STABILITY ANALYSIS OF QUOTA ALLOCATION ACCESS PROTOCOLS IN RING NETWORKS WITH SPATIAL REUSE *

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Abstract

We consider a slotted ring that allows simultaneous transmissions of messages by different nodes, known as ring with *spatial reuse.* To alleviate fairness problems that arise in such networks, policies have been proposed that operate in cycles and guarantee that certain number of packets, not exceeding a given number called a *quota,* will be transmitted by every node in every cycle. In this paper, we provide sufficient and necessary stability conditions that implicitly characterize the stability region for such rings. These conditions are derived by extending a technique developed by two of us for some networks of queues satisfying a monotonicity property. Our approach to instability is novel and its peculiar property is that it is derived from the instability of a dominant system. Interestingly, the stability region depends on the entire distribution of the message arrival process and the steady-state average cycle lengths of lower dimensional systems, leading to a region with nonlinear boundaries, the exact computation of which is in general intractable. Next, we introduce the notions of *essential* and *absolute* stability region. An arrival rate vector belongs to the former region if the system is stable under any arrival distribution with this arrival vector, while it belongs to the later if there exists some distribution with this rate vector for which the system is stable. Using a linear programming approach, we derive bounds for these stability regions that depend only on conditional average cycle lengths. For the case of two nodes, we provide closed-form expressions for the essential stability region.

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1 Introduction

We consider a unidirectional ring with spatial reuse, i.e., a ring in which multiple simultaneous transmissions are allowed as long as they take place over different links (cf. [8, 12, 15]). While rings with spatial reuse have higher throughput than standard token passing rings, they also introduce the possibility that some overloaded nodes may block other nodes from accessing the ring. To avoid this problem, the following policy is proposed in [8, 12] for the operation of the ring: Each node is assigned a number called "quota". The policy operates in cycles. A node is allowed to transmit packets generated locally during a cycle, as long as the number of these packets that have already been transmitted does not exceed its assigned quota. A cycle ends when the quota of all nodes are delivered to their destinations. In this way, the operation of a node with regular traffic requirements is not adversely affected by nodes that may become overloaded. The policy requires a distributed mechanism by which every node realizes that all the other nodes completed their quota and thus a cycle ends. Such a mechanism is provided in [12]. An analysis of the throughput characteristics of this policy is presented in [15]. It should be pointed out that in [15] it was assumed that all nodes were overloaded, i.e., they had infinite queues and the measure of interest was the average number of packets originated at a node (throughput) that could eventually be delivered to their destination. It was shown that by appropriately picking the quotas, all feasible throughput vectors could be achieved. Therefore, the situation where a node is blocked from transmitting its own packets because of transmissions from other nodes can be effectively eliminated. In the current work we assume a stochastic input with arrival rates of packets to each node that are finite, and we are interested in the region of arrival rates for which the queues of all nodes have proper probability distributions (stability region).

The primary goal of this work is to obtain the stability region of the ring network with finite quota and to compare it with the maximum achievable stability region for such ring networks derived in [15, 18, 29]. We demonstrate (cf. Example 1 in Section 3.1) that the stability region of the system with the fixed quota mechanism is reduced relative to the stability region of other policies that lack the fairness properties that the quota mechanism provides (cf: [18, 29]). The second motivation is to extend the stability approach of Georgiadis and Szpankowski [16, 17] and Szpankowski [27, 28] to ring networks with spatial reuse, and other queueing networks that operate in cycles and satisfy a monotonicity property. The sufficient conditions for stability are derived by means of a technique that is based on an application of mathematical induction, stochastic monotonicity properties and Loynes stability criteria. A special technique, based on the structure of the complement of the stability region and the construction of a dominant system, permits the derivation of the necessary stability conditions from the instability condition of the *dominant* system. In the process, we provide a decomposition and characterization of the instability region of the system. While this decomposition has been used before on an intuitive basis, it is not as obvious as might seem at first. We l illustrate this point by a simple example (see the discussion following Proposition 4 in Section 3.1). The general steps of the above stability analysis have been applied to the analysis of other systems as well (cf. [16, 17, 28]). It should be stressed, however, that this general construction of [16, 28] requires detailed and subtle modifications for almost every queueing network which may be far from trivial, and this paper is a typical example.

As it turns out, the exact computation of the stability region for the ring with spatial reuse depends on the distribution of the arrival processes as well as the steady-state average cycle lengths of lower dimensional systems and this often renders this computation intractable. This leads us to the introduction of the notions of the *Essential* and *Absolute Stability Regions*. The first contains any arrival vector such that for every distribution with this arrival rate vector the network is stable. The second contains any arrival rate vector for which there exists some distribution with this arrival rate vector under which the network is stable. In this paper, we present a method based on linear programming (cf. also [19] for another usage of linear programming to stability problems) that permits the development of upper and lower bounds on the Absolute and Essential Stability Regions using only the knowledge of the *conditional average* cycle lengths. For the case of two nodes, we provide a closed-form expression for the Essential Stability Region in terms of the conditional average cycle lengths. The conditional average cycle lengths are fundamental quantities of the system operation, whose statistics depend only on the packet destination probabilities, and *not* on the steady-state on steady-state quantities or the packet arrival distributions. In this sense, they are the simplest quantities on which the stability analysis can be based. We note, however, that due to the complicated expression for the cycle lengths (cf. formula (1)), even these quantities cause computational difficulties. For small number of nodes the conditional average cycle lengths can be computed directly, while asymptotic results for large number of nodes can be found in [15].

Stability criteria for Markov chains and more general queueing systems have a long tradition. In recent years, resurgence of interest in these problem arose due to novel applications. It resulted in an excellent book of Meyn and Tweedie [24]. This book as well as most research in this area is based on the so called Lyapunov or test function approach. Construction of this function is quite troublesome for multidimensional Markov chains. A general approach to such a construction was suggested in 1981 by Malyshev and Mensikov [22]. This general construction still fails for many important distributed systems, however, recently some progress has been achieved (cf. [6, 14, 19, 25]). Our approach is non-standard and it is based on different philosophy, but it has some similarities with the *faces* and *induced Markov chains* of Malyshev and Menshikov [22]. As mentioned above, in our analysis we apply mathematical induction (that recently became very popular in stability analysis [7, 16, 17, 23, 26, 28]), Loynes stability criteria (cf. [1, 5, 3] for extensions and other applications), and stochastic monotonicity. Monotonicity was recently used in [4, 7, 13, 23] to establish stability regions for other multidimensional queueing systems and computer networks. Finally, we should mention a recent new development in this area suggested by Dai [10], and Dai and Meyn [11] who used fluid approximation to derive general stability criteria for queueing networks. For more exhaustive discussion of the existing literature on stability criteria the reader is referred to [5, 11, 16, 24, 27, 28].

The paper is organized as follows. In the next section we formulate a stochastic model for the network under consideration. Section 3 contains our main results: In Subsection 3.1 we present the construction of the exact stability region. Bounds on the stability region are provided in Section 3.2. For a ring with two nodes we present in Section 3.3 the derivation of the Essential Stability Region. Finally, Section 4 contains proofs of the results needed for establishing the necessary conditions for stability, and describes a novel approach to the instability analysis.

2 Model Description and Preliminary Results

We consider a unidirectional ring network consisting of a set of $\mathcal{M}$ nodes with cardinality, $|\mathcal{M}| = M$. Node $i \in \mathcal{M}$ transmits in its outgoing link, either packets arriving to this node from the outside world (i.e., "external" packets) or packets that were originated at some other node and have to cross node i in order to reach their destination. Time is divided in slots, packets are of fixed size and each slot is equal to the length of a packet. We assume zero propagation delay. A node can transmit a packet on the outgoing link at the same time that it receives another packet in the incoming link. A node receiving a packet whose destination is another node in the ring (ring packet) may relay the packet in the outgoing link *in the same slot*, i.e., the ring has *cut-through* capabilities. Moreover, a ring packet has non-preemptive priority over the packets that exist in the node queue. Packets are removed from the ring by their destination (not by the source as in standard token rings). We study the following policy, which is a generalization of a policy proposed in the literature (cf. [8, 12]).

(A1) The system works in cycles, and the kth cycle starts at time τ_k. We write $\mathbf{N}(k) = (N_1(k), \ldots, N_M(k))$ to denote the number of packets in the node buffers at the beginning of cycle $k = 1, \ldots$. The number of external packets that node i is allowed to transmit during cycle k is $Q_i(k) = \min\left\{f_i(N_i(k)), \widehat{Q}_i\right\}$, $\widehat{Q}_i > 0$, where $f_i(\cdot)$ is a nondecreasing and contractive function, i.e., $f_i(s_1) - f_i(s_2) \leq s_1 - s_2$ whenever $s_1 > s_2$. The quantity $\widehat{Q}_i$ is called the (maximum) *quota*. At each time slot during a cycle, a node may transmit packets that are either generated locally or by some upstream node, according to the policy Π_1^* described in Section IV of [15]: The kth cycle ends when all $Q_i(k)$ packets, $1 \leq i \leq M$, are delivered to their destinations. Algorithmic and implementation details can be found in [12, 8, 15], however, of interest to our discussion here is only the statistics of the length of time needed to complete a cycle (cf. (1) below). The standard ring network operating with the quota allocation policy corresponds to the case when $f_i(s) = s$.

As part of the technique used in the proof of the stability conditions we need to analyze a system $\Theta^{\mathcal{M},\mathcal{U}}$, where in addition to the set, $\mathcal{M}$, of regular nodes there is also a set, $\mathcal{U}$, of "persistent nodes", which operate as follows.

(A2) (i) There are no external packet arrivals at node $i \in \mathcal{U}$, and (ii) a node $i \in \mathcal{U}$ participates in the policy described in (A1) by generating locally and transmitting exactly $\widehat{Q}_i$ "dummy" packets in a cycle (in addition to the packets that may have originated in some other node but have to be retransmitted by node i in its outgoing link in order to reach their destination). While the nodes in $\mathcal{U}$ affect the duration of the cycles, by definition they do not have queues and we are interested only in the stability of the queue length process of the nodes in $\mathcal{M}$. As will be seen in the next section, the introduction of persistent nodes assures that when some regular nodes behave like persistent ones, the system consisting of the rest of the nodes is a copy of the original system, but of *lower dimension*. This property permits the application of mathematical induction. The case $\mathcal{U} = \emptyset$ corresponds to the ring we are interested in.

Next, we make an assumption regarding the statistics of external packet arrival process at the ring nodes.

(A3) We denote by $R_i(t)$ the number of external packets arriving at station $i \in \mathcal{M}$ in slot $t \geq 1$. The nth packet originated at node $i \in \mathcal{M} \cup \mathcal{U}$ has destination $D_i(n) \in \mathcal{M} \cup \mathcal{U}$. The processes $\{R_i(t)\}_{t=1}^{\infty}$, $i \in \mathcal{M}$ and $\{D_i(n)\}_{n=1}^{\infty}$, $i \in \mathcal{M} \cup \mathcal{U}$ consist of i.i.d. random variables and are independent of each others. We set $\lambda_i = ER_i(1)$, $i \in \mathcal{M}$ and $p_{ij} = \Pr\{D_i(1) = j\}$, $i, j \in \mathcal{M} \cup \mathcal{U}$. Clearly, $\sum_{j \in \mathcal{M} \cup \mathcal{U}} p_{ij} = 1$ for $i \in \mathcal{M} \cup \mathcal{U}$.

