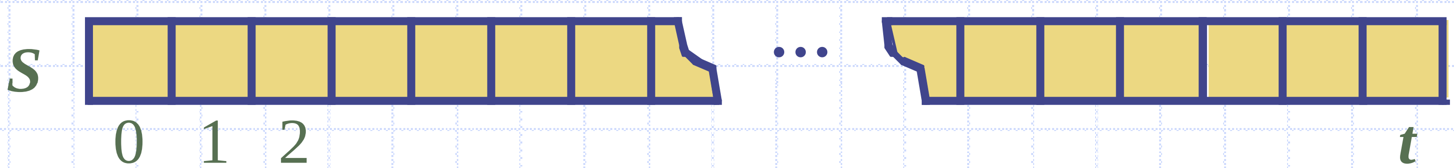
Algorithm *pop*()
  if *isEmpty*() **then**
    throw *EmptyStackException*
  else
    $t \leftarrow t - 1$
    return $S[t + 1]$



Array-based Stack (cont.)

- The array storing the stack elements may become full
- A push operation will then throw a FullStackException
  - Limitation of the array-based implementation
  - Not intrinsic to the Stack ADT

Algorithm *push*(*o*)
  if $t = S.length - 1$ **then**
    throw *FullStackException*
  else
    $t \leftarrow t + 1$
    $S[t] \leftarrow o$

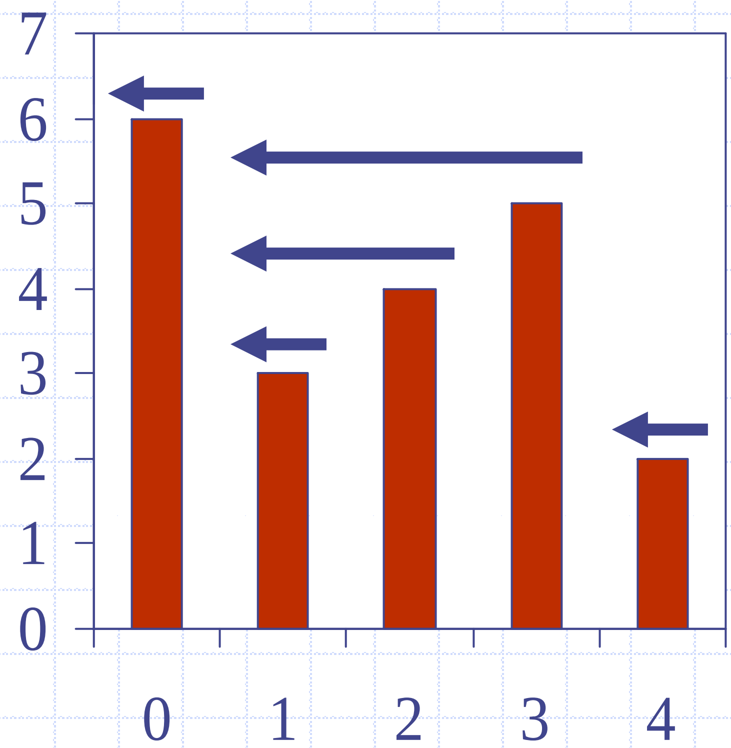


Performance and Limitations

- Performance
  - Let n be the number of elements in the stack
  - The space used is $O(n)$
  - Each operation runs in time $O(1)$
- Limitations
  - The maximum size of the stack must be defined *a priori* , and cannot be changed
  - Trying to push a new element into a full stack causes an implementation-specific exception

Computing Spans

- We show how to use a stack as an auxiliary data structure in an algorithm
- Given an array X, the span $S[i]$ of $X[i]$ is the maximum number of consecutive elements $X[j]$ immediately preceding $X[i]$ and such that $X[j] \leq X[i]$
- Spans have applications to financial analysis
  - E.g., stock at 52-week high