Before proceeding, we must introduce some new notations. Boldface letters denote vectors, while calligraphic ones denote sets of nodes. Our main goal is to study the ergodicity of the imbedded Markov chain $\mathbf{N}(k) = (N_1(k), \ldots, N_M(k))$ for $k = 0, 1, \ldots$. We write $\mathcal{M}_{\mathcal{A}} = \mathcal{M} - \mathcal{A}$ (while nonstandard, this notation simplifies the presentation significantly). We will often consider the partition $(\mathcal{M}_{\mathcal{V}}, \mathcal{V})$ of the set $\mathcal{M}$, where $\mathcal{V} \subseteq \mathcal{M}$. For a vector $\mathbf{x} = (x_1, \ldots, x_M)$ we set $\mathbf{x}^{\mathcal{A}} = \{x_i\}_{i \in \mathcal{A}}$. In particular, we write $\mathbf{N}(k) = (\mathbf{N}^{\mathcal{M}_{\mathcal{A}}}(k), \mathbf{N}^{\mathcal{A}}(k))$. For M-dimensional vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \leq \mathbf{y}$ reads $x_i \leq y_i$ for all $1 \leq i \leq M$.

As already observed in [12, 15], the behavior of the network depends crucially on the cycle length $T_k = \tau_{k+1} - \tau_k$ which is also called the *evacuation time.* Let $T(\mathbf{q})$ (or $T_k(\mathbf{q})$) be the length of cycle k when the quota vector is $\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(k) = \mathbf{q} = (q_1, \ldots, q_M, \ldots, q_{M+|\mathcal{U}|})$. It was shown in [15] (cf. [18]) that[1]

$$T_k(\mathbf{q}) = \max\left\{ \max_{i \in \mathcal{M}\cup\mathcal{U}} H_i(\mathbf{q}), 1 \right\} \tag{1}$$

where $H_i(\mathbf{q})$ is the total number of packets out of $\sum_{i \in \mathcal{M}\cup\mathcal{U}} q_i$ originated in a cycle at any node (i.e., regular or persistent) that have to pass through the outgoing link of node i in order to reach their destination. Note that $H_i(\mathbf{q})$ includes the packets originated at node i. Also, note that the statistics of $T(\mathbf{q})$ depend only on the vector $\mathbf{q}$ and the packet destination probabilities p_{ij}.

In passing, we should mention that in order for all nodes to realize the end of a cycle, a distributed mechanism is needed [12]. The implementation of this mechanism increases the evacuation time by two slots and the results in this paper can be directly applied by simply replacing $T(\mathbf{q})$ with $T(\mathbf{q})+2$. We also mention that (1) holds under any work-conserving policy, i.e., any policy that instructs each node never to idle whenever it can transmit packet in its outgoing link (see [18]). Therefore, the order by which packets are served at a node is immaterial.

Below we establish a monotonicity property of the cycle lengths. As we will see, this is a relevant property of the cycle lengths from the stability point of view. In fact, our analysis holds for any other system which, in addition to operating in cycles during which a certain quota can be transmitted by each node and satisfying the statistical assumptions in (A3), has the property that the cycle lengths are independent of the past history given $\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(k) = \mathbf{q}$ and satisfy the monotonicity property presented in the next proposition.

[1] In [15] it was assumed that $q_i \geq 1$ for some node i. In our model, however, it is possible that all nodes have empty queues at the beginning of a cycle, in which case $T_k(\mathbf{0}) = 1$. For this reason we include the maximum with 1 in (1).

Proposition 1 *Let* $\mathbf{q}_1 \leq \mathbf{q}_2$. *Then*

$$T_k(\mathbf{q}_1) \leq_{st} T_k(\mathbf{q}_2)$$

where $\leq_{st}$ *means "stochastically smaller".*

Proof. Follows easily from formula (1). ∎

Let us now consider a modified system in which a set $\mathcal{V} \subset \mathcal{M}$ of users becomes persistent, that is, every user in $i \in \mathcal{V}$ transmits $\widehat{Q}_i$ packets (i.e., it transmits "dummy" packets when their queues are empty or possess less than $\widehat{Q}_i$ packets). From the point of view of the nodes in the set $\mathcal{M}_{\mathcal{V}}$, the nodes in $\mathcal{V}$ behave exactly as the *persistent* nodes in the set $\mathcal{U}$. Note, however, that there is a difference between the nodes in $\mathcal{U}$ and $\mathcal{V}$ in that the nodes in $\mathcal{V}$ receive external packets and therefore have queues formed. We denote such a system as $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V}),\mathcal{U}}$. Define $\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V})}(k) = (\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k), \widetilde{\mathbf{N}}^{\mathcal{V}}(k))$ as the queue length vector in the system $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}},\mathcal{V}),\mathcal{U}}$. In the next result, we prove that the queues in the modified system dominate stochastically the queue length in the original system. This property is crucial to applications of our method.

Proposition 2 *Consider two partitions* $(\mathcal{M}_{\mathcal{V}_1}, \mathcal{V}_1)$ *and* $(\mathcal{M}_{\mathcal{V}_2}, \mathcal{V}_2)$ *such that* $\mathcal{V}_1 \subseteq \mathcal{V}_2$. *Then for every* $k = 0, 1, \ldots$

$$\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_1},\mathcal{V}_1)}(k) \leq_{st} \widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_2},\mathcal{V}_2)}(k) \tag{2}$$

provided $\widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_1},\mathcal{V}_1)}(0) = \widetilde{\mathbf{N}}^{(\mathcal{M}_{\mathcal{V}_2},\mathcal{V}_2)}(0)$.

Proof. The proof follows the steps of the proof of the monotonicity property of the queue lengths in token passing rings. For details, the reader is referred to Theorem 4 of Georgiadis and Szpankowski [16]. ∎

3 Main Results

This section presents our main results. In the sequel, we construct the stability region for the network, derive some bounds on the stability region, and finally provide in a closed-form the essential stability region of a ring with two nodes.

3.1 Construction of the Stability Region

Consider the system $\Theta^{\mathcal{M},\mathcal{U}}$ consisting of a set $\mathcal{M}$ of regular nodes and a set $\mathcal{U}$ of persistent nodes. Our goal is to establish stability conditions for the queue length vector $\mathbf{N}^{\mathcal{M}}(k)$. By stability we mean the existence of the limiting distribution.

The process $\mathbf{N}^{\mathcal{M}}(k)$ is an imbeded Markov chain. Indeed, we have for every $i \in \mathcal{M}$,

$$N_i(k+1) = N_i(k) - Q_i(k) + \sum_{t=\tau_k}^{\tau_{k+1}-1} R_i(t). \tag{3}$$

Under (A1)-(A3) and equation (1), the above set of stochastic equations forms an M-dimensional Markov chain defined on a countable state space.

In the sequel, we will use the following property of multidimensional Markov chains defined on a countable state space: *To establish ergodicity of* $\mathbf{N}^{\mathcal{M}}(k)$ *it suffices to show that every component* $N_i(k)$, $i \in \mathcal{M}$, *of* $\mathbf{N}^{\mathcal{M}}(k)$ *is substable* (i.e., the one dimensional process $N_i(k)$ is bounded in probability as $k \to \infty$). This fact is easy to prove on a countable state space, and the reader is referred to [16, 28]. On a general state space, the situation is more complicated, and one should consult Meyn and Tweedie [24]. This fact, called *isolation lemma* in [27, 28], permits the study of the stability of each queue in isolation.

We now begin the construction of the stability region (i.e., the set of node arrival rates) of system $\Theta^{\mathcal{M},\mathcal{U}}$ based on the knowledge of the stability region and the steady-state average cycle lengths of lower dimensional systems. We denote stability region of a whole system as $S^{\mathcal{M},\mathcal{U}}$. We write $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ to denote the stability region of the dominant system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ which arose by making the ith node behave like a persistent one. Note that while node i in the dominant system behaves like a persistent one, this node still has a queue formed, and therefore region $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$ consists of M-dimensional (*not* of $(M-1)$-dimensional as the region $S^{\mathcal{M}_{\{i\}},\mathcal{U}\cup\{i\}}$) vectors. For simplicity, whenever there is no possibility for confusion, we omit the set $\mathcal{U}$ from the notation in Θ, $\overline{\Theta}$, S, or $\overline{S}$. For example, unless otherwise specified, $\Theta^{\mathcal{M}} \equiv \Theta^{\mathcal{M},\mathcal{U}}$, $\Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} \equiv \Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{U}\cup\mathcal{V}}$, $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} \equiv \overline{S}^{(\mathcal{M}_{\{i\}},\{i\}),\mathcal{U}}$.

The construction of the stability region follows the steps developed in [16, 28]. We will therefore skip the details of a rigorous derivation and instead we will explain in some detail the main idea behind each step. The construction is done inductively as follows:

Step 1: Derive the (sufficient) stability condition for a ring with one regular node and an arbitrary set $\mathcal{U}$ of persistent nodes. In this case, we have a single queue (at the regular node) and the derivation of the stability condition is easy. Specifically, let i be a single regular node in the ring. Then, since the queue length $N_i(k)$ is a Markov chain, using the Lyapunov test function method (cf. [24]) one directly proves that the chain is ergodic if,

$$\lambda_i < \frac{\widehat{Q}_i}{ET(\widehat{\mathbf{Q}})} \tag{4}$$

where, we recall that $T(\widehat{\mathbf{Q}})$ denotes the cycle length when $\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(1) = \widehat{\mathbf{Q}}$ at the beginning of a cycle. That is, $ET(\widehat{\mathbf{Q}}) \equiv E[T_1|\mathbf{Q}^{\mathcal{M}\cup\mathcal{U}}(1) = \widehat{\mathbf{Q}}]$.

Step 2: Assume that we derived the stability region for a ring with $M-1$ regular nodes and an arbitrary set $\mathcal{U}$ of persistent nodes. We next seek to define the stability region of a ring with a set $\mathcal{M}$, $|\mathcal{M}| = M$, regular nodes, and an arbitrary set $\mathcal{U}$ of persistent nodes, in terms of the stability regions and the steady-state average cycle lengths of *lower dimensional systems.* This is done by taking a set $\mathcal{V} \subseteq \mathcal{M}$, $\mathcal{V} \neq \emptyset$, of regular nodes and making them behave like persistent ones, i.e., by considering system $\overline{\Theta}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. By Proposition 2, $\mathbf{N}^{\mathcal{M}}(k) \equiv \tilde{\mathbf{N}}^{(\mathcal{M},\emptyset)}(k) \leq_{st} \tilde{\mathbf{N}}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}(k)$, provided that $\mathbf{N}^{\mathcal{M}}(0) = \tilde{\mathbf{N}}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}(0)$. Therefore, $\Theta^{\mathcal{M}}$ is stable whenever $\overline{\Theta}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$ is, i.e., $S^{\mathcal{M}} \supseteq \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. Since $\mathcal{V}$ is arbitrary, we conclude that $S^{\mathcal{M}} \supseteq \bigcup_{\mathcal{V}\subseteq\mathcal{M}} \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$, where $\mathcal{V} \neq \emptyset$. However, it follows again from Proposition 2 that if $i \in \mathcal{V}$, then $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} \supseteq \overline{S}^{(\mathcal{M}_\mathcal{V},\mathcal{V})}$. Therefore,

$$S^{\mathcal{M}} \supseteq \bigcup_{i\in\mathcal{M}} \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}.$$

In fact, it turns out (see Theorem 2 below) that for the problem at hand, we have equality (with the possible exception of boundaries) in the previous subset relation. With a slight abuse of notation, to avoid the introduction of new symbols we will also denote the set $\bigcup_{i\in\mathcal{M}} \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ by $S^{\mathcal{M}}$.