X	6	3	4	5	2
S	1	1	2	3	1

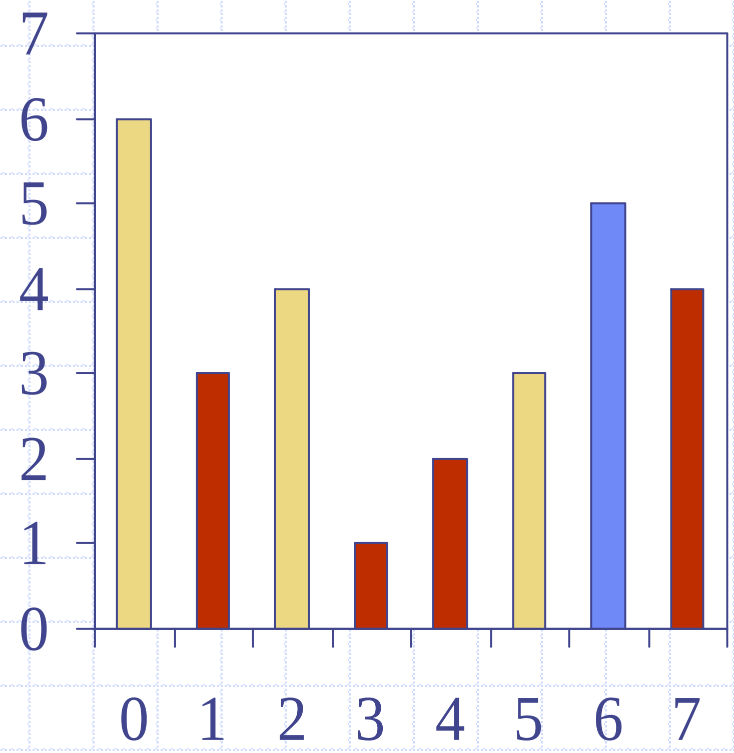
Quadratic Algorithm

Algorithm *spans1*(X, n)	
Input array X of n integers	
Output array S of spans of X	#
$S \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow 1$	n
while $s \leq i \wedge X[i - s] \leq X[i]$	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + 1$	$1 + 2 + \ldots + (n - 1)$
$S[i] \leftarrow s$	n
return S	1

- Algorithm ***spans1*** runs in $O(n^2)$ time

Computing Spans with a Stack

- We keep in a stack the indices of the elements visible when "looking back"
- We scan the array from left to right
  - Let i be the current index
  - We pop indices from the stack until we find index j such that $X[i] < X[j]$
  - We set $S[i] \leftarrow i - j$
  - We push i onto the stack



Linear Algorithm

- Each index of the array
  - Is pushed into the stack exactly one
  - Is popped from the stack at most once
- The statements in the while-loop are executed at most n times
- Algorithm ***spans2*** runs in $O(n)$ time

Algorithm *spans2*(X, n)	#
$S \leftarrow$ new array of n integers	n
$A \leftarrow$ new empty stack	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
while $(\neg A.isEmpty() \wedge$	
$X[top()] \leq X[i]$) **do**	n
$j \leftarrow A.pop()$	n
if $A.isEmpty()$ **then**	n
$S[i] \leftarrow i + 1$	n
else	
$j \leftarrow top()$	n
$S[i] \leftarrow i - j$	n
$A.push(i)$	n
return S	1

Growable Array-based Stack

- In a push operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one
- How large should the new array be?
  - incremental strategy: increase the size by a constant *c*
  - doubling strategy: double the size

Algorithm *push*(*o*)
  if $t = S.length - 1$ **then**
    $A \leftarrow$ new array of size …
    for $i \leftarrow 0$ **to** t **do**
      $A[i] \leftarrow S[i]$
      $S \leftarrow A$
  $t \leftarrow t + 1$
  $S[t] \leftarrow o$

Comparison of the Strategies

- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations
- We assume that we start with an empty stack represented by an array of size 1
- We call amortized time of a push operation the average time taken by a push over the series of operations, i.e., $T(n)/n$

Incremental Strategy Analysis

- We replace the array $k = n/c$ times
- The total time $T(n)$ of a series of n push operations is proportional to

$$n + c + 2c + 3c + 4c + \ldots + kc$$
$$= n + c(1 + 2 + 3 + \ldots + k)$$
$$= n + ck(k + 1)/2$$

- Since c is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$
- The amortized time of a push operation is
  - $O(n)$