Step 2a: Next, we determine the stability region $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ of system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, in terms of the stability region and the steady-state average cycle lengths of the system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$ which is of dimension $M-1$, and therefore, its stability region has been determined by the inductive assumption. It should be noted that this is the point where the "fixed but arbitrary" set $\mathcal{U}$ is used in the proof since now we can claim that $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$ is a "smaller copy" of the original system. Specifically, in system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$ the set of persistent nodes is $\mathcal{U}$. However, in system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$, the set of persistent nodes is $\mathcal{U} \cup \{i\} \neq \mathcal{U}$. Therefore we could not have applied the inductive hypothesis if the assumption in this hypothesis did not involve an arbitrary set $\mathcal{U}$. To determine $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$ we apply the isolation lemma, i.e., we look for conditions under which each queue in the set $\mathcal{M}$, under system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, is substable. For this, we look first at the queues in the set $\mathcal{M}_{\{i\}}$, which evolve exactly as in system $\Theta^{\mathcal{M}_{\{i\}},\{i\}}$. Therefore, these queues are stable as long as $\boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}}$, which by the inductive hypothesis is known.

Step 2b: It remains to determine conditions under which the queue at node $\{i\}$ is (sub)stable in system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$, which is done as follows. Assuming that $\boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}}$, we can construct a stationary and ergodic version of the queue length vector $\tilde{\mathbf{N}}^{\mathcal{M}_{\{i\}}}(k)$ by starting it from the stationary distribution. Provided that this is done, the cycles in $\overline{S}^{(\mathcal{M}_{\{i\}},\{i\})}$, denoted as $T^{\mathcal{M}_{\{i\}}}(k)$, form a stationary and ergodic sequence (by similar arguments to the one presented in our papers [16, 28]). We set $T^{\mathcal{M}_{\{i\}}} = T^{\mathcal{M}_{\{i\}}}(1)$. More generally, in the following we denote by $T^{\mathcal{M}_{\mathcal{V}}}$ the steady-state cycle length in the system $\Theta^{\mathcal{M}_{\mathcal{V}},\mathcal{V}}$, provided that this system is stable.[2] Since the queue length of node i satisfies (3) and the cycle lengths are stationary, an application of Loynes' criterion [21] shows that the queue at node i is stable if

$$\lambda_i < \widehat{Q}_i / ET^{\mathcal{M}_{\{i\}}} \ .$$

which completes the construction of the stability region. □

In summary, system $\overline{\Theta}^{(\mathcal{M}_{\{i\}},\{i\})}$ is stable when

$$\boldsymbol{\lambda}^{\mathcal{M}} \in \overline{S}^{(\mathcal{M}_{\{i\}},\{i\})} = \left\{ \boldsymbol{\lambda} : \ \boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}},\{i\}} \ \text{ and } \lambda_i < \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\{i\}}}} \right\}.$$

Repeating the previous argument for all $i \in \mathcal{M}$, we finally have the following result.

Theorem 1 *Let*

$$S^{\mathcal{M}} = \bigcup_{i \in \mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_i} \in S^{\mathcal{M}_{\{i\}},\{i\}} \qquad \textit{and} \qquad \lambda_i < \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\{i\}}}} \right\}. \tag{5}$$

Then, system $\Theta^{\mathcal{M}}$ is stable if $\boldsymbol{\lambda} \in S^{\mathcal{M}}$.

Using the stability condition of the one-dimensional system as described above and iterating the recursive formula (5), we obtain a more explicit form for the stability region.

Corollary 1 *Let Σ be the set of permutations of the set $\mathcal{M} = \{1, 2, \ldots, M\}$, and let $\sigma = (\sigma(1), \ldots, \sigma(M)) \in \Sigma$. System $\Theta^{\mathcal{M}}$ is stable if $\boldsymbol{\lambda} \in S^{\mathcal{M}}$, with*

$$S^{\mathcal{M}} = \bigcup_{\sigma \in \Sigma} \left\{ \boldsymbol{\lambda} : \ \lambda_{\sigma(l)} < \frac{\widehat{Q}_{\sigma(l)}}{ET^{\mathcal{M}_{\sigma}(l)}}, \ \ l \in \mathcal{M} \right\}, \tag{6}$$

where $\mathcal{M}_{\sigma}(l) = \bigcup_{n=1}^{l-1} \{\sigma(n)\}$ (by convention $\bigcup_{n=1}^{0} \{\sigma(n)\} = \emptyset$) .

Theorem 1 provides sufficient conditions for the stability region of the M-dimensional system in terms of the sufficient conditions for the stability region (through $S^{\mathcal{M}_{\{i\}},\mathcal{U}\cup\{i\}}$) and the steady-state average cycle lengths (through $ET^{\mathcal{M}_{\{i\}}}$) of $M-1$-dimensional systems. As we will see below, with the exception of the boundaries, these conditions are also necessary.

We now start discussing the necessity of the conditions in (5). The following decomposition is crucial for the analysis. Its proof is presented in Section 4.

[2] Note that by definition, $T^{\emptyset}$ is the steady-state cycle length in the system $\Theta^{\emptyset,\mathcal{M}}$, that is in the system where all nodes behave like persistent nodes, i.e., node $i \in \mathcal{U} \cup \mathcal{M}$ generates $\widehat{Q}_i$ packets during a cycle. Therefore, in this case we have $T^{\emptyset} \equiv T(\widehat{\mathbf{Q}})$.

Proposition 3 *Let $S_c^{\mathcal{M}}$ be the complement of the stability region $S^{\mathcal{M}}$. Then, the following decomposition holds*

$$S_c^{\mathcal{M}} = \bigcup_{\mathcal{V} \subseteq \mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \quad , \quad \lambda_j \geq \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all } j \in \mathcal{V} \right\} \tag{7}$$

where $\mathcal{V}$ ranges over all nonempty subsets of $\mathcal{M}$.

In order to prove the necessary stability condition we need the following general result that is of its own interest. Its proof can be found in Section 4.

Proposition 4 *Let $\mathbf{X}^{\mathcal{M}}(n)$, $n = 1, \ldots,$ be an M-dimensional Markov chain (not necessarily denumerable). Assume that it is known that if the process starts from state $\mathbf{u} \in \Re^M$, then for all $i \in \mathcal{V} \subseteq \mathcal{M}$,*

$$\lim_{n \to \infty} X_i(n) = \infty.$$

Then, given any bounded one-dimensional set A, there is a state $\mathbf{c} \in \Re^M$ such that $c_i \notin A$ for all $i \in \mathcal{V}$ and

$$\Pr \{ X_i(n) \notin A, \ i \in \mathcal{V}, \ n \geq 1 \mid \mathbf{X}(1) = \mathbf{c} \} > 0 \ ,$$

that is, with positive probability all components of $\mathbf{X}(n)$ with indices belonging to $\mathcal{V}$ never return to the set A.

We are now ready to show that with the exception of the boundaries, condition (5) is necessary for the stability of the ring with spatial reuse. In addition, we provide a characterization of the instability region. Specifically, we show that with the exception of the boundaries, when the system is unstable, we can identify regions where some queues are sub-stable and the remaining queues tend to infinity with positive probability. Note that while it is easy to show that instability of one queue leads to instability of the whole system (for a formal proof see for example [28]), in general, instability of a multidimensional Markov chain, does not imply that at least one of the components converges to infinity. It is easy to construct multidimensional systems where fluctuations of the queue lengths between large and small values occur when the system is unstable (cf. [22]). Consider for example the case of two queues with packets of unit length, served by a single server and assume that the server serves exhaustively the queue that it visits. If $\lambda_i < 1$ for $i = 1, 2$ but $\lambda_1 + \lambda_2 > 1$, then the system is unstable while the queue sizes of both queues return to zero infinitely often with probability one, for all initial states.

The next theorem completes the construction of the stability region for the ring with spatial reuse.

Theorem 2 *System $\Theta^{\mathcal{M}}$ is unstable if $\boldsymbol{\lambda} \in \widehat{S}_c^{\mathcal{M}}$, where $\widehat{S}_c^{\mathcal{M}}$ is the complement of $S^{\mathcal{M}}$ minus the boundary points, that is,*

$$\widehat{S}_c^{\mathcal{M}} = \bigcup_{\mathcal{V} \subseteq \mathcal{M}} \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \quad , \quad \lambda_j > \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all} \quad j \in \mathcal{V} \right\} , \tag{8}$$

where $\mathcal{V}$ ranges over all nonempty subsets of $\mathcal{M}$. Furthermore, in the region

$$\widehat{S}_c^{\mathcal{M}}(\mathcal{V}) = \left\{ \boldsymbol{\lambda} : \quad \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \quad , \quad \lambda_j > \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \quad \text{for all} \quad j \in \mathcal{V} \right\} \tag{9}$$

all queues $j \in \mathcal{M}_{\mathcal{V}}$ are substable while all queues $i \in \mathcal{V}$ tend to infinity with positive probability.

Proof. Consider the dominant system $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}}, \mathcal{V})}$ and let $\boldsymbol{\lambda} \in \widehat{S}_c^{\mathcal{M}}(\mathcal{V})$. Since $\boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}}$, the queue lengths $\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k)$ constitute an ergodic Markov chain and starting from any state we have

$$\lim_{k \to \infty} \frac{\sum_{m=1}^{k} T_m^{\mathcal{M}_{\mathcal{V}}}}{k} = ET^{\mathcal{M}_{\mathcal{V}}}.$$

Since, in addition $\lambda_j > \widehat{Q}_j / ET^{\mathcal{M}_{\mathcal{V}}}$, an application of Loynes method [21] for instability shows that starting from any state, $\lim_{k \to \infty} \widetilde{N}_j(k) = \infty$ for all $j \in \mathcal{V}$. Setting $A = \left[0, \max_{j \in \mathcal{V}} \widehat{Q}_j\right]$ in Proposition 4, we conclude that there is a state $\mathbf{c} \in \Re^M$ such that if the process $\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k)$ starts from state $\mathbf{c}$, then there is a set of sample paths, Ω_c, of positive probability such that $\widetilde{N}_j(k) \geq \widehat{Q}_j, \; j \in \mathcal{V}$ for all $k = 1, 2, \ldots$. Observe now that by definition, on the set Ω_c the queues in the original system Θ and in the dominant system $\overline{\Theta}^{(\mathcal{M}_{\mathcal{V}}, \mathcal{V})}$ are identical and therefore $\lim_{k \to \infty} N_j(k) = \infty$ for all $j \in \mathcal{V}$. This implies that the Markov chain $\mathbf{N}^{\mathcal{M}_{\mathcal{V}}}(k)$ is transient. The fact that all queues in $\mathcal{M}_{\mathcal{V}}$ are substable follows directly from Proposition 2 and the ergodicity of $\widetilde{\mathbf{N}}^{\mathcal{M}_{\mathcal{V}}}(k)$. ■

We present next in some detail an example that illustrates the complications involved in the calculation of the exact stability region of the system and the strong dependence of the stability region on the distribution of the arrival rates.

Example 1. *Stability Region of a Two Node Ring with Quotas 1 and 2*

Consider the ring with $\mathcal{U} = \emptyset$, $\widehat{Q}_1 = 2$, $\widehat{Q}_2 = 1$ and $f_i(s) = s$, $i = 1, 2$ ($f_i(s)$ is defined in condition (A1) in Section 2). The stability region can be expressed as follows

$$\begin{aligned} S^{\{1,2\}} &= \left\{ \lambda_1 < \frac{2}{ET^{\{2\}}} \quad , \; \lambda_2 < \frac{1}{ET^{\emptyset}} \right\} \cup \left\{ \lambda_1 < \frac{2}{ET^{\emptyset}} \quad , \; \lambda_2 < \frac{1}{ET^{\{1\}}} \right\} \\ &= S_1^{\{1,2\}} \cup S_2^{\{1,2\}} \end{aligned} \tag{10}$$

where $T^{\{i\}}$ is the steady-state cycle length in a system with node $\{i\}$ being regular and the other one persistent.

Let us assume the simplest destination probabilities, namely $p_{12} = p_{21} = 1$. Then the computation of the first set on the right hand side in (10) is straightforward. Indeed, observe that by the choice of the destination probabilities, a node transmits in its outgoing link only packets originated at itself. The interaction between the two nodes in this case is due only to the fact that one node may have to wait until the other one completes transmission of its quota packets. Recalling the definitions after formula (1) we have,

$$H_1\left(Q_1(k), Q_2(k)\right) = Q_1(k), \quad H_2\left(Q_1(k), Q_2(k)\right) = Q_2(k),$$

where $H_i(Q_1(k), Q_2(k))$ represents the number of packets out of $Q_1(k) + Q_2(k)$ that will pass through the outgoing link of node i to reach their destination. If nodes 1 and 2 are persistent, we have $H_1(\widehat{Q}_1, \widehat{Q}_2) = 2$ and $H_2(\widehat{Q}_1, \widehat{Q}_2) = 1$. Therefore,

$$T^{\emptyset} = T(\widehat{Q}_1, \widehat{Q}_2) = \max\{2, 1\} = 2 \ .$$

If on the other hand node 1 is persistent while node 2 is regular, then since $Q_2(k) \leq \widehat{Q}_2 = 1$ we again have

$$ET^{\{2\}} = \lim_{k\to\infty} E\left\{\max\left\{H_1\left(\widehat{Q}_1, Q_2(k)\right), H_2\left(\widehat{Q}_1, Q_2(k)\right)\right\}\right\} = \lim_{k\to\infty} E\left\{\max\left\{2, Q_2(k)\right\}\right\} = 2.$$

Therefore,

$$S_1^{\{1,2\}} = \left\{\lambda_1 < \frac{2}{ET^{\{2\}}} \ \ , \ \lambda_2 < \frac{1}{ET^{\{\emptyset\}}}\right\} = \left\{\lambda_1 < 1 \ \ , \ \lambda_2 < 0.5\right\}.$$

We consider now the second set $S_2^{\{1,2\}}$. The quantity that needs to be determined in this case is the expected cycle length in steady-state, when node 2 is persistent and node 1 is regular, that is,

$$ET^{\{1\}} = \lim_{k\to\infty} E\left\{\max\left\{Q_1(k), 1\right\}\right\}. \tag{11}$$

Let $N_k = N_1(k)$ be the queue size at node 1 at the beginning of the kth cycle. Let $R(z)$ be the z-transform of $R_1(1)$, the number of arrivals to node 1 in the first slot (recall that we assume that $\{R_1(k)\}_{k=1}^{\infty}$ are i.i.d.). Let $l = \tau_k$, be the time when the kth cycle starts. Then, it is easy to see from the definition of $Q_1(k)$ that $\max\{Q_1(k), 1\} = 1$ when $N_k \leq 1$ and $\max\{Q_1(k), 1\} = 2$ otherwise. Therefore,

$$\begin{aligned} N_{k+1} &= (N_k - 2)^+ + R_1(l)\mathbf{1}_{\{N_k \leq 1\}} + (R_1(l) + R_1(l+1))\mathbf{1}_{\{N_k \geq 2\}} \\ &= (N_k - 2)^+ + R_1(l) + R_1(l)\mathbf{1}_{\{N_k \geq 2\}}. \end{aligned} \tag{12}$$

The sequence N_k, $k = 1, 2, \ldots$ constitutes a one-dimensional Markov chain which by construction is stable in the region $S_2^{\{1,2\}}$. Indeed, in this special case, the condition for stability of N_k is $\lambda_1 < 1$ which is guaranteed since by the definition of region $S_2^{\{1,2\}}$,

$$S_2^{\{1,2\}} = \left\{\lambda_1 < \frac{2}{ET^{\emptyset}},\ \lambda_2 < \frac{1}{ET^{\{1\}}}\right\} = \left\{\lambda_1 < 1,\ \lambda_2 < \frac{1}{ET^{\{1\}}}\right\} ,$$

and one must determine $ET^{\{1\}}$ to characterize the stability region.

In order to estimate $ET^{\{1\}}$, let π_n denote the steady-state probability that there are n packets in the queue of node 1 at the beginning of a cycle. Taking z-transforms in (12) and considering the steady-state, we have

$$\begin{aligned} N(z) &= R(z)\left(\Pr\{N \le 1\} + E\left(z^{N-2}|N \ge 2\right) R(z) \Pr\{N \ge 2\}\right) \\ &= R(z)\left(\pi_0 + \pi_1 + R(z)\sum_{n=2}^{\infty} \pi_n z^{n-2}\right) \\ &= (\pi_0 + \pi_1)R(z) + R^2(z)z^{-2}N(z) - \left(\pi_0 z^{-2} + \pi_1 z^{-1}\right) R^2(z), \end{aligned} \tag{13}$$

where $N(z)$ is the generating function of N_k in the steady-state. From the above we conclude that

$$N(z) = \frac{(\pi_0 + \pi_1)z^2 R(z) - (\pi_0 + \pi_1 z)R^2(z)}{z^2 - R^2(z)}. \tag{14}$$

Using standard arguments based on the analyticity of $N(z)$, we find that the probabilities $\pi_0,\ \pi_1$, are determined by the system of equations,

$$(2 - \lambda_1)\pi_0 + (1 - \lambda_1)\pi_1 = 2 - 2\lambda_1, \tag{15}$$

$$\pi_0(z_a + 1) + 2z_a\pi_1 = 0, \tag{16}$$

where z_a is the unique root in $[-1, 0]$ of the equation

$$R(z) = -z, \tag{17}$$

Since the cycle length is either 1 if there is 0 or 1 packet in the queue of node 1 at the beginning of a cycle, or 2 otherwise, we can easily compute the average steady-state cycle length as follows

$$ET^{\{1\}} = \pi_0 + \pi_1 + 2(1 - \pi_0 - \pi_1) = \frac{4z_a}{2z_a + (1 - \lambda_1)z_a - (1 - \lambda_1)} ,$$

and therefore we finally obtain

$$S_2^{\{1,2\}} = \left\{\lambda_1 < 1,\ \lambda_2 < \frac{2z_a + (1 - \lambda_1)z_a - (1 - \lambda_1)}{4z_a}\right\}$$

Note that one should not conclude that in the previous formula the limit of

$$\frac{2z_a + (1 - \lambda_1)z_a - (1 - \lambda_1)}{4z_a}$$

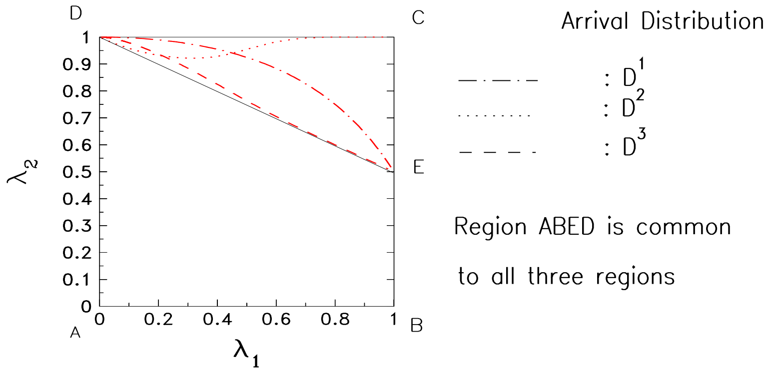

Figure 1: Stability region for the ring of Example 1.

as $\lambda_1 \to 1$, is $\frac{1}{2}$. In fact, z_a depends implicitly on λ_1 and therefore, other limits are also possible.

The root z_a of (17) depends on the distribution of the arrival process to node 1 and as a result the same is true for the stability region. To demonstrate this strong dependence, we plotted in Figure 1 the stability regions for the following arrival distributions to node 1:

1. $D^1 = \{\Pr\{R_1(1) = 0\}, \Pr\{R_1(1) = 1\}, \Pr\{R_1(1) = 2\}\} = \{\alpha^2,\ \ 2\alpha(1-\alpha),\ \ (1-\alpha)^2\}$, $0 \le \alpha \le 1$. This is binomial with parameters $(\alpha, 2)$;

2. $D^2 = \{1 - \alpha^2 - \alpha e^{-1/(1-\alpha)},\ \ \alpha^2,\ \ \alpha e^{-1/(1-\alpha)}\},\ \ 0 \le \alpha \le 1;$

3. $D^3 = \{1 - \alpha e^{-1/(1-\alpha)} - 0.5\alpha^2,\ \ \alpha e^{-1/(1-\alpha)},\ \ 0.5\alpha^2\},\ \ 0 \le \alpha \le 1.$

Note the strange at first sight behavior of the region corresponding to distribution D^2. This curve implies that for certain values of (λ_1, λ_2) it is possible to make an unstable system stable by keeping λ_2 constant and *increasing* λ_1. The physical explanation of this behavior is that as $\lambda_1 \to 1$, the probability that a single packet arrives at a slot dominates quickly over the probability that 2 packets arrive at a slot. This results in the queue at node 1 to be more likely of size 1, and therefore the cycle lengths are also more likely to be 1 than 2. But then there are fewer slots wasted by node 2 waiting for node 1 to complete its quota during a cycle and therefore the unstable queue at node 2 may become stable.