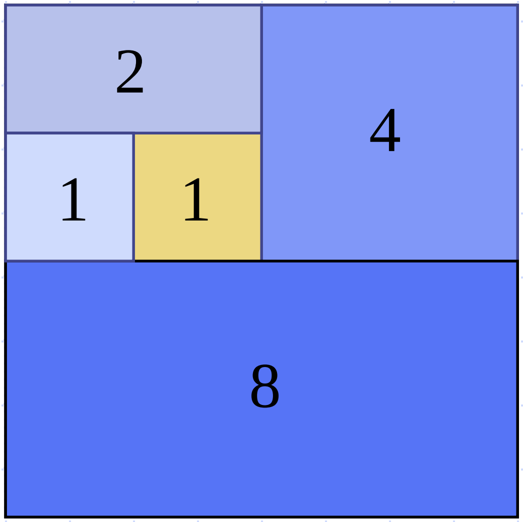
Doubling Strategy Analysis

- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n push operations is proportional to

$$n + 1 + 2 + 4 + 8 + \ldots + 2^k$$
$$= n + 2^{k+1} - 1$$
$$= 2n - 1$$

- $T(n)$ is $O(n)$
- The amortized time of a push operation is
  - $O(1)$

geometric series



Stack Interface in C++

- Interface corresponding to our Stack ADT
- Requires the definition of class EmptyStackException
- Most similar STL construct is vector

```
template <typename Object>
class Stack {
public:
    int size();
    bool isEmpty();
    Object& top()
        throw(EmptyStackException);
    void push(Object o);
    Object pop()
        throw(EmptyStackException);
};
```

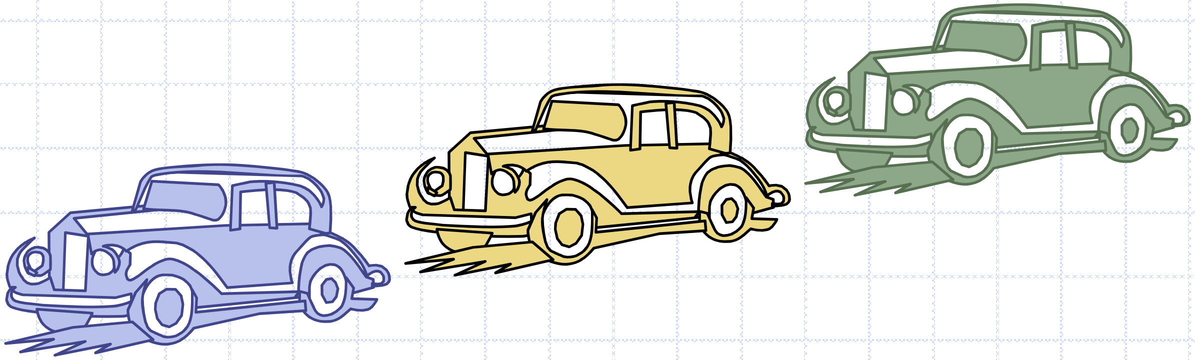
Array-based Stack in C++

```
template <typename Object>
class ArrayStack {
private:
    int capacity;       // stack capacity
    Object *S;          // stack array
    int top;            // top of stack
public:
    ArrayStack(int c) {
        capacity = c;
        S = new Object[capacity];
        t =  –1;
    }
```

```
    bool isEmpty()
      {  return (t < 0);  }

    Object pop()
          throw(EmptyStackException) {
        if(isEmpty())
          throw EmptyStackException
            (“Access to empty stack”);
          return S[t--];
    }
// … (other functions omitted)
```

Queues



Outline and Reading

- The Queue ADT
- Implementation with a circular array
- Growable array-based queue
- Queue interface in C++

The Queue ADT

- The Queue ADT stores arbitrary objects
- Insertions and deletions follow the first-in first-out scheme
- Insertions are at the rear of the queue and removals are at the front of the queue
- Main queue operations:
  - enqueue(Object o): inserts an element o at the end of the queue
  - dequeue(): removes and returns the element at the front of the queue
- Auxiliary queue operations:
  - front(): returns the element at the front without removing it
  - size(): returns the number of elements stored
  - isEmpty(): returns a Boolean indicating whether no elements are stored
- Exceptions
  - Attempting the execution of dequeue or front on an empty queue throws an EmptyQueueException

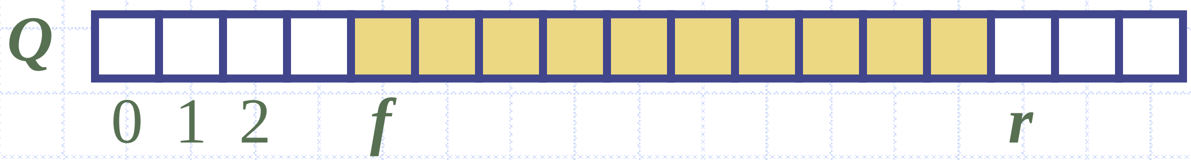
Applications of Queues

- Direct applications
  - Waiting lists, bureaucracy
  - Access to shared resources (e.g., printer)
  - Multiprogramming
- Indirect applications
  - Auxiliary data structure for algorithms
  - Component of other data structures

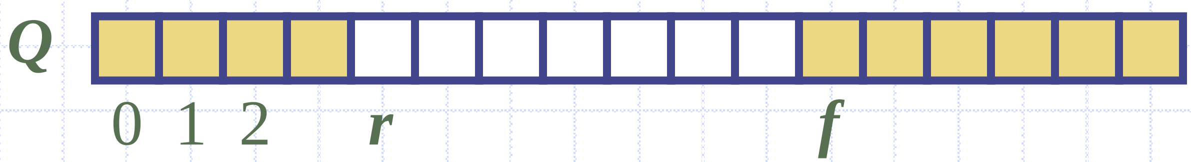
Array-based Queue

- Use an array of size N in a circular fashion
- Two variables keep track of the front and rear
  - f index of the front element
  - r index immediately past the rear element
- Array location r is kept empty