As one can observe, the region $ABED$ is common for the three arrival distributions. However, the rest of the region depends strongly on the arrival distribution. From (17) it can be seen that if the number of arrivals in a slot is always even, that is, if $\Pr\{R_1(1) = 2k+1\} = 0$ for all integer $k \geq 1$, then $z_a = -1$ and the stability region is $ABED$. On the other hand, when the number of arrivals during a slot is either 0 or 1, i.e., $\Pr\{R_1(1) = 0\} + \Pr\{R_1(1) = 1\} = 1$, it can be easily seen that the stability region is $ABCD$. As we will see in the next section, the region $ABED$ is a subset of the stability region for *any* arrival distribution. □

The previous example also shows the price that has to be paid in order to achieve fairness with the quota mechanism. The maximal stability region of the ring with spatial reuse (i.e., the region inside which there is always at least one policy that can stabilize the system) is determined by (cf. [15, 18])

$$S = \{\boldsymbol{\lambda} : \sum_{i=1}^{M} \lambda_i a_{ij} < 1 \text{ for } \; j \in \mathcal{M}\}$$

where $a_{ij} = \Pr\{\text{a packet generated by node } i \text{ has to cross node } j\}$. In [18] we presented a policy whose stability region is S. Under the latter policy, node are assigned quota dynamically by setting $Q_i(k) = N_i(k)$ (i.e., at the beginning of the kth cycle the quota assigned to node i is equal to the queue length in this node at the beginning of the cycle). In Example 1, region S corresponds to the area $ABCD$. We see that the stability region under the fixed quota policy is a strict subset of S. It should be mentioned, however, that under the policy that dynamically adapts the node quota, an overloaded node will cause an overload to all other nodes, a situation that does not occur under the fixed quota policy.

3.2 Bounds on the Stability Region Through Linear Programming

Example 1 demonstrates that even in the simplest case the stability region of the system depends strongly on the distribution of the arrival process. While in this case the computations are feasible, as the number of nodes and/or the quota sizes increase the computation of the exact stability region quickly becomes intractable.

The strong dependence of the stability region on the arrival rate distribution as well as the steady-state average cycle lengths of lower dimensional systems, makes it worthwhile to search for the following regions of arrival rates.

Essential Stability Region (ESR): The set of arrival rates $\boldsymbol{\lambda}^{\mathcal{M}}$ with the property that the system is stable under *any* arrival distributions as long as the nodes have the corresponding arrival rates $\boldsymbol{\lambda}^{\mathcal{M}}$.

Absolute Stability Region (ASR): The set of arrival rates $\boldsymbol{\lambda}^{\mathcal{M}}$ with the property that there is *at least one* set of node arrival distributions with corresponding rates $\boldsymbol{\lambda}^{\mathcal{M}}$ such that the system is stable.

The ESR is the intersection of the stability regions under all arrival distribution while the ASR is the union of these stability regions. Clearly, ESR$\subseteq$ASR. The ESR is useful in situations where the arrival distributions are not known *a priori*, a common situation in many practical systems. Besides the theoretical interest of the ASR as the region outside which the system cannot be stabilized under any arrival distribution with the given arrival rates, it might also have practical implications when the input traffic can be controlled before entering the network.

Based on Theorem 1 and Corollary 1, we will now develop bounds for the ESR and ASR respectively, that depend only on the *conditional average* cycle lengths $ET(\mathbf{q})$ which are easier to compute than the steady-state average cycle lengths appearing in Theorem 1. We must emphasize that the conditional average cycle lengths are fundamental quantities which can be computed without resorting to steady-state quantities. The computation of $ET(\mathbf{q})$ depends only on the packet destination probabilities p_{ij} and $\mathbf{q}$, and for small number of nodes can be estimated directly based on (1). For large number of nodes and large quota, computing even $ET(\mathbf{q})$ is not easy, however, asymptotic results for these quantities exist [15]. The bounds are derived by associating the stability of the system to a solution of some linear programming optimization problems whose constraints are derived from the flow balance equations.

Our first goal is to find an upper bound on the average steady-state cycle length $ET^{\mathcal{G}}$ in system $\overline{\Theta}^{(\mathcal{G},\mathcal{V})}$ where (for simplicity of notations we set) $\mathcal{G} = \mathcal{M}_{\mathcal{V}}$, that is independent of the arrival distribution. As will be seen, this leads to a subset of the ESR. When $\boldsymbol{\lambda} \in S^{\mathcal{G}}$, then by definition the nodes in the set $\mathcal{G}$ constitute a stable system. Let $\pi(\mathbf{n}),\ \mathbf{n} = \{n_j,\ j \in \mathcal{G}\}$ be the steady-state probability of the process of node queue lengths at the beginning of a cycle and for $l \in \mathcal{G}$ define a one dimensional distribution as

$$\pi_l(n) = \sum_{\mathbf{n},\ n_l = n} \pi(\mathbf{n}). \tag{18}$$

Standard arguments based on the regenerative theorem can be used (see e.g., [2, 16] for similar results) to show that the following flow equations are satisfied for the system consisting of the nodes in $\mathcal{G}$

$$\lambda_l ET^{\mathcal{G}} = \sum_{n=0}^{\infty} q_l(n)\pi_l(n), \qquad l \in \mathcal{G}, \tag{19}$$

where $q_l(n) = \min\left\{f_l(n), \widehat{Q}_l\right\}$, is the number of local packets transmitted by node l in a cycle when the number of packets at that node at the beginning of the cycle is n. Equation (19) states simply that in steady-state the average number of arrivals to node l in a cycle is equal to the average number of external packets transmitted by node l during a cycle.

Let now $ET^{\mathcal{G}}(\mathbf{q})$, $\mathbf{q} = \{q_j : j \in \mathcal{G}\}$ be the conditional average cycle length in system $\overline{\Theta}^{(\mathcal{G},\mathcal{V})}$ when node $j \in \mathcal{G}$ transmits q_j packets in a cycle (and by definition a node $j \in \mathcal{U} \cup \mathcal{V}$ transmits $\widehat{Q}_j$ packets). Then, the average steady-state cycle length satisfies,

$$ET^{\mathcal{G}} = \sum_{\mathbf{n}} ET^{\mathcal{G}}(\mathbf{q}(\mathbf{n}))\pi(\mathbf{n}), \tag{20}$$

where $\mathbf{q}(\mathbf{n}) = \{q_j(n_j),\ j \in \mathcal{G}\}$. Define next for $\mathbf{m} = \{m_j,\ j \in \mathcal{G}\}$, $0 \le m_j \le \widehat{Q}_j$,

$$x(\mathbf{m}) = \begin{cases} \pi(\mathbf{m}) & \text{if } \ 0 \le m_j \le \widehat{Q}_j - 1, \ \text{ for all } \ j \in \mathcal{G} \\ \sum_{m_j \ge \widehat{Q}_j,\, j\in\mathcal{K}} \pi(\mathbf{m}) & \text{if } \ m_j = \widehat{Q}_j,\ j \in \mathcal{K} \subseteq \mathcal{G} \ \text{ and } \ 0 \le m_j \le \widehat{Q}_j - 1,\ \ j \in \mathcal{G}_{\mathcal{K}}. \end{cases}$$

In terms of these variables, and based on the fact that $q_j(n_j) = \widehat{Q}_j$ when $n_j \ge \widehat{Q}_j$, we can rewrite the equations in (19) and (20) as follows.

$$\left.\begin{array}{rcl} \lambda_l \sum_{\mathbf{m}} ET^{\mathcal{G}}(\mathbf{q}(\mathbf{m}))x(\mathbf{m}) & = & \sum_{n=0}^{\widehat{Q}_l} q_l(n) \sum_{\mathbf{m},\, m_l=n} x(\mathbf{m}), \quad l \in \mathcal{G} \\ \sum_{\mathbf{m}} x(\mathbf{m}) & = & 1 \\ x(\mathbf{m}) & \ge & 0 \end{array}\right\}, \tag{21}$$

where $\mathbf{m} = \{m_j,\ j \in \mathcal{G}\}$, $0 \le n_j \le \widehat{Q}_j$.

¿From the above discussion we see that with every partition $(\mathcal{G},\mathcal{V})$ of $\mathcal{M}$ we can associate a polytope $\wp^{(\mathcal{G},\mathcal{V})}$ defined by the constraints in (21). Let us define $T^{\mathcal{G}}_{\max}$ as the solution of the following linear programming optimization problem

$$T^{\mathcal{G}}_{\max} = \max_{x(\mathbf{m}) \in \wp^{(\mathcal{G},\mathcal{V})}} \left\{ \sum_{\mathbf{m}} ET^{\mathcal{G}}(\mathbf{q}(\mathbf{m}))x(\mathbf{m}) \right\}, \tag{22}$$

Notice that the solution to this optimization problem requires only the knowledge of the values of the *conditional average* cycle lengths which are usually easier to compute than the corresponding steady-state quantities.

Using the notation from Corollary 1, let us define the following two regions

$$\begin{array}{c} S^{\mathcal{M}}_{\sigma} = \left\{ \boldsymbol{\lambda} : \ \lambda_{\sigma(l)} < \frac{\widehat{Q}_{\sigma(l)}}{ET^{\mathcal{M}_{\sigma}(l)}},\ l \in \mathcal{M} \right\} \\ L^{\mathcal{M}}_{\sigma} = \left\{ \boldsymbol{\lambda} : \ \lambda_{\sigma(l)} < \frac{\widehat{Q}_{\sigma(l)}}{T^{\mathcal{M}_{\sigma}(l)}_{\max}},\ l \in \mathcal{M} \right\} \end{array} \tag{23}$$

But, from the definition of $T^{\mathcal{G}}_{\max}$ we have

$$T^{\mathcal{M}_{\sigma}(l)}_{\max} \ge ET^{\mathcal{M}_{\sigma}(l)}, \quad l \in \mathcal{M},$$

which implies that for any permutation $\sigma(\cdot)$,

$$L_\sigma^{\mathcal{M}} \subseteq S_\sigma^{\mathcal{M}}. \tag{24}$$

Defining next

$$T_{\min}^{\mathcal{G}} = \min_{x(\mathbf{n}) \in \wp(\mathcal{G},\mathcal{V})} \left\{ \sum_{\mathbf{n}} ET^{\mathcal{G}}(\mathbf{q}(\mathbf{n}))x(\mathbf{n}) \right\}, \tag{25}$$

and using similar arguments we have that

$$S_\sigma^{\mathcal{M}} \subseteq U_\sigma^{\mathcal{M}},$$

where

$$U_\sigma^{\mathcal{M}} = \left\{ \boldsymbol{\lambda} : \ \ \lambda_{\sigma(l)} < \frac{\widehat{Q}_{\sigma(l)}}{T_{\min}^{\mathcal{M}_\sigma(l)}}, \ l \in \mathcal{M} \right\}. \tag{26}$$

In conclusion we have the following theorem.

Theorem 3 *Let us define $T_{\max}^{\mathcal{G}}$ and $T_{\min}^{\mathcal{G}}$ as in (22) and (25) respectively. Then, we obtain a lower bound $L^{\mathcal{M}}$ on the stability region $S^{\mathcal{M}}$ as follows*

$$L^{\mathcal{M}} = \bigcup_\sigma L_\sigma^{\mathcal{M}} \subseteq \bigcup_\sigma S_\sigma^{\mathcal{M}} = S^{\mathcal{M}},$$

where $L_\sigma^{\mathcal{M}}$ is defined in (23). Similarly, an upper bound is,

$$S^{\mathcal{M}} = \bigcup_\sigma S_\sigma^{\mathcal{M}} \subseteq \bigcup_\sigma U_\sigma^{\mathcal{M}} = U^{\mathcal{M}},$$

where $U_\sigma^{\mathcal{M}}$ is defined in (26).

Since by construction any arrival vector $\boldsymbol{\lambda}$ that belongs to $L^{\mathcal{M}}$ results in a stable system, we conclude that $L^{\mathcal{M}}$ is in fact a subset of ESR. Similarly, $U^{\mathcal{M}}$ is a superset of ASR.

Example 2. *Again the Ring From Example 1.*

Consider the ring from Example 1. Referring to Figure 1, it can be easily checked that in this case the lower bound on the ESR is the region ABED, while the upper bound is the region ABCD. For the same ring, assume now that the destination probabilities are $p_{12} = p_{21} = 0.75$. Referring to Figure 2, the bounds on the ESR and ASR are the regions ABED and ABEFD respectively. For arrivals rates in ABED the system is stable irrespective of the distribution of the arrivals. For rates outside the region ABEFD, there is no distribution of arrivals that can stabilize the system. As we will see in the next section, the region ABED is in fact the ESR for the system with two nodes. □

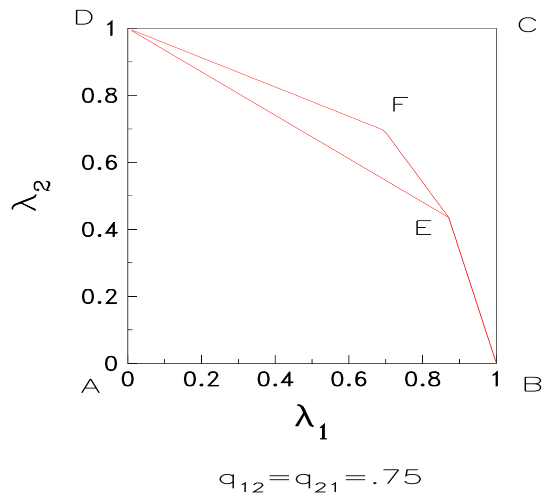

Figure 2: ESR and ASR bounds.

3.3 ESR for the Ring with Two Nodes

In this section we derive explicitly the ESA region for two-node ring network , that is, $\mathcal{M} = \{1, 2\}$ and $\mathcal{U} = \emptyset$, with $f_i(s) = s$ (the most interesting case in practice) and arbitrary quotas sizes. Although in this case the ESA is simply the union of two polytopes, the exact calculation still requires the computation of the conditional average cycle lengths, not an easy task in general.

To establish just announced result, we need the following lemma that we prove in the Appendix:

Lemma 1 *The functions*

$$\varphi(n) := \frac{ET(\widehat{Q}_1, n) - ET(\widehat{Q}_1, 0)}{n}, \qquad 1 \le n \le \widehat{Q}_2,$$

$$\psi(n) := \frac{ET(n, \widehat{Q}_2) - ET(0, \widehat{Q}_2)}{n}, \qquad 1 \le n \le \widehat{Q}_1,$$

are nondecreasing. ■

Based on Lemma 1 we can now determine the ESR for the ring with two nodes.

Theorem 4 *The Essential Stability Region for system $\Theta^{\{1,2\}}$ coincides with the lower bound in Theorem 3 and is given by*

$$L = \left\{ \lambda_1 < \frac{\widehat{Q}_1}{ET(\widehat{Q}_1,\widehat{Q}_2)},\ \left(\frac{ET(\widehat{Q}_1,\widehat{Q}_2)}{\widehat{Q}_1} - \frac{\widehat{Q}_2}{\widehat{Q}_1}\right)\lambda_1 + \lambda_2 < 1, \right\} \tag{27}$$

$$\cup \left\{ \lambda_2 < \frac{\widehat{Q}_2}{ET(\widehat{Q}_1,\widehat{Q}_2)},\ \lambda_1 + \left(\frac{ET(\widehat{Q}_1,\widehat{Q}_2)}{\widehat{Q}_2} - \frac{\widehat{Q}_1}{\widehat{Q}_2}\right)\lambda_2 < 1, \right\} \tag{28}$$

Proof. The subset of the lower bound in Theorem 3 determined by the permutation $\sigma(1) = 1,\ \sigma(2) = 2$ is

$$L_\sigma^{\{1,2\}} = \left\{ \lambda_1 < \frac{\widehat{Q}_1}{ET(\widehat{Q}_1,\widehat{Q}_2)},\ \ \lambda_2 < \frac{\widehat{Q}_2}{T_{\max}^{\{1\}}} \right\},$$

where

$$T_{\max}^{\{1\}} = \max_{\{x(n)\}\in\wp^{(\{1\},\{2\})}} \left\{ \sum_{n=1}^{\widehat{Q}_1} ET(n,\widehat{Q}_2)x(n) \right\},$$

and the polytope $\wp^{(\{1\},\{2\})}$ is defined by the constraints

$$\left.\begin{array}{rcl} \lambda_1 \sum_{n=0}^{\widehat{Q}_1} ET(n,\widehat{Q}_2)x(n) & = & \sum_{n=0}^{\widehat{Q}_1} nx(n), \\ \sum_{n=0}^{\widehat{Q}_1} x(n) & = & 1 \\ x(n) & \geq & 0,\ \ 0 \leq n \leq \widehat{Q}_1. \end{array}\right\} \tag{29}$$

We will show that the solution to the above maximization problem is obtained at the point x^* defined as $x^*(n) = 0,\ \ 1 \leq n \leq \widehat{Q}_1 - 1$, and

$$x^*(0) = \frac{\widehat{Q}_1 - ET(\widehat{Q}_1,\widehat{Q}_2)\lambda_1}{\widehat{Q}_1 + \widehat{Q}_2\lambda_1 - ET(\widehat{Q}_1,\widehat{Q}_2)\lambda_1}, \tag{30}$$

$$x^*(\widehat{Q}_1) = \frac{\widehat{Q}_2\lambda_1}{\widehat{Q}_1 + \widehat{Q}_2\lambda_1 - ET(\widehat{Q}_1,\widehat{Q}_2)\lambda_1}. \tag{31}$$

It will follow that

$$T_{\max}^{\{1\}} = ET(0,\widehat{Q}_2)x^*(0) + ET(\widehat{Q}_1,\widehat{Q}_2)x^*(\widehat{Q}_1) = \frac{\widehat{Q}_1\widehat{Q}_2}{\widehat{Q}_1 + \widehat{Q}_2\lambda_1 - ET(\widehat{Q}_1,\widehat{Q}_2)\lambda_1},$$

and therefore,

$$\lambda_2 < \frac{\widehat{Q}_2}{T_{\max}^{\{1\}}} = 1 + \lambda_1\frac{\widehat{Q}_2}{\widehat{Q}_1} - \frac{ET(\widehat{Q}_1,\widehat{Q}_2)}{\widehat{Q}_1}\lambda_1 \tag{32}$$

which is equivalent to the second inequality in (27).

Since entirely analogous arguments hold for the permutation $\sigma(1) = 2,\ \sigma(2) = 1$, we conclude that region L is a subset of ESR. To show that it is indeed equal to the ESR, it is sufficient to provide arrival distributions under which the described region is actually the stability region of the ring. But this can easily be done, by considering that the number of packets arriving at node i in a slot is either 0 or $\widehat{Q}_i$. In this case, the number of packets at node i at the beginning of a cycle is either 0 or a multiple of $\widehat{Q}_i$. For the permutation $\sigma(1) = 1$, $\sigma(2) = 2$, this implies that $\pi_1(n) = 0$, $1 \le n \le \widehat{Q}_1 - 1$. But since the variables $x(n) = \pi_1(n)$, $0 \le n \le \widehat{Q}_1 - 1$, and $x(\widehat{Q}_1) = \sum_{n=\widehat{Q}_1}^{\infty} \pi_1(n)$ have to satisfy the flow equations for node 1, i.e., constraints (29), we conclude that $\pi_1(0) = x^*(0)$ and $\sum_{n=\widehat{Q}_1}^{\infty} \pi_1(n) = x^*(\widehat{Q}_1)$. It follows that $T_{\max}^{\{1\}} = ET^{\{1\}}$, which implies that the ESR for this system is the one described in the theorem.

We now show that x^* is the solution to the maximization problem (29) by considering the associated Kuhn-Tucker conditions (cf. [20]). We need to show the existence of unique $u_n \ge 0$, $0 \le n \le \widehat{Q}_1$, (inequality constraints) and ℓ_1, ℓ_2 (equality constraints) such that

1. $u_n x^*(n) = 0$
2. $-ET(n, \widehat{Q}_2) - u_n + \ell_1(\lambda_1 ET(n, \widehat{Q}_2) - n) + \ell_2 = 0$.

Since $\lambda_1 < (\widehat{Q}_1 / ET(\widehat{Q}_1, \widehat{Q}_2)$, it follows that $x^*(0) > 0$, $x^*(\widehat{Q}_1) > 0$ and therefore, $u_0 = u_{\widehat{Q}_1} = 0$. This implies that ℓ_i, $i = 1, 2$ are determined uniquely by the solution of the system:

$$\begin{array}{rcccl} \lambda_1 ET(0, \widehat{Q}_2)\ell_1 & + & \ell_2 & = & ET(0, \widehat{Q}_2) \\ \left(\lambda_1 ET(\widehat{Q}_1, \widehat{Q}_2) - \widehat{Q}_1\right)\ell_1 & + & \ell_2 & = & ET(\widehat{Q}_1, \widehat{Q}_2). \end{array} \tag{33}$$

Next, u_n, $0 < n < \widehat{Q}_1$, are determined from

$$\left(\lambda_1 ET(n, \widehat{Q}_2) - n\right)\ell_1 + \ell_2 = u_n + ET(n, \widehat{Q}_2) \tag{34}$$

Substituting the values of ℓ_i determined from (33) in (34), we find that the condition that the u_n are nonnegative is equivalent to the condition

$$\frac{ET(Q_1, Q_2) - ET(0, Q_2)}{Q_1} \ge \frac{ET(n, Q_2) - ET(0, Q_2)}{n}, \; 0 < n < Q_1. \tag{35}$$

The truth of (35) follows from Lemma 1. ∎

4 Proof of Instability Results

In this section we prove two auxiliary results, namely, Proposition 3 (cf. Subsection 4.1) and Proposition 4 (cf. Subsection 4.2) that are crucial for our main instability result, namely Theorem 2. These propositions allow us to conclude the instability of the system from the instability of the dominant one and may be useful in other situation as well.

4.1 The Decomposition Result

We start with the decomposition formula (7) of Proposition 3. First, we need two simple facts.