normal configuration



wrapped-around configuration

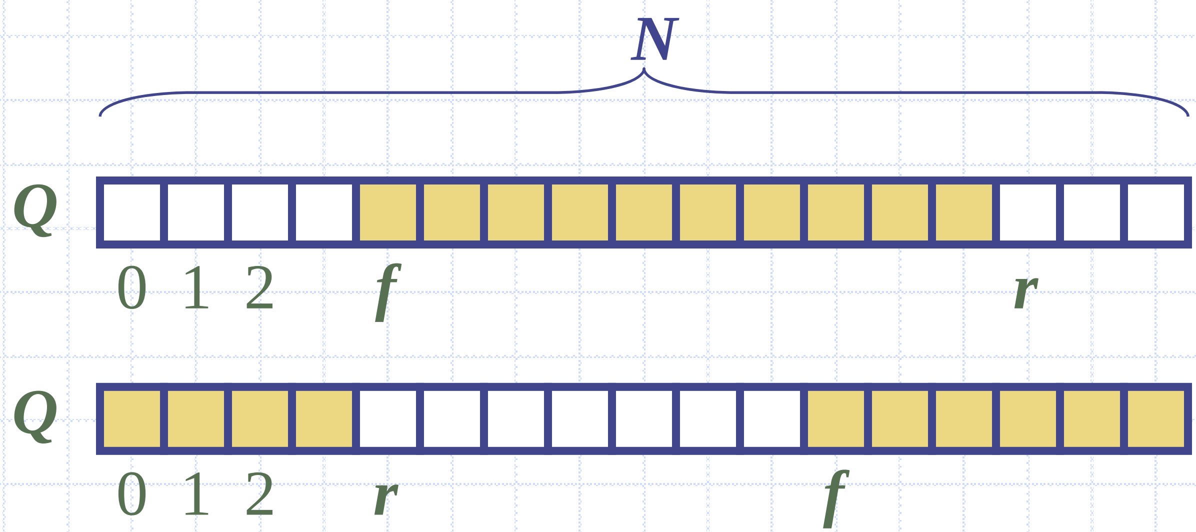


Queue Operations

- Hint: use the modulo operator

e.g. N=17, f=4, r=14
size=(17-4+14) mod 17
size=27 mod 17
size=10

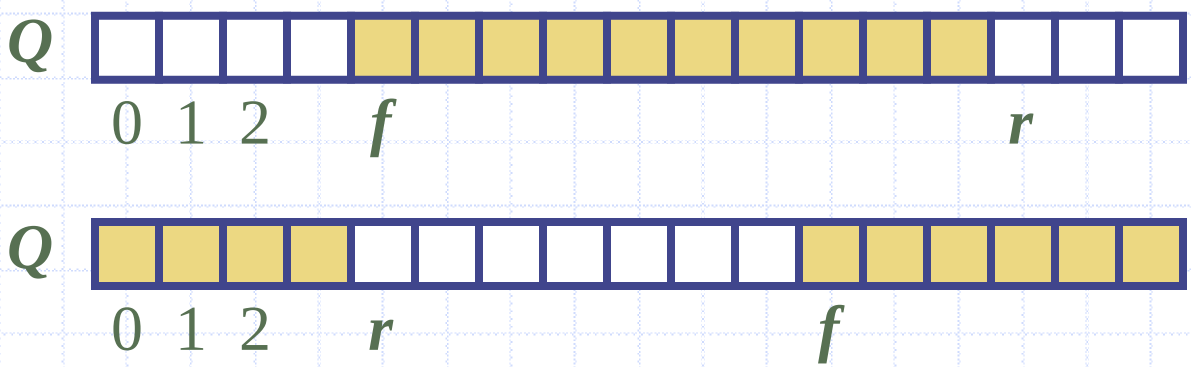
Algorithm ***size***()
  return $(N - f + r) \bmod N$
    [or $= ((N+r)-f) \bmod N$]
Algorithm ***isEmpty***()
  return $(f = r)$



Queue Operations (cont.)

- Operation enqueue throws an exception if the array is full
- This exception is implementation-dependent

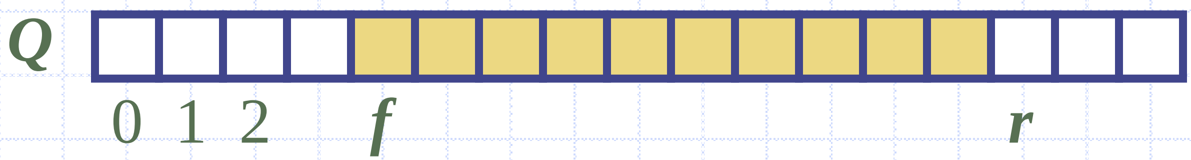
Algorithm *enqueue*(*o*)
  if *size*() = $N - 1$ **then**
    throw *FullQueueException*
  else
    $Q[r] \leftarrow o$
    $r \leftarrow (r + 1) \bmod N$

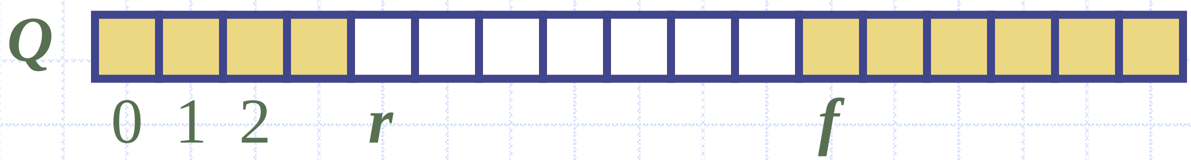


Queue Operations (cont.)

- Operation dequeue throws an exception if the queue is empty
- This exception is specified in the queue ADT

Algorithm *dequeue*()
  if *isEmpty*() **then**
    throw *EmptyQueueException*
  else
    $o \leftarrow Q[f]$
    $f \leftarrow (f + 1) \bmod N$
    return o





Growable Array-based Queue

- In an enqueue operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one
- Similar to what we did for an array-based stack
- The enqueue operation has amortized running time
  - $O(n)$ with the incremental strategy
  - $O(1)$ with the doubling strategy

Informal C++ Queue Interface

- Informal C++ interface for our Queue ADT
- Requires the definition of class EmptyQueueException
- No corresponding built-in STL class

```
template <typename Object>
class Queue {
public:
    int size();
    bool isEmpty();
    Object& front()
        throw(EmptyQueueException);
    void enqueue(Object o);
    Object dequeue()
        throw(EmptyQueueException);
};
```