Lemma 2 *Let $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{M}$. Then,*

$$ET^{\mathcal{M}_{\mathcal{V}_2}} \geq ET^{\mathcal{M}_{\mathcal{V}_1}}.$$

Proof. The proof follows directly from Propositions 1 and 2. ∎

Lemma 3 *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subsets of $\mathcal{M}$ such that $\mathcal{V}_1 - \mathcal{V}_2 \neq \emptyset$. Then,*

$$B_{\mathcal{V}_1} \cap S^{\mathcal{M}_{\mathcal{V}_2}} = \emptyset,$$

where

$$B_{\mathcal{V}_1} = \left\{ \boldsymbol{\lambda} : \ \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}_1}} \in S^{\mathcal{M}_{\mathcal{V}_1}}, \ \lambda_j \geq \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}_1}}} \ , \ j \in \mathcal{V}_1 \right\}.$$

Proof. Let $\mathcal{V}_0 = \mathcal{V}_1 - \mathcal{V}_2$. By (6) of Corollary 1 we can write $S^{\mathcal{M}_{\mathcal{V}_2}}$ as a union of sets,

$$S^{\mathcal{M}_{\mathcal{V}_2}} = \bigcup_{\sigma} C_\sigma,$$

where each C_σ contains the following constraint for some $i \in \mathcal{V}_0$

$$\lambda_i < \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\mathcal{V}}}}, \tag{36}$$

where $\mathcal{V} \supseteq \mathcal{V}_1 \cup \mathcal{V}_2$. But, since by Lemma 2, $ET^{\mathcal{M}_{\mathcal{V}}} \geq ET^{\mathcal{M}_{\mathcal{V}_1}}$, constraint (36) contradicts the constraint

$$\lambda_i \geq \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\mathcal{V}_1}}}$$

which all rate vectors in $B_{\mathcal{V}_1}$ must satisfy. ∎

Now we are ready to establish our decomposition formula (7).

Proposition 3 *Let $S_c^{\mathcal{M}}$ be the complement of the stability region $S^{\mathcal{M}}$. Then, the following decomposition holds*

$$S_c^{\mathcal{M}} = \bigcup_{\mathcal{V} \subseteq \mathcal{M}} \left\{ \boldsymbol{\lambda} : \ \ \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \ \ , \ \ \lambda_j \geq \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \ \ \text{for all} \ \ j \in \mathcal{V} \right\} \ .$$

where $\mathcal{V}$ ranges over all nonempty subsets of $\mathcal{M}$.

Proof. Let Φ_n be the set of all subsets $\mathcal{V}$ of $\mathcal{M}$ with cardinality $|\mathcal{V}| = n \leq M$ and $\overline{\Phi}_n = \bigcup_{k=1}^{n} \Phi_k$. We will show that we can write

$$S_c^{\mathcal{M}} = \left(\bigcap_{\mathcal{V} \in \Phi_n} S_c^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \right) \bigcup \left(\bigcup_{\mathcal{V} \in \overline{\Phi}_n} B_{\mathcal{V}} \right), \tag{37}$$

where

$$B_{\mathcal{V}} = \left\{ \boldsymbol{\lambda} : \ \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}}} \in S^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}}, \ \ \lambda_j \geq \frac{\widehat{Q}_j}{ET^{\mathcal{M}_{\mathcal{V}}}} \text{ for all } \ j \in \mathcal{V} \right\}.$$

Notice that setting $n = M$ in (37) is equivalent to the desired result. The proof of (37) will be by induction on n.

For $n = 1$, taking complements of (5) in Theorem 1 we have

$$S_c^{\mathcal{M}} = \bigcap_{i \in \mathcal{M}} \{D_{i,1} \cup D_{i,2}\} = \bigcup_{\mathbf{s}} \left\{ \bigcap_{i \in \mathcal{M}} D_{i,s_i} \right\} = \left\{ \bigcap_{i \in \mathcal{M}} D_{i,1} \right\} \bigcup_{\mathbf{s} \neq \mathbf{1}} \left\{ \bigcap_{i \in \mathcal{M}} D_{i,s_i} \right\}, \tag{38}$$

where $\mathbf{s} = (s_1, \cdots, s_M)$, $s_i = 1$ or 2, and $\mathbf{1} = (1, 1, \ldots, 1)$,

$$D_{i,1} = S_c^{\mathcal{M}_{\{i\}}, \{i\}}$$

and

$$D_{i,2} = \left\{ \boldsymbol{\lambda} : \ \boldsymbol{\lambda}^{\mathcal{M}_{\{i\}}} \in S^{\mathcal{M}_{\{i\}}, \{i\}}, \ \lambda_i \geq \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\{i\}}}} \right\}.$$

Now, by Lemma 3 we have that if $i \neq j$, then $D_{i,2} \cap S^{\mathcal{M}_{\{j\}}, \{j\}} = \emptyset$. This implies that for $i \neq j$, $D_{i,2} \cap D_{j,2} = \emptyset$ and $D_{i,2} \cap D_{j,1} = D_{i,2}$. This in turn implies that if $s_i = 2$ and $s_j = 1$ for all $j \neq i$, then $\bigcap_i D_{i,s_i} = D_{i,2}$ while if $s_i = 2$ and $s_j = 2$ for at least one $j \neq i$, then $\bigcap_i D_{i,s_i} = \emptyset$. Therefore, we can write (38) as follows.

$$S_c^{\mathcal{M}} = \left(\bigcap_{i \in \mathcal{M}} D_{i,1} \right) \cup \left(\bigcup_{i \in \mathcal{M}} D_{i,2} \right)$$

The last equality is equivalent to (37) for $n = 1$.

Assume now that (37) is true for $n < M$. We will show that

$$\bigcap_{\mathcal{V} \in \Phi_n} S_c^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} = \left(\bigcap_{\mathcal{V} \in \Phi_{n+1}} S_c^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} \right) \cup \left(\bigcup_{\mathcal{V} \in \Phi_{n+1}} B_{\mathcal{V}} \right)$$

which implies (37) for $n+1$. Exactly as in the case $n = 1$, we can write

$$S_c^{\mathcal{M}_{\mathcal{V}}, \mathcal{V}} = E_{\mathcal{V},1} \cup E_{\mathcal{V},2},$$

where

$$E_{\mathcal{V},1} = \bigcap_{i \in \mathcal{M}_{\mathcal{V}}} S_c^{\mathcal{M}_{\mathcal{V} \cup \{i\}}, \{i\} \cup \mathcal{V}}$$

and

$$E_{\mathcal{V},2} = \bigcup_{i \in \mathcal{M}_{\mathcal{V}}} \left\{ \boldsymbol{\lambda} : \ \boldsymbol{\lambda}^{\mathcal{M}_{\mathcal{V}\cup\{i\}}} \in S^{\mathcal{M}_{\mathcal{V}\cup\{i\}},\{i\}\cup\mathcal{V}} \ , \ \lambda_i \geq \frac{\widehat{Q}_i}{ET^{\mathcal{M}_{\mathcal{V}\cup\{i\}}}} \right\}.$$

Therefore, we again have,

$$\bigcap_{\mathcal{V}\in\Phi_n} S_c^{\mathcal{M}_{\mathcal{V}},\mathcal{V}} = \bigcup_{\mathbf{s}} \left\{ \bigcap_{\mathcal{V}\in\Phi_n} E_{\mathcal{V},s_{m(\mathcal{V})}} \right\} = \left\{ \bigcap_{\mathcal{V}\in\Phi_n} E_{\mathcal{V},1} \right\} \bigcup_{\mathbf{s}\neq\mathbf{1}} \left\{ \bigcap_{\mathcal{V}\in\Phi_n} E_{\mathcal{V},s_{m(\mathcal{V})}} \right\},$$

where $\mathbf{s} = (s_1, \ldots, s_{\binom{M}{n}})$, $s_i = 1$ or 2 and $m(\mathcal{V})$ is a one-to-one mapping from Φ_n to $\left\{1, \cdots, \binom{M}{n}\right\}$. It is easy to see that

$$\bigcap_{\mathcal{V}\in\Phi_n} E_{\mathcal{V},1} = \bigcap_{\mathcal{V}\in\Phi_{n+1}} S_c^{\mathcal{M}_{\mathcal{V}},\mathcal{V}}.$$

It remains to show that

$$\bigcup_{\mathbf{s}\neq\mathbf{1}} \left\{ \bigcap_{\mathcal{V}\in\Phi_n} E_{\mathcal{V},s_{m(\mathcal{V})}} \right\} = \bigcup_{\mathcal{V}\in\Phi_{n+1}} B_{\mathcal{V}}$$

For this we can use arguments similar to the case $n = 1$ after observing that for $\mathcal{V} \in \Phi_{n+1}$

$$\bigcap_{i\in\mathcal{V}} E_{\mathcal{V}-\{i\},2} = B_{\mathcal{V}}.$$

which completes the proof. ∎

4.2 A General Result For Unstable Markov Chains

Here we establish Proposition 4 concerning the probabilistic behavior of an unstable Markov chain. For convenience, we repeat below the proposition.

Proposition 4 *Let $\mathbf{X}^{\mathcal{M}}(n)$, $n = 1, \ldots,$ be an M-dimensional Markov chain (not necessarily denumerable). Assume that it is known that if the process starts from state $\mathbf{u} \in \Re^M$, then for all $i \in \mathcal{V} \subseteq \mathcal{M}$,*

$$\lim_{n\to\infty} X_i(n) = \infty.$$

Then, given any bounded one-dimensional set A, there is a state $\mathbf{c} \in \Re^M$ such that $c_i \notin A$ for all $i \in \mathcal{V}$ and

$$\Pr\{X_i(n) \notin A, \ i \in \mathcal{V}, \ n \geq 1 \mid \mathbf{X}(1) = \mathbf{c}\} > 0 \ ,$$

that is, with positive probability all components of $\mathbf{X}(n)$ with indices belonging to $\mathcal{V}$ never return to the set A.

Proof. Let $B = \left\{\mathbf{s} \in \Re^M : \ s_i \in A, \text{ for some } i \in \mathcal{V}\right\}$. Assume that

$$\Pr\{\mathbf{X}(n) \notin B, \text{ for all } n \geq 1 \mid \mathbf{X}(1) = \mathbf{s}\} = 0$$

for all states $\mathbf{s} \notin B$. This implies that for *all* states $\mathbf{s}$,

$$\Pr\{\mathbf{X}(n) \in B, \text{ for some } \ n \geq 1 \mid \mathbf{X}(1) = \mathbf{s}\} = 1. \tag{39}$$

We will show now that (39) implies that for any state $\mathbf{s}$,

$$\Pr\{\mathbf{X}(n) \in B, \text{ i.o.} \mid \mathbf{X}(1) = \mathbf{s}\} = 1.$$

Let $C_l = \{\omega : \mathbf{X}(n) \in B \text{ for at least } l \text{ times}\}$. Since $\{\mathbf{X}(n) \in B, \text{ i.o.}\} = \bigcap_{l=1}^{\infty} C_l$, we have that

$$\Pr\{X_i(n) \in B, \text{ i.o.} \mid \mathbf{X}(1) = \mathbf{s}\} = \lim_{l \to \infty} \Pr\{C_l \mid \mathbf{X}(1) = \mathbf{s}\}.$$

Therefore, it suffices to show that $\Pr\{C_l \mid \mathbf{X}(1) = \mathbf{s}\} = 1, \ l \geq 1$. From (39) we see that this is true for $l = 1$. Assume now that it is true for l. Define the random time T as the first time that the process visits the set B for the lth time. Since $\Pr\{C_l \mid \mathbf{X}(1) = \mathbf{s}\} = 1$, we conclude that T is finite almost surely. Therefore,

$$\begin{aligned} \Pr\{C_{l+1} \mid \mathbf{X}(1) = \mathbf{s}\} \ &= \ \Pr\{\mathbf{X}(T+n) \in B \text{ for some } n \geq 1 \mid \mathbf{X}(1) = \mathbf{s}\} \\ &= \int \Pr\{\mathbf{X}(T+n) \in B \text{ for some } n \geq 1 \mid \mathbf{X}(T+1) = \mathbf{z}, \ \mathbf{X}(1) = \mathbf{s}\} \, d\Pr\{\mathbf{z} \mid \mathbf{X}(1) = \mathbf{s}\} \end{aligned} \tag{40}$$

Since T is a stopping time and $\mathbf{X}(n)$ is Markov, we conclude that

$$\begin{aligned} &\Pr\{\mathbf{X}(T+n) \in B \text{ for some } n \geq 1 \mid \mathbf{X}(T+1) = \mathbf{z}, \ \mathbf{X}(1) = \mathbf{s}\} \\ &= \Pr\{\mathbf{X}(n) \in B, \text{ for some } \ n \geq 1 \mid \mathbf{X}(1) = \mathbf{z}\} \\ &= 1. \end{aligned}$$

This together with (40) implies that $\Pr\{C_{l+1} \mid \mathbf{X}(1) = \mathbf{s}\} = 1$.

Since $\Pr\{\mathbf{X}(n) \in B, \text{ i.o.} \mid \mathbf{X}(1) = \mathbf{s}\} = 1$ and $|\mathcal{V}| < \infty$, starting from any state at least one of the components $j \in \mathcal{V}$ of the Markov chain visits the set A infinitely often . But this contradicts the assumption that starting from state u,

$$\lim_{n \to \infty} X_i(n) = \infty, \ i \in \mathcal{V}.$$

This completes the proof. ■

5 Conclusions

We derived the necessary and sufficient conditions for the stability of a ring with partial reuse. These conditions define implicitly the stability region of the system. Specifically, the stability region of an M-dimensional system is defined in terms of the stability regions and the steady-state average cycle lengths of $(M-1)$-dimensional systems. Therefore, in principle, if the stability region of the system with 1 node is known, the stability region of higher dimensional systems can be determined. It should be stressed, however, that the calculation of the steady-state average cycle lengths of lower dimensional stable systems is also required. While knowledge of the stability region of lower-dimensional systems is necessary for this calculation to be meaningful, the calculation itself is a whole new problem which quickly becomes intractable. As a result the stability regions are in general very complicated. We developed bounds on the stability region using a Linear Programming approach, where only the conditional average cycle lengths (not steady-state) are used.

The results presented here can be directly applied to any multi-dimensional queueing system that operates in cycles during which certain quota of packets can be transmitted by each node. The only quantity that will change is the formula for the conditional average cycle lengths which is the fundamental quantity determined by the operation of the policy for serving the various queues. Whether the developed bounds are easy to calculate depends on how easy it is to calculate the conditional average cycle lengths.

We saw that the algorithm operating with fixed quota results in reduced stability region relative to the algorithm studied in [18], where the quota vary dynamically. However, as a result of keeping the quota fixed, no node is ever blocked for a long time from transmitting its locally generated packets. In practice this is significant enough to justify some reduction in the stability region. Besides, it has been shown in [15] that if the statistics of packet destination probabilities p_{ij} are known, then the quota can be chosen so that each node acquires its required throughput. In the absence of such knowledge, the problem becomes more difficult. In [9], mechanisms have been proposed by which the nodes adjust their quota according to ring load conditions. Simulation results show that these mechanisms result in increase in throughput while still guaranteeing that a node is not blocked for a long time from transmitting its locally generated packets.

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APPENDIX: Proof of Lemma 1

In this appendix we prove Lemma 1 which we repeat below for convenience.

Lemma 1 *The functions*

$$\varphi(n) := \frac{ET(\widehat{Q}_1, n) - ET(\widehat{Q}_1, 0)}{n}, \qquad 1 \le n \le \widehat{Q}_2,$$

$$\psi(n) := \frac{ET(n, \widehat{Q}_2) - ET(0, \widehat{Q}_2)}{n}, \qquad 1 \le n \le \widehat{Q}_1,$$

are nondecreasing.

Proof. Let $d_{i,k}$ be the destination of the kth packet transmitted by node i, $i = 1, 2$, and let

$$X_k := 1_{\{d_2(k)=2\}}, \quad Y_k := 1_{\{d_1(k)=1\}}.$$

In other words, X_k is 1 if the kth packet generated by node 2 will have to go through node 1 and similarly for Y_k. We know that (see (1))

$$T(\widehat{Q}_1, n) = \max\left\{\widehat{Q}_1 + \sum_{k=1}^{n} X_k, \quad n + \sum_{k=1}^{\widehat{Q}_1} Y_k\right\} \tag{41}$$

and therefore,

$$\begin{aligned} T(\widehat{Q}_1, n) - T(\widehat{Q}_1, 0) &= \max\left\{\sum_{k=1}^{n} X_k, \quad n - \widehat{Q}_1 + \sum_{k=1}^{\widehat{Q}_1} Y_k\right\} \\ &= \max\left\{\sum_{k=1}^{n} X_k, \quad n - V\right\}, \end{aligned}$$

where $V = \widehat{Q}_1 - \sum_{k=1}^{\widehat{Q}_1} Y_k$, (therefore, $0 \le v \le \widehat{Q}_1$). Let $Z = \sum_{k=1}^{n} X_k$ and

$$W = n \max\{Z + X_{n+1}, \quad n + 1 - V\} - (n+1)\max\{Z, \quad n - V\}.$$

To show that $\varphi(\cdot)$ is nondecreasing, it is sufficient to show that for $0 \le v \le \widehat{Q}_1$,

$$E\{W | V = v\} \ge 0, \quad 1 < n < \widehat{Q}_2. \tag{42}$$

Notice first that if $V = v$ and $Z > n - v$, then

$$\begin{aligned} W &= n \max\{Z + X_{n+1}, \quad (n+1) - v\} - (n+1)Z \\ &= n \max\{X_{n+1}, \quad n - v - Z\} - Z \\ &\ge nX_{n+1} - Z \end{aligned} \tag{43}$$

On the other hand, if $V=v$, $n-v\geq 0$ and $0\leq Z\leq n-v$,

$$\begin{aligned} W &= n\max\{Z+X_{n+1},\quad n+1-v\}-(n+1)(n-v) \\ &= \max\{nX_{n+1}+n(Z-(n-v))-(n-v),\quad v\} \\ &\geq v. \end{aligned} \tag{44}$$

If $n-v\leq -1$, taking into account that $Z\geq 0$ and V is independent of $X_k,\ k=1,\cdots,\widehat{Q}_2$, we conclude from (43) that,

$$\begin{aligned} E\{W|V=v\} &= E\{W|V=v,\ Z>n-v\} \\ &\geq nEX_{n+1}-EZ \\ &= nq_{22}-nq_{22}=0 \end{aligned}$$

Assume now that $n-v\geq 0$. Then, from (43), (44) we have that

$$\begin{aligned} E\{W|V=v\} &\geq E\left(\left\{(nX_{n+1}-Z)\mathbf{1}_{\{Z>n-v\}}\right\}|V=v\right)+vE\left(\left\{\mathbf{1}_{\{0\leq Z\leq n-v\}}\right\}|V=v\right) \\ &= nq_{22}\Pr\{Z>n-v\}-E\left(Z\mathbf{1}_{\{Z>n-v\}}\right)+v\Pr\{0\leq Z\leq n-v\}, \end{aligned} \tag{45}$$

where in the last equality we used the fact that the random variable V, Z and X_{n+1} are independent. To simplify the notation set $q_{22}=q$. Since by definition $EZ=nq$, we have from (45)

$$E\{W|V=v\}\geq E\left(Z1_{\{0\leq Z\leq n-v\}}\right)-(nq-v)\Pr\{0\leq Z\leq n-v\},$$

or, setting $m=n-v$,

$$E\{W|V=v\}\geq E\left(Z1_{\{0\leq Z\leq m\}}\right)-(nq-n+m)\Pr\{0\leq Z\leq m\}, \tag{46}$$

To show that $E\{W|V=v\}\geq 0$, it is sufficient to have

$$E\left(Z1_{\{0\leq Z\leq m\}}\right)\geq(nq-n+m)\Pr\{0\leq Z\leq m\}, \tag{47}$$

or equivalently,

$$E(Z|\{0\leq Z\leq m\})\geq nq-n+m. \tag{48}$$

Recalling that $Z=\sum_{k=1}^n X_k$ and that X_k take only the values $0,1$ we have

$$\begin{aligned} E(Z|\{0\leq Z\leq m\}) &= nE\left(X_1|\left\{\sum_{k=1}^n X_i\leq m\right\}\right) \\ &= n\Pr\left\{X_1=1\middle|\left\{\sum_{k=1}^n X_k\leq m\right\}\right\} \\ &= n\frac{\Pr\{X_1=1,\quad \sum_{k=2}^n X_k\leq m-1\}}{\Pr\{\sum_{k=1}^n X_k\leq m\}} \\ &= n\frac{q\Pr\left\{\sum_{k=1}^{n-1} X_k\leq m-1\right\}}{\Pr\{\sum_{k=1}^n X_k\leq m\}} \end{aligned}$$

In the last equality we used again the fact that $X_k,\ k = 1,\dots,n$ are i.i.d. The probabilities in the last equality are related as follows:

$$\begin{aligned}
\Pr\left\{\sum_{k=1}^{n} X_k \le m\right\} &= \Pr\left\{\sum_{k=1}^{n} X_k \le m,\ \sum_{k=1}^{n-1} X_k \le m-1\right\} + \Pr\left\{\sum_{k=1}^{n} X_k \le m,\ \sum_{k=1}^{n-1} X_k \ge m\right\} \\
&= \Pr\left\{\sum_{k=1}^{n-1} X_k \le m-1\right\} + \Pr\left\{X_n = 0,\ \sum_{k=1}^{n-1} X_k = m\right\} \\
&= \Pr\left\{\sum_{k=1}^{n-1} X_k \le m-1\right\} + (1-q)\Pr\left\{\sum_{k=1}^{n-1} X_k = m\right\}.
\end{aligned}$$

Using these facts we find after some simple calculations that in order to prove (48) it is sufficient to prove

$$\left(1 - \frac{m}{n}\right)\Pr\left\{\sum_{k=1}^{n} X_k \le m\right\} \ge q(1-q)\Pr\left\{\sum_{k=1}^{n-1} X_k = m\right\} \tag{49}$$

To show that (49) holds argue as follows

$$\begin{aligned}
\frac{n-m}{n}\Pr\left\{\sum_{k=1}^{n} X_k \le m\right\} &\ge \frac{n-m}{n}\Pr\left\{\sum_{k=1}^{n} X_k = m\right\} \\
&= \frac{n-m}{n}\frac{n!}{m!(n-m)!}q^m(1-q)^{n-m} \\
&= (1-q)\frac{(n-1)!}{m!(n-m-1)!}q^m(1-q)^{n-m-1} \\
&= (1-q)\Pr\left\{\sum_{k=1}^{n-1} X_k = m\right\} \\
&\ge q(1-q)\Pr\left\{\sum_{k=1}^{n-1} X_k = m\right\}
\end{aligned}$$

which completes the proof. ∎

